Jørgensen's inequality for metric spaces with application to the octonions

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Abstract

Jørgensen's inequality gives a necessary condition for the discreteness of a non-elementary group of isometries of hyperbolic 3-space. The main idea of the proof may be generalised widely but the statement is quite specialised. Here we give a scheme for restating Jørgensen's inequality for Möbius transformations of a metric space. This unifies many previously published versions of Jørgensen's inequality. We then show how this scheme may be applied by giving a version of Jørgensen's inequality for the octonionic hyperbolic plane.

1 Introduction

In [7], Jørgensen gave a famous inequality giving a necessary condition on a pair of matrices in $\mathrm{SL}(2,\mathbb{C})$ that generate a non-elementary discrete group. This result and its proof have been generalised to other matrix groups, see for example [9], [14], [6], [11]. These matrix groups are all isometry groups of rank 1 symmetric spaces of non-compact type. In this paper we give a scheme for generalising Jørgensen's inequality that unifies these results. We go on to demonstrate that this leads to a generalisation of Jørgensen's inequality to $F_{4(-20)}$, the isometry group of the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$. In order to do so we cannot use matrix methods because of the non-associativity of the octonions.

Our approach is to consider Jørgensen's inequality not as a statement about matrices, but as a statement about Möbius transformations acting on the Riemann sphere. It is well known that a Möbius transformation can be decomposed into (orientation preserving) Euclidean isometries, dilations $z \longmapsto d^2 z$ for $d > 0$ and the involution $z \longmapsto -1/z$. The latter map preserves the unit circle and interchanges its interior and exterior. (Alternatively one may use both orientation preserving and reversing Euclidean isometries, dilations and the inversion in the unit circle $z \longmapsto 1/\overline{z}$.) The following identities are obvious.

$$|d^2 z - d^2 w| = d^2 |z - w|, \tag{1}$$

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}, \tag{2}$$

$$\left|\frac{1}{z} - \frac{1}{w}\right| = \frac{|z-w|}{|z|\,|w|}, \tag{3}$$

$$|az - z| = |a-1|\,|z|. \tag{4}$$

From the metric space viewpoint, the proof of Jørgensen's inequality only uses these four identities and well known facts about the Euclidean metric on $\mathbb{C}$.

In Section 2 we consider more general metric spaces X with dilations and an inversion satisfying conditions analogous to these identities. The group generated by the isometries of X, the dilations and the inversion will be called the *Möbius group* of X, written $\text{Möb}(X)$. We show how to extend Jørgensen's inequality to subgroups of $\text{Möb}(X)$ with a loxodromic generator, Theorem 2.4. We outline how this approach unifies several previously known versions of Jørgensen's inequality in a variety of settings [5], [6], [7], [9] and [14].

In Section 3 we concentrate on the octonions. The framework of Section 2 gives us a context to generalise Jørgensen's inequality to discrete groups of octonionic hyperbolic isometries. The boundary of the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$ is the one point compactification of a 15 dimensional nilpotent Lie group $\mathcal{H}^{15}$. Moreover, by giving $\mathcal{H}^{15}$ a suitable metric, we may realise the group $F_{4(-20)}$ as $\text{Möb}(\mathcal{H}^{15})$. This enables us to give a version of Jørgensen's inequality for subgroups of $F_{4(-20)}$ with a certain type of loxodromic generator, Theorem 3.6, which is the main result of Section 3.

2 Jørgensen's inequality

2.1 Möbius transformations of a metric space

Let X be a complete metric space with metric ρ_0. Let $\text{Aut}(X) \subset \text{Isom}(X)$ be a group of isometries of X. This may be either the full isometry group or a sufficiently large subgroup that preserves some extra structure on X. For example, $\text{Aut}(X)$ may be the group of orientation preserving or holomorphic isometries. We will suppose that $\text{Aut}(X)$ acts transitively on X. Let o be a distinguished point of X. (Since $\text{Aut}(X)$ acts transitively, in fact we may take o to be any point of X.) Suppose that the stabiliser of o in $\text{Aut}(X)$ is compact (with respect to the compact-open topology). We make some more assumptions about X that allow us to extend $\text{Aut}(X)$ to the group of Möbius transformations on X.

Suppose that for each $d \in \mathbb{R}_+$ there is a map $D_d : X \longrightarrow X$ so that $D_d o = o$ and for all $z,\, w \in X$ we have

$$\rho_0(D_d z, D_d w) = d^2 \rho_0(z, w). \tag{5}$$

We call D_d the *dilation* with *dilation factor* $d^2 \in \mathbb{R}_+$. (It may seem more natural to have taken d rather than d^2. However that would have introduced square roots into our formulae, such as (10) below.) This formula is the natural generalisation of (1) and holds for a wide class of metric spaces, such as Carnot-Carathéodory spaces; see page 39 of [12].

