RESEARCH STATEMENT

MODULAR FORMS AND CYCLES IN LOCALLY SYMMETRIC SPACES AND SHIMURA VARIETIES:

INTERPLAY BETWEEN AUTOMORPHIC FORMS, REPRESENTATION THEORY, AND ARITHMETIC AND DIFFERENTIAL GEOMETRY

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1. Introduction

1.1. **Modular forms and number theory.** Modular forms are (holomorphic) functions on the upper half plane $\mathbb{H}$, which transform nicely (with a certain 'weight') under the action of a suitable subgroup of finite index of $SL_2(\mathbb{Z})$. Here $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ via linear fractional transformations. A famous example is the classical j-function, $j(z) = e^{-2\pi iz} + 744 + 196884e^{2\pi iz} + \dots$, which we encounter in a course on complex analysis.

During the course of the last decades modular forms have played an increasingly central role in modern number theory.

The most striking example for this hypothesis is Wiles' proof of Fermat's last theorem as a consequence of his proof of the Taniyama-Shimura conjecture. This conjecture establishes a close relationship between modular forms of weight 2 and elliptic curves over $\mathbb{Q}$ (which we can view as an equation $y^2 = P(x)$, where P is a rational monic cubic polynomial with distinct complex roots). To both a modular form f and an elliptic curve E one can associated a so-called L-function $L(f, s)$ and $L(E, s)$, and one way of stating Taniyama-Shimura is that for any rational elliptic curve there exists a modular form of weight 2 such that their L-functions coincide.

Gross and Zagier [15] relate the *derivative* $L'(E, 1)$ at the 'critical' value $s = 1$ to the so-called 'canonical height' of a certain 'Heegner point' on $E(\mathbb{Q})$. One of the spectacular consequences of this result is the completion of the solution to Gauss's class number problem for imaginary quadratic fields. More generally, the Birch-Swinnerton-Dyer conjecture relates the order of vanishing of $L(E, s)$ to the rank of $E(\mathbb{Q})$ as a finitely generated abelian group.

This kind of relationship is conjecturally part of a much more general connection between 'automorphic' objects such as modular forms and 'arithmetic' or 'motivic' objects which arise in arithmetic algebraic geometry. The investigation of this relationship is part of the far reaching Langlands program and any progress holds great promise for applications in number theory.

The arithmetic interpretation of Fourier coefficients of modular forms or the realization of formal generating series as modular forms has been traditionally of particular interest. Whenever one can realize a generating series as a modular form, one can expect significant arithmetic consequences.

A classical example is the theta series attached to the lattice $\mathbb{Z}^m \subset \mathbb{Q}^m$ with the standard positive definite bilinear form (,). For $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$, the theta series

$$\theta_{\mathbb{Z}^m}(\tau) = \sum_{x \in \mathbb{Z}^m} e^{2\pi i(x,x)\tau} = \sum_{N=1}^{\infty} r_m(N) q^N, \tag{1.1}$$

is a modular form of weight $m/2$. Here $r_m(N) = \#\{x \in \mathbb{Z}^m : (x, x) = N\}$ is the representation number of N as the sum of m squares. This is an example of a so-called theta lift from the orthogonal group $O(m)$ to SL_2. This relation has been used extensively to study the arithmetic of quadratic forms.

Generally speaking, my research is concerned with finding analogues for (1.1) for *indefinite* quadratic forms (where representation numbers in a naive sense do not make sense) with geometric and arithmetic interpretations and applications similarly to the ones described above.

1.2. **Generating series in arithmetic and differential geometry.** My interest is in modular and automorphic forms whose Fourier coefficients have interpretations in differential and arithmetic algebraic geometry. My focus is on using the theta correspondence and theta lifts for $O(p, q)$ and $U(p, q)$ to study (the arithmetic properties of) certain 'special' cycles in locally symmetric spaces and Shimura varieties of orthogonal and unitary type. The cycles arise from subsymmetric spaces of codimension nq by embeddings of orthogonal/unitary groups of signature $(p - n, q)$. In particular, I am interested in generating series of cohomological intersection numbers in locally symmetric spaces and of intersection numbers in the arithmetic Chow groups of Shimura varieties.

This subject originates from the celebrated work of Hirzebruch and Zagier [17] in the case of a Hilbert modular surface, where the generating series of the now-called Hirzebruch-Zagier divisors is realized as a modular form of weight 2.