Let $X \cup \{\infty\}$ be the one point compactification of X. Suppose that there is an involution R interchanging o and ∞ and so that if $z,\, w \in X - \{o\}$ then

$$\rho_0(Rz, o) = \frac{1}{\rho_0(z, o)}, \tag{6}$$

$$\rho_0(Rz, Rw) = \frac{\rho_0(z, w)}{\rho_0(z, o)\rho_0(w, o)}. \tag{7}$$

We may think of R as reflection in the unit sphere of centre $o \in X$. There certainly exists such a map R if the one point compactification of X is the boundary of a rank 1 symmetric space of non-compact type. For $\mathbb{R}^n$ and the usual Heisenberg group we give details in the examples in 2.2 and for $\mathcal{H}^{15}$ we prove this in Proposition 3.5. When X is the generalised Heisenberg group with three dimensional centre whose one point compactification gives the boundary of quaternionic hyperbolic space, then it is easy to show that when R is the involution given on page 296 of [10], the identities (6) and (7) hold with the natural Cygan metric. It is not clear to us which other metric spaces (if any) admit such an inversion R.

The conditions (5), (6) and (7) generalise the identities (1), (2) and (3) given in the introduction. Let $\text{Möb}(X)$ be the group generated by $\text{Aut}(X)$, D_d and R for all $d \in \mathbb{R}_+$. We call $\text{Möb}(X)$ the group of *Möbius transformations* of X.

Proposition 2.1 *Let X be a metric space and $\text{Möb}(X)$ be the group generated by $\text{Aut}(X)$, D_d and R satisfying (5), (6) and (7).*

(i) Let A be any element of $\text{Möb}(X)$ for which $A\infty = \infty$. Then there exists a positive number d_A so that for all $z,\, w \in X$

$$\rho_0(Az, Aw) = {d_A}^2 \rho_0(z, w).$$

(ii) Let B be any element of $\text{Möb}(X)$ for which $B\infty \neq \infty$. Then there exists a positive number r_B so that for all $z,\, w \in X - \{B^{-1}\infty\}$

$$\rho_0(Bz, Bw) = \frac{{r_B}^2 \rho_0(z, w)}{\rho_0(z, B^{-1}\infty)\rho_0(w, B^{-1}\infty)},$$

$$\rho_0(Bz, B\infty) = \frac{{r_B}^2}{\rho_0(z, B^{-1}\infty)}.$$

Proof: Any element of Möb(X) can be written as a word in isometries of X, dilations D_d and the inversion R. We prove the result by induction on the length of these words. From (5), (6), (7) above we see that the result holds for all $A \in \mathrm{Aut}(X)$ (with $d_A = 1$), for D_d (with $d_{D_d} = d$) and for R (with $r_R = 1$) since $o = R^{-1}\infty$.

Suppose that $A \in$ Möb(X) with $A\infty = \infty$ and suppose that A satisfies the condition (i) above. Then it is clear that AD_d and AR satisfy the conditions of (i) with $d_{AD_d} = d_A d$ and (ii) with $r_{AR} = d_A$ respectively.

Now suppose that $B \in$ Möb(X) with $B\infty \neq \infty$ and suppose that B satisfies the conditions of (ii) above. Consider BA, BD_d, BR for $A \in \mathrm{Aut}(X)$ and $d > 0$. Clearly $BA\infty \neq \infty$ and $BD_d\infty \neq \infty$. If z, $w \neq {D_d}^{-1}B^{-1}\infty$ we have

$$\begin{aligned}
\rho_0(BD_dz, BD_d\infty) &= \rho_0(BD_dz, B\infty) \\
&= \frac{{r_B}^2}{\rho_0(D_dz, B^{-1}\infty)} \\
&= \frac{(r_B/d)^2}{\rho_0(z, {D_d}^{-1}B^{-1}\infty)}
\end{aligned}$$

Similarly,

$$\rho_0(BD_dz, BD_dw) = \frac{(r_B/d)^2\rho_0(z,w)}{\rho_0(z, {D_d}^{-1}B^{-1}\infty)\rho_0(w, {D_d}^{-1}B^{-1}\infty)}.$$

In the same way, if z, $w \neq A^{-1}B^{-1}\infty$ we have

$$\begin{aligned}
\rho_0(BAz, BA\infty) &= \frac{{r_B}^2}{\rho_0(z, A^{-1}B^{-1}\infty)}, \\
\rho_0(BAz, BAw) &= \frac{{r_B}^2\rho_0(z,w)}{\rho_0(z, A^{-1}B^{-1}\infty)\rho_0(w, A^{-1}B^{-1}\infty)}.
\end{aligned}$$