In [19], Kudla and Millson define (in greater generality) a theta lift from the orthogonal group $O(p,q)$ to the symplectic group $Sp(n)$. This map realizes generating series of intersection numbers of the special cycles as holomorphic Siegel modular forms. In joint work with Millson [13], I considered the more general situation where the cycles have local coefficients. We now obtain a lift from the cohomology with non-trivial coefficients in suitable representation spaces for $O(p,q)$ to *vector-valued* Siegel modular forms. Furthermore, this correspondence can be explicitly given in terms of the highest weights of $O(p,q)$ and $Sp(n)$. In recent work [14], we systematically studied the boundary behavior of these cohomology classes. This will play a major role in extending the Kudla-Millson lift to the full cohomology of the locally symmetric classes in question.

Borcherds [1] introduced a 'singular' theta lift from SL_2 to $O(p,2)$, which gives rise to remarkable product expansions of automorphic forms for $O(p,2)$ ('Borcherds products'). In joint work with Bruinier [2], I obtained an adjointness result between the Borcherds and the Kudla-Millson lift, and in [4] we use this relationship to discuss surjectivity and injectivity for these lifts. Recently [10], I have constructed an analogous singular theta lift for the unitary group $U(p,q)$.

Kudla, Rapoport and Yang (see e.g., [18, 20, 21]) have established a general program to realize generating series in arithmetic geometry for certain Shimura varieties associated to $O(p,2)$ as modular forms. This program has received considerable attention and can be partly be viewed as a systematic strategy to find analogues and generalizations of the Gross-Zagier formula. In joint work with Bruinier [3], I have used a certain theta lift to recover the result of [22] on the generating series of the Faltings heights of CM points in a modular curve as the derivative of Zagier's well-known Eisenstein series of weight 3/2 [23].

2. Traces of CM values of modular functions

To illustrate some of my work in this area, we focus for the remainder on one particular case, namely an isotropic quadratic space (V,Q) of three variables with signature $(1,2)$. This case is quite accessible, while already providing several interesting features of the subject.

Assume the quadratic form is given by $Q(a,b,c)=4ac-b^2$. Let L be a lattice in V and let Γ be the stabilizer of L in the orthogonal group of V. By the special isomorphism $\mathrm{Spin}(1,2)\simeq SL_2(\mathbb{R})$, we can think of $\Gamma\subseteq SL_2(\mathbb{R})$. Now L_D, the set of lattice vectors of length D, is no longer finite (in contrast to the positive definite case). However, $\Gamma\backslash L_D$, the set of Γ-equivalence classes of L_D, is finite for $D\neq 0$, and its order $|\Gamma\backslash L_D|$ is the analogue of representation numbers. In fact, these numbers are generalized class numbers since for $L=\mathbb{Z}^3$ (and $\Gamma=SL_2(\mathbb{Z})$), we obtain the class number $H(D)$ of positive definite integral binary quadratic forms $aX^2+bXY+cY^2$ of discriminant $-D=b^2-4ac$. In [8], I show that the generating series $\sum_{D=0}^{\infty}|\Gamma\backslash L_D|q^D$ is the holomorphic part of a nonholomorphic modular form of weight 3/2. In particular, as a special case, I recovered Zagier's famous weight 3/2 Eisenstein series $\mathcal{F}(\tau,s)$ at $s=1/2$.

Theorem 2.1 ([23, 8])**.** *Set* $H(0)=-1/12$. *Then*

$$\sum_{D=0}^{\infty}H(D)q^D+\frac{v^{-1/2}}{16\pi}\sum_{D=-\infty}^{\infty}\beta(4\pi D^2v)q^{-D^2}=\mathcal{F}(\tau,\tfrac{1}{2}) \tag{2.1}$$

is a non-holomorphic modular form of weight 3/2. *Here* $\tau=u+iv\in\mathbb{H}$ *and* $\beta(s)=\int_1^{\infty}t^{-3/2}e^{-st}\frac{dt}{t}$.

My approach to this result is by studying the theta integral

$$I(\tau,1)=\int_{\Gamma\backslash\mathbb{H}}1\cdot\theta_L(\tau,z,\varphi)\,\frac{dx\,dy}{y^2}, \tag{2.2}$$

integrating the constant function 1 against a theta series associated to L (and a certain Schwartz function φ coming from [19]). Here $z=x+iy\in\mathbb{H}$. This can be thought of as the analogue of the positive definite theta series (1.1). This integral gives rise to a form of weight 3/2 in τ and for a general lattice L, these modular forms take the same form as in (2.1). As an arithmetic application one can easily obtain from this work Gauss' famous formula $r_3(N)=12(H(4N)-2H(N))$.

CM points are quadratic irrationalities on the upper half plane $\mathbb{H}$. In our context, a vector $(a, b, c) \in L_D$ of positive length D gives rise to a CM point $\frac{-b+i\sqrt{D}}{2a}$ (which is the zero in $\mathbb{H}$ of the quadratic equation $aX^2 + bX + C$ with discriminant $-D < 0$). We let $Z(D)$ be a set of the Γ-equivalence classes of CM points of discriminant D. The name is arising from the fact, that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is also the moduli space of elliptic curves over $\mathbb{C}$, and the CM points correspond to those curves with complex multiplication.