Suppose that $BR\infty \neq \infty$ and z, $w \neq RB^{-1}\infty$. Then

$$\begin{aligned}
\rho_0(BRz, BR\infty) &= \rho_0(BRz, Bo) \\
&= \frac{{r_B}^2\rho_0(Rz, o)}{\rho_0(Rz, B^{-1}\infty)\rho_0(o, B^{-1}\infty)} \\
&= \frac{\big(r_B\rho_0(RB^{-1}\infty, o)\big)^2\rho_0(z,o)}{\rho_0(z,o)\rho_0(z, RB^{-1}\infty)} \\
&= \frac{\big(r_B\rho_0(RB^{-1}\infty, o)\big)^2}{\rho_0(z, RB^{-1}\infty)}.
\end{aligned}$$

Similarly

$$\rho_0(BRz, BRw) = \frac{\big(r_B\rho_0(RB^{-1}\infty, o)\big)^2\rho_0(z,w)}{\rho_0(z, RB^{-1}\infty)\rho_0(w, RB^{-1}\infty)}.$$

Finally, suppose that $BR\infty = \infty$, that is $Bo = \infty$. Then for z, $w \neq o$ we have

$$\begin{aligned}
\rho_0(BRz, BRo) &= \rho_0(BRz, B\infty) \\
&= \frac{{r_B}^2}{\rho_0(Rz, B^{-1}\infty)} \\
&= \frac{{r_B}^2}{\rho_0(Rz, o)} \\
&= {r_B}^2 \rho_0(z, o).
\end{aligned}$$

Similarly,

$$\rho_0(BRz, BRw) = {r_B}^2 \rho_0(z, w).$$

□

Using Proposition 2.1(ii) we see that for all $B \in \text{Möb}(X)$ with $B\infty \neq \infty$ we have

$$\frac{\rho_0(Bz, z)}{\rho_0(Bz, B\infty)} = \frac{\rho_0(z, B^{-1}z)}{\rho_0(B^{-1}z, B^{-1}\infty)}. \tag{8}$$

Also, we may regard B as inversion in a sphere of radius r_B centred at $B^{-1}\infty$ followed by an isometry taking $B^{-1}\infty$ to $B\infty$ (compare with Proposition 2.4 of [8]).

Lemma 2.2 *If* $\text{Aut}(X)$ *acts transitively on* X *then* $\text{Möb}(X)$ *acts 2-transitively on* $X \cup \infty$*. That is, given any two pairs* p_1*,* q_1*;* p_2*,* q_2 *of points in* $X \cup \{\infty\}$ *then there exists* $B \in G$ *so that* $B(p_2) = p_1$ *and* $B(q_2) = q_1$*.*

Proof: It suffices to consider the case of $p_2 = \infty$ and $q_2 = o$. Choose A_1 and A_2 in $\text{Aut}(X)$ so that $A_1 o = p_1$ and $A_2 o = R{A_1}^{-1}q_1$. Taking $B = A_1 R A_2$ gives

$$\begin{aligned}
&B\infty = A_1 R A_2 \infty = A_1 R \infty = A_1 o = p_1, \\
&Bo = A_1 R A_2 o = A_1 R(R{A_1}^{-1} q_1) = q_1.
\end{aligned}$$

□

Suppose that $A \in \text{Möb}(X)$ fixes ∞. From Proposition 2.1 (i), there exists ${d_A}^2 > 0$, which we call the *dilation factor* of A, so that $\rho_0(Az, Aw) = {d_A}^2 \rho_0(z, w)$. Then A is said to be *loxodromic* if and only if $d_A \neq 1$. Clearly $d_{A^{-1}} = 1/d_A$. By the contraction mapping theorem (replacing A by A^{-1} if necessary) we see that such an A must have a unique fixed point in X. Using transitivity we may conjugate A so that this fixed point is $o \in X$.

An element B of $\text{Möb}(X)$ is said to be *loxodromic* if it is conjugate in $\text{Möb}(X)$ to a loxodromic map $A \in \text{Möb}(X)$ with $A\infty = \infty$.

Lemma 2.3 *Suppose that* $B \in \text{Möb}(X)$ *fixes distinct points* p*,* $q \in X \cup \{\infty\}$*. Then* B *is conjugate to* $A \in \text{Möb}(X)$ *with fixed points* o *and* ∞*. Moreover, the dilation factor* ${d_A}^2$ *of* A *is independent of the conjugating map.*

Proof: The first part follows from the 2-transitivity of the action of Möb(X) on $X \cup \{\infty\}$. For the second part, suppose that $C_1 B {C_1}^{-1} = A_1$ and $C_2 B {C_2}^{-1} = A_2$ where A_j fixes o and ∞, for $j = 1, 2$. Since $C_1^{-1}\infty = C_2^{-1}\infty$ we see that the map $C = C_2 C_1^{-1}$ fixes ∞. So C has a dilation factor, denoted ${d_C}^2$, and its inverse C^{-1} has dilation factor ${d_{C^{-1}}}^2 = 1/{d_C}^2$. Then $A_2 = C_2 {C_1}^{-1} A_1 C_1 {C_2}^{-1} = C A_1 C^{-1}$ and arguing as in the proof of Proposition 2.1 we see that A_2 has dilation factor ${d_{A_2}}^2 = {d_C}^2 {d_{A_1}}^2 {d_{C^{-1}}}^2 = {d_{A_1}}^2$. □

Define the *cross-ratio* of quadruples of points in $X \cup \{\infty\}$ by

$$\begin{aligned} \mathbb{X}(z_1, z_2; w_1, w_2) &= \frac{\rho_0(w_1, z_1)\rho_0(w_2, z_2)}{\rho_0(w_2, z_1)\rho_0(w_1, z_2)}, \\ \mathbb{X}(z_1, z_2; \infty, w_2) &= \frac{\rho_0(w_2, z_2)}{\rho_0(w_2, z_1)}. \end{aligned}$$

The cross-ratio is a familiar object for $X = \mathbb{C}$ and may be defined similarly for $\mathbb{R}^n$; see §1.1 of Kellerhals [9]. This definition has been generalised to the boundaries of other rank 1 symmetric spaces of non-compact type. For example, when X is the standard Heisenberg group then the definition we give above may be found in Basmajian and Miner [4]; and when X is the generalised Heisenberg group whose one point compactification is the boundary of quaternionic hyperbolic space, the definition is given in §4.1 of [10]. Using Proposition 2.1 it is not hard to show that the cross-ratio of four points is preserved by the action of Möb(X).

Let A be a loxodromic element of Möb(X) fixing $p, q \in X \cup \{\infty\}$ and with dilation factor ${d_A}^2$. Suppose that m_A is a positive number so that for all points $z \in X \cup \{\infty\} - \{p, q\}$ we have

$$\mathbb{X}(p, Az; q, z) \leq d_A m_A. \tag{9}$$

This is a conjugation invariant statement of the following inequality in the special case when $p = o$ and $q = \infty$:

$$\rho_0(z, Az) \leq d_A m_A \rho_0(o, z). \tag{10}$$

This inequality generalises the identity (4) in the introduction. Observe that combining (10) with Proposition 2.1 gives

$$\rho_0(z, A^{-1}z) \leq {d_A}^{-1} m_A \rho(z, o)$$

and so $m_{A^{-1}} = m_A$. We remark that such an m_A always exists. For example using $Ao = o$ and the triangle inequality, we obtain

$$\rho_0(z, Az) \leq \rho_0(z, o) + \rho_0(o, Az) = d_A(d_A + 1/d_A)\rho_0(z, o).$$

Thus one may always take $m_A = d_A + 1/d_A \geq 2$. However, for Jørgensen's inequality we will only be interested in loxodromic maps with $m_A < 1$.

2.2 The main theorem

The main result of Section 2 is the following version of Jørgensen's inequality for groups of Möbius transformations on a metric space X, as defined above. We then demonstrate that this theorem unifies the loxodromic case of many versions of Jørgensen's inequality.

Theorem 2.4 *Let X be a complete metric space and suppose that* $\mathrm{Aut}(X)$ *acts transitively on X with compact stabilisers. Suppose that* $\mathrm{M\ddot{o}b}(X)$*, the group of Möbius transformations on X, satisfies hypotheses (5), (6) and (7). Let A be a loxodromic element of* $\mathrm{M\ddot{o}b}(X)$ *with fixed points p and q and let m_A be a positive number satisfying (9). If Γ is a discrete subgroup of* $\mathrm{M\ddot{o}b}(X)$ *containing A, then for all $B \in \Gamma$ so that $\{Bp,\, Bq\} \cap \{p,\, q\} = \emptyset$ we have*

$$m_A{}^2 \Big(\mathbb{X}(Bp, q; p, Bq) + 1 \Big) \geq 1. \tag{11}$$

We now show that Theorem 2.4 generalises the case of a loxodromic generator in Jørgensen's original inequality and some of its generalisations.