For a Γ-invariant function f on $\mathbb{H}$, we define its modular trace of index D by

$$\mathbf{t}_f(D) = \sum_{z\in Z(D)} \frac{1}{|\Gamma_z|} f(z), \tag{2.3}$$

where Γ_z is the stabilizer of $z \in \mathbb{H}$ in Γ. Note $\mathbf{t}_1(D) = \deg Z(D) = |\Gamma\backslash L_D|$.

Zagier [24] considers the so-called Hauptmodul $J(z) = j(z) - 744$ for $\Gamma = SL_2(\mathbb{Z})$. The values of j at the CM points are known as singular moduli. They play an important role in many branches of number theory. For example, they are algebraic integers and generate the Hilbert class field for $\mathbb{Q}(\sqrt{-D})$. Zagier shows that the generating series

$$-q^{-1} + 2 + \sum_{D=1}^{\infty} \mathbf{t}_J(D)q^D = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 + \dots \tag{2.4}$$

is a modular form of weight 3/2 (with poles only at the cusps). We generalize Zagier's work by considering $I(\tau, f)$, see (2.2), for any modular function f of weight 0 on $\mathbb{H}$ for Γ. For example, let p be a prime and $L = \{(a, b, c) : p|a\}$. Then we can take $\Gamma = \Gamma_0^*(p)$ be the extension of the Hecke subgroup $\Gamma_0(p)$ by the Fricke involution W_p. Let $\sigma_1(0) = -\frac{1}{24}$ and $\sigma_1(n) = \sum_{t|n} t$ for $n \in \mathbb{Z}_{>0}$.

Theorem 2.2 (Bruinier-F.[3])**.** *Let $f(z) = \sum_{n\gg-\infty} a(n)e^{2\pi i n z}$ be a modular function for $\Gamma_0^*(p)$. Assume that the constant coefficient $a(0)$ vanishes. Then*

$$I(\tau, f) = \sum_{D>0} \mathbf{t}_f(D)q^D + \sum_{n\geq 0} \big(\sigma_1(n) + p\sigma_1(n/p)\big)a(-n) - \sum_{m>0}\sum_{n>0} m a(-mn)q^{-m^2}$$

is a meromorphic modular form of weight 3/2 with poles concentrated at the cusps.

Bruinier and I also study $I(\tau, \log\|\Delta(z)\|)$, where $\|\Delta(z)\| = |y^6\Delta(z)|$ is the Petersson metric of the famous discriminant function, $\Delta(z) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$, which is a modular form of weight 12. Using the Kronecker limit formula one obtains $-\frac{1}{12}I(\tau, \log\|\Delta(z)\|) = \mathcal{F}'(\tau, \frac{1}{2})$, where $\mathcal{F}'(\tau, \frac{1}{2})$ is the *derivative* of Zagier's Eisenstein series at $s = \frac{1}{2}$. On the other hand, $I(\tau, \log\|\Delta\|)$ can be interpreted in terms of arithmetic geometry recovering the main result of [22]. We let $\mathcal{M}$ be the Deligne-Rapoport compactification of the moduli stack over $\mathbb{Z}$ of elliptic curves, so that $\mathcal{M}(\mathbb{C})$ is $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\cup\infty$. For $D \in \mathbb{Z}$ and $v > 0$, Kudla, Rapoport and Yang [20, 22] construct cycles $\widehat{\mathcal{Z}}(D, v)$ in the extended arithmetic Chow group of $\mathcal{M}$ with real coefficients $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$, see [5]. For $D > 0$, the complex points of the underlying divisor of $\widehat{\mathcal{Z}}(D, v)$ are the CM points $Z(D)$ in $\mathbb{H}$. We let $\widehat{\omega}$ be the normalized metrized Hodge bundle on $\mathcal{M}$, which defines an element $\widehat{c}_1(\widehat{\omega}) = \frac{1}{12}(\infty, -\log\|\Delta(z)\|^2)$ in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$. Finally, we let $\langle\,,\,\rangle$ be the Gillet-Soulé intersection pairing. The D-th Fourier coefficient of $-\frac{1}{12}I(\tau, \log\|\Delta(z)\|)$ turns out to be equal to $4\langle\widehat{\mathcal{Z}}(D, v), \widehat{\omega}\rangle$.