Examples

(i) Let X be $\mathbb{C}$ and ρ_0 be the Euclidean metric. We take $\mathrm{Aut}(\mathbb{C})$ to be $\mathbb{C} \rtimes \mathrm{U}(1)$, the group of holomorphic (or equivalently orientation preserving) Euclidean isometries. Then $\mathbb{C} \cup \{\infty\}$ is the Riemann sphere and $\mathrm{M\ddot{o}b}(\mathbb{C})$ is $\mathrm{PSL}(2, \mathbb{C})$ acting by Möbius transformations. This is the group of orientation preserving isometries of (real) hyperbolic 3-space $\mathbf{H}^3_{\mathbb{R}}$. The conditions of Proposition 2.1 follow easily, as in Section 1. If $\lambda \in \mathbb{C} - \{0\}$ then set

$$Az = \frac{\lambda z}{\lambda^{-1}}, \quad Bz = \frac{az + b}{cz + d}$$

with $ad - bc = 1$ then $d_A = |\lambda|$ and $m_A = |\lambda - \lambda^{-1}|$. The condition (11) of Theorem 2.4 becomes

$$\begin{aligned} 1 &\leq |\lambda - \lambda^{-1}|^2 \left(\frac{|b/d|}{|b/d - a/c|} + 1 \right) \\ &= |\lambda - \lambda^{-1}|^2 \big(|bc| + 1 \big) \\ &= \big|\mathrm{tr}(A)^2 - 4\big| + \big|\mathrm{tr}[A, B] - 2\big|. \end{aligned}$$

This is just the original statement of Jørgensen [7] (see also equation (1.21) of [9]).

(ii) Let X be the quaternions $\mathbb{H}$ with the Euclidean metric ρ_0. Kellerhals gives conditions on quaternions a, b, c, d so that $Bz = (az+b)(cz+d)^{-1}$ is a Möbius transformation (Lemma 1 of [9]). The group of such Möbius transformations forms the orientation preserving isometry group of real hyperbolic 5-space $\mathbf{H}^5_{\mathbb{R}}$. Let μ and ν be any purely imaginary quaternions, $l > 1$ be a real number and α, β be angles in $[0, \pi]$ then take

$$Az = e^{l/2} \exp(\mu\alpha) z \big(e^{-l/2} \exp(-\nu\beta) \big)^{-1}, \quad Bx = (az + b)(cz + d)^{-1}.$$

Clearly $d_A = e^{l/2}$ and Kellerhals shows (in the proofs of Propositions 3 and 4 of [9]) that $m_A = \big(2\cosh(l) - 2\cos(\alpha+\beta)\big)^{1/2}$ and $\mathbb{X}(B\infty, o; \infty, Bo) = |bc|$. The condition (11) becomes

$$\big(2\cosh(l) - 2\cos(\alpha+\beta)\big)\big(|bc|+1\big) \geq 1$$

which is equation (1.23) of [9]. Thus Theorem 2.4 is a restatement of [9] Proposition 3.

(iii) Let $X = \mathbb{R}^n$ with the Euclidean metric ρ_0 and let $\mathrm{Aut}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$ be the group of orientation preserving Euclidean isometries. Then we may represent elements of $\mathrm{M\ddot{o}b}(\mathbb{R}^n)$ as $Bz = (az+b)(cz+d)^{-1}$ where B is in $\mathrm{SL}(\Gamma_n)$, the group of 2×2 matrices with elements in the Clifford group Γ_n satisfying Ahlfors' conditions, Definition 2.1 of [1] (see also Waterman [14]). These Möbius transformations are orientation preserving isometries of $\mathbf{H}^{n+1}_{\mathbb{R}}$. For $\lambda \in \Gamma_n - \{0\}$ and a, b, c, d satisfying Ahlfors' conditions, take

$$Az = \lambda z \lambda^*, \quad Bz = (az+b)(cz+d)^{-1}.$$

Then, as above $d_A = |\lambda|$ and $m_A = \sup\big|\lambda - x\lambda^{*-1}x^{-1}\big|$ where the supremum is taken over all $x \in \mathbb{R}^n - \{0\}$ (compare Lemma 6.20 of [5]). Waterman (Theorem 9 of [14]) defines this quantity to be $\big|\lambda - \widetilde{\lambda}/|\lambda|^2\big|$. Moreover, (6) and (7) are just restatements of Theorem 7(iv) and Lemma 6(i) of [14], respectively. Also, using Ahlfors' conditions, it is easy to show that

$$\mathbb{X}(B\infty, o; \infty, Bo) = \frac{|bd^{-1}|}{|ac^{-1} - bd^{-1}|} = \frac{|bd^{-1}|}{|(ad^* - bc^*)c^{*-1}d^{-1}|} = |bc|.$$

Then the condition (11) of Theorem 2.4 is

$$\Big|\lambda - \widetilde{\lambda}/|\lambda|^2\Big|^2\big(|bc|+1\big) \geq 1$$

which is Theorem 9(i) of [14] (see also Theorem 7.4 of [5]).

(iv) Let X be the Heisenberg group $\mathcal{N}$ and ρ_0 be the Cygan metric. We take $\mathrm{Aut}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(1)$ (which has index 2 in the full group of Heisenberg isometries). Then elements of $\mathrm{M\ddot{o}b}(\mathcal{N})$ lie in $\mathrm{PU}(2,1)$, which is the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. Let

$$A(\zeta, v) = \big(\lambda\zeta, |\lambda|^2 v\big)$$

for $\lambda \in \mathbb{C} - \{0\}$. Then we have $d_A = |\lambda|^{1/2}$ and $m_A = \big(|\lambda - 1| + |\lambda^{-1} - 1|\big)^{1/2}$, see Lemma 2.1 of [13]. Also, (6) and (7) are Proposition 2.4 of [8]. Then the condition (11) of Theorem 2.4 becomes

$$\big(|\lambda - 1| + |\lambda^{-1} - 1|\big)\big(\mathbb{X}(B\infty, o; \infty, Bo) + 1\big) \geq 1$$

which is the first statement of Theorem 4.1 of [6], noting that the cross-ratio $\mathbb{X}(z_1, z_2; w_1, w_2)$ defined above is the square root of $\big|[z_1, z_2; w_1, w_2]\big|$ defined in [6] (see also [4] where the cross-ratio is defined in terms of the Cygan metric).

2.3 Proof of the main theorem

Using Lemma 2.3, we suppose that A is a loxodromic map fixing o and ∞. Suppose that B is any element of Möb(X) for which $\{Bo,\, B\infty\} \cap \{o,\, \infty\} = \emptyset$. Define a sequence $\{B_n\}$ of elements of G as follows:

$$B_0 = B, \quad B_{n+1} = B_n A {B_n}^{-1}.$$

For simplicity let r_n denote r_{B_n}, the constant from Proposition 2.1 (ii).

Let $p_n = B_n o$ and $q_n = B_n \infty$ denote the fixed points of B_{n+1}. Assume for the moment that $\{p_n,\, q_n\} \cap \{o,\, \infty\} = \emptyset$ for all n. Define

$$\mathbb{X}_n = \mathbb{X}(q_n, o; \infty, p_n) = \frac{\rho_0(o, p_n)}{\rho_0(p_n, q_n)}.$$

Our goal is to show that if ${m_A}^2(1 + \mathbb{X}_0) < 1$ then there is a sequence of distinct elements in $\langle A, B\rangle$ tending to the identity.

First, we relate $\mathbb{X}_{n+1}$ to $\mathbb{X}_n$ and use this to show that $\mathbb{X}_n$ tends to zero as n tends to infinity.

Lemma 2.5 *With the above notation*

$$\begin{aligned} \rho_0(o, p_{n+1}) &\leq m_A d_A \rho_0(p_{n+1}, q_n)\, \frac{\rho_0(o, p_n)}{\rho_0(p_n, q_n)}, \\ \frac{1}{\rho_0(p_{n+1}, q_{n+1})} &\leq \frac{m_A}{d_A \rho_0(p_{n+1}, q_n)}\, \frac{\rho_0(o, q_n)}{\rho_0(p_n, q_n)}. \end{aligned}$$

Proof: Using Proposition 2.1 and (10) we have

$$\begin{aligned} \rho_0(o, p_{n+1}) &= \rho_0(o, B_n A {B_n}^{-1} o) \\ &= \frac{{r_n}^2 \rho_0({B_n}^{-1} o, A {B_n}^{-1} o)}{\rho_0(A{B_n}^{-1} o, {B_n}^{-1}\infty)\rho_0({B_n}^{-1} o, {B_n}^{-1}\infty)} \\ &\leq \frac{d_A m_A {r_n}^2 \rho_0(o, {B_n}^{-1} o)}{\rho_0(A{B_n}^{-1} o, {B_n}^{-1}\infty)\rho_0({B_n}^{-1} o, {B_n}^{-1}\infty)} \\ &= \frac{d_A m_A \rho_0(B_n A B_n^{-1} o, B_n\infty)\rho_0(o, B_n o)}{\rho_0(B_n o, B_n \infty)} \\ &= m_A d_A \rho_0(p_{n+1}, q_n)\, \frac{\rho_0(o, p_n)}{\rho_0(p_n, q_n)}. \end{aligned}$$

We have used (8) on the penultimate line. Similarly, we have

$$\begin{aligned}
\frac{1}{\rho_0(p_{n+1}, q_{n+1})} &= \frac{1}{\rho_0(B_n A {B_n}^{-1} o, B_n A {B_n}^{-1} \infty)} \\
&= \frac{\rho_0(A {B_n}^{-1} o, {B_n}^{-1} \infty)\rho_0(A {B_n}^{-1} \infty, {B_n}^{-1} \infty)}{{r_n}^2 \rho_0(A {B_n}^{-1} o, A {B_n}^{-1} \infty)} \\
&\leq \frac{d_A m_A \rho_0(A {B_n}^{-1} o, {B_n}^{-1} \infty)\rho_0(o, {B_n}^{-1} \infty)}{{d_A}^2 {r_n}^2 \rho_0({B_n}^{-1} o, {B_n}^{-1} \infty)} \\
&= \frac{m_A \rho_0(o, B_n \infty)}{d_A \rho_0(B_n A B_n^{-1} o, B_n \infty)\rho_0(B_n o, B_n \infty)} \\
&= \frac{m_A}{d_A \rho_0(p_{n+1}, q_n)} \frac{\rho_0(o, q_n)}{\rho_0(p_n, q_n)}.
\end{aligned}$$