Theorem 2.3 ([22, 3])**.** *We have*

$$\sum_{D\in\mathbb{Z}} \langle\widehat{\mathcal{Z}}(D, v), \widehat{\omega}\rangle q^D = \frac{1}{4}\mathcal{F}'(\tau, \frac{1}{2}). \tag{2.5}$$

Note that for $D > 0$ the trace $\mathbf{t}_D(\log\|\Delta\|)$ is the Faltings height of the cycles $\widehat{\mathcal{Z}}(D, v)$ and is the main piece of $\langle\widehat{\mathcal{Z}}(D, v), \widehat{\omega}\rangle$. Our proof is quite different from the one given in [22] which relies on the explicit calculation and comparison of the Fourier coefficients of both sides of (2.5). For the analogue in the Shimura curve case (which is considerably more complicated), see [20].

The *norm* $\mathbf{n}_j(D) = \prod_{z\in Z(D)} j(z)$ of the singular moduli is also of particular interest and has been studied by Gross-Zagier [16]. Since $\log|\mathbf{n}_j(D)| = \mathbf{t}_{\log|j|}(D)$, we study $I(\tau, \log|j|)$. This involves $\mathbf{t}_{\log|j|}(D)$, however with one delicate twist: If $D = 3m^2$, then $\frac{1}{3}\{\omega := e^{2\pi i/3}\} \in Z(D)$, where j vanishes. Hence $\log|j(z)|$ is not defined at that point. We only describe the D-th Fourier coefficient for $D > 0$ with $D \neq 3m^2$.

Theorem 2.4 (F.[9])**.**

$$I(\tau, \log|j|) = \sum_{\substack{D>0 \\ D\neq 3m^2}} \log|\mathbf{n}_j(D)|q^D + \dots$$

Applying the holomorphic projection principle to $I(\tau, \log|j|)$ should yield the formulas of Gross-Zagier for the singular moduli (but we haven't carried this out yet).

3. Current and future plans and projects

I outline several projects I am currently engaged in.

Project I: The theta lift of a Borcherds form

The discriminant function Δ and the j-invariant both occur in Borcherds theory as a Borcherds lift $\Psi(z,f)$ for the dual pair $SL_2 \times O(1,2)$ of a modular form f of weight 1/2.

In joint work with Bruinier, we study the theta lift $I(\tau, \log\|\Psi(f)\|)$ of the Petersson metric of the Borcherds product $\Psi(f)$. This lift should be of considerable interest in view of Kudla's program, since the Fourier coefficients of the resulting modular form involve a priori (part of) the Archimedean height pairing of certain cycles. In particular, it will be of particular interest in the Shimura curve case to compare $I(\tau, \log\|\Psi(f)\|)$ with the generating series of Kudla, Rapoport and Yang [20]. We will also study analogous lifts in the higher dimensional case for $O(p,2)$.

Project II: Generalization of Zagier's Eisenstein series to higher Siegel genus

This is a joint project with Kudla. In order to generalize Zagier's Eisenstein series (2.1), we will consider the analogous theta integral as in [8] for a non-compact quotient X of the Hermitian domain attached to $O(n,2)$. This gives rise to a Siegel modular form of genus n of weight $\frac{n+2}{2}$. This integral, while being convergent, is not absolutely termwise convergent, and its computation will require some 'regularization' process as for $O(1,2)$. Again, this should lead to interesting non-holomorphic terms associated to the cusps, while the positive definite Fourier coefficients we expect to be the degree of certain 0-cycles in X, which are analogues of CM points.

Project III: A singular theta lift and arithmetic generating series for unitary groups

Motivated by the results of my joint paper [2] with Bruinier, I used its main idea - utilization of the Kudla-Millson lift - to introduce an additive singular theta lift of Borcherds type for the dual pair $U(p,q) \times U(1,1)$ [10]. In particular, this new singular theta lift gives rise to Green currents for the special cycles (which now are no longer divisors, but of complex codimension q). In joint work with Bruinier we will further investigate this lift. In particular, we will compute its Fourier-Jacobi expansion. Furthermore, for $U(p,1)$, we can expect to have a Borcherds product when the cycles are divisors. We expect this work to be significant in extending Kudla's program on generating series in arithmetic geometry to Shimura varieties associated to unitary groups.

Project IV: 0-cycles in hyperbolic manifolds and modular forms

In joint work with O. Imamoglu [11], we consider certain 0-cycles in arithmetic quotients of hyperbolic manifolds, which are the exact analogs of the CM points in the upper half plane. We consider an analogous theta lift as in section 2, again with varying input. In particular, we obtain analogues of Cohen's Eisenstein series of half-integral weight [6] whose Fourier coefficients are given by special values of L-series. We show that these values can be interpreted as class numbers of these 0-cycles in the hyperbolic manifold. Furthermore, we consider the question of equi-distribution of these cycles, generalizing work of Duke [7] to this situation. Note that in this case the powerful ergodic theoretic methods do not apply which recently have been developed to address these kind of questions.

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