□

Combining these two estimates and using the triangle inequality we have

$$\begin{aligned}
\mathbb{X}_{n+1} &= \frac{\rho_0(o, p_{n+1})}{\rho_0(p_{n+1}, q_{n+1})} \\
&\leq {m_A}^2 \frac{\rho_0(o, p_n)\rho_0(o, q_n)}{\rho_0(p_n, q_n)^2} \\
&\leq {m_A}^2 \frac{\rho_0(o, p_n)\big(\rho_0(o, p_n) + \rho_0(p_n, q_n)\big)}{\rho_0(p_n, q_n)^2} \\
&= {m_A}^2 \mathbb{X}_n (1 + \mathbb{X}_n).
\end{aligned}$$

Thus, if ${m_A}^2(1 + \mathbb{X}_0) < 1$ we have

$$\mathbb{X}_n \leq \big({m_A}^2(1 + \mathbb{X}_0)\big)^n \mathbb{X}_0$$

and so $\mathbb{X}_n$ tends to zero exponentially as n tends to infinity.

The second step in the proof is to estimate $\rho_0(o, p_{n+1})$ and $\rho_0(p_{n+1}, q_{n+1})$ directly. Using Lemma 2.5 we see that we must estimate $\rho_0(p_{n+1}, q_n)$. Since $\mathbb{X}_n$ tends to zero as n tends to infinity, we may take n large enough that $\mathbb{X}_n$ is as small as we please.

Lemma 2.6 *Suppose that n is sufficiently large that $d_A m_A \mathbb{X}_n < 1$. Then*

$$\frac{\rho_0(o, q_n)}{1 + d_A m_A \mathbb{X}_n} \leq \rho_0(p_{n+1}, q_n) \leq \frac{\rho_0(o, q_n)}{1 - d_A m_A \mathbb{X}_n}.$$

Proof: We prove the lower bound. The upper bound is similar. Using the triangle inequality and Lemma 2.5 we have

$$\begin{aligned}
\rho_0(o, q_n) &\leq \rho_0(p_{n+1}, q_n) + \rho_0(p_{n+1}, o) \\
&\leq \rho_0(p_{n+1}, q_n)\big(1 + d_A m_A \mathbb{X}_n\big).
\end{aligned}$$

□

Combining the inequalities from Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}
\rho_0(o,p_{n+1}) &\le \frac{d_A m_A}{1-d_A m_A \mathbb{X}_n}\,\frac{\rho_0(o,q_n)}{\rho_0(p_n,q_n)}\,\rho_0(o,p_n) \\
&\le \frac{d_A m_A(1+\mathbb{X}_n)}{1-d_A m_A \mathbb{X}_n}\,\rho_0(o,p_n), \\
\frac{1}{\rho_0(p_{n+1},q_{n+1})} &\le \frac{m_A(1+d_A m_A \mathbb{X}_n)}{d_A}\,\frac{1}{\rho_0(p_n,q_n)}.
\end{aligned}$$

Define

$$\delta_n = \frac{m_A(1+\mathbb{X}_n)}{1-d_A m_A \mathbb{X}_n}$$

and observe that $m_A(1+d_A m_A \mathbb{X}_n) < \delta_n$. Since $m_A < 1$ and $\mathbb{X}_n$ tends to zero as n tends to infinity, we can find an integer N so that $\delta_N < 1$. Thus for all $n \ge N$ we have $\delta_n < \delta_N$ and so

$$\rho_0(o,p_{n+1}) \le \delta_N d_A \rho_0(o,p_n), \qquad \text{and} \qquad \rho_0(p_{n+1},q_{n+1}) \ge {\delta_N}^{-1} d_A \rho_0(p_n,q_n). \tag{12}$$

The third and final step is to produce a convergent sequence of distinct elements of $\langle A,B\rangle$. Following Jørgensen, we define $C_n = A^{-n}B_{2n}A^n$. Observe that (12) implies that the B_n, and hence the C_n, are distinct. Denote the fixed points of C_n by $p_n' = A^{-n}B_{2n-1}o$ and $q_n' = A^{-n}B_{2n-1}\infty$. Then

$$\rho_0(p_n',o) = {d_A}^{-2n}\rho_0(p_{2n-1},o), \quad \rho_0(p_n',q_n') = {d_A}^{-2n}\rho_0(p_{2n-1},q_{2n-1}).$$

Hence, using (12), for all $n \ge N$ we have

$$\begin{aligned}
\rho_0(p_{n+1}',o) &= {d_A}^{-2n-2}\rho_0(p_{2n+1},o) \\
&\le \delta_N {d_A}^{-2n-1}\rho_0(p_{2n},o) \\
&\le {\delta_N}^2 {d_A}^{-2n}\rho_0(p_{2n-1},o) \\
&= {\delta_N}^2 \rho_0(p_n',o) \\
&\le {\delta_N}^{2(n+1-N)}\rho_0(p_N',o).
\end{aligned}$$

Likewise

$$\rho_0(p_{n+1}',q_{n+1}') \ge {\delta_N}^{-2}\rho_0(p_n',q_n') \ge {\delta_N}^{-2(n+1-N)}\rho_0(p_N',q_N').$$

In particular,

$$\begin{aligned}
\rho_0(q_{n+1}',o) &\ge \rho_0(p_{n+1}',q_{n+1}') - \rho_0(p_{n+1}',o) \\
&\ge {\delta_N}^{-2}\rho_0(p_n',q_n') - {\delta_N}^2\rho_0(p_n',o) \\
&\ge {\delta_N}^{-2(n+1-N)}\rho_0(p_N',q_N') - {\delta_N}^{2(n+1-N)}\rho_0(p_N',o).
\end{aligned}$$

Hence for $n \ge N$ the fixed points of C_n tend exponentially to o and ∞. We claim that the C_n lie in a compact subset of Möb(X). Hence (a subsequence of) the C_n

tend to the identity (compare [4]). Since the C_n are distinct, we see that $\langle A, B\rangle$ is not discrete. This will prove the main theorem in the case where p_n, $q_n \neq o, \infty$.

In order to verify the claim, observe that we may choose D_n lying in a compact subset of Möb(X) so that $D_n C_n {D_n}^{-1}$ fixes both o and ∞. Secondly, since C_n is conjugate to A, using Lemma 2.3 we see that the dilation factor of $D_n C_n {D_n}^{-1}$ is ${d_A}^2$. Thus for all z, $w \in X$ we have

$$\rho_0(D_n C_n {D_n}^{-1} A^{-1} z, D_n C_n {D_n}^{-1} A^{-1} w) = {d_A}^2 \rho_0(A^{-1} z, A^{-1} w) = \rho_0(z, w).$$

Hence $D_n C_n {D_n}^{-1} A^{-1}$ is in Aut(X) and fixes o. By hypothesis the stabiliser of o in Aut(X) is compact. Hence C_n lies in a compact subset of Möb(X) as claimed.

We need to treat the case where there is an $N \geq 0$ for which either p_N or q_N is o or ∞, and so $p_{N+1} = o$ or $q_{N+1} = \infty$. Without loss of generality, suppose $q_{N+1} = \infty$ and hence $q_n = \infty$ for all $n \geq N + 1$.

Suppose $p_n \neq o$ for all n. We will not use (11) but only the fact that $\langle A, B\rangle$ is discrete. (Note that taking $N = 0$ this shows that if $\langle A, B\rangle$ is discrete then $\{Bo, B\infty\} \cap \{o, \infty\}$ cannot be just one point.) Consider the sequence B_n as defined above. Since B_n is conjugate to A and fixes ∞ for $n \geq N + 1$, we have

$$\begin{aligned}
\rho_0(p_{n+1}, o) &= \rho_0(B_n A {B_n}^{-1} o, o) \\
&= {d_A}^2 \rho_0(A {B_n}^{-1} o, {B_n}^{-1} o) \\
&\leq {d_A}^3 m_A \rho_0({B_n}^{-1} o, o) \\
&\leq d_A m_A \rho_0(B_n o, o) \\
&\leq \delta {d_A}^{2k} \rho_0(p_n, o)
\end{aligned}$$

for some integer k and some $\delta < 1$. Then $A^{-kn} B_{n+1} A^{kn}$ fixes $p''_n = A^{-kn} p_n$ and ∞. Thus $\rho_0(p''_n, o) = {d_A}^{-2kn} \rho_0(p_n, o)$ and, arguing as above,

$$\rho_0(p''_{n+1}, o) \leq \delta \rho_0(p''_n, o) \leq \delta^{n+1-N} \rho_0(p''_N, o).$$

Thus (a subsequence of) the $A^{-kn} B_{n+1} A^{kn}$ converges and are distinct elements of $\langle A, B\rangle$. Again, $\langle A, B\rangle$ cannot be discrete.

Finally, suppose $p_{N+1} = o$ and $q_{N+1} = \infty$ for some $N \geq 0$. Thus B_{n+1} fixes both o and ∞ for all $n \geq N + 1$. Again we will not use (11), but this time we only use the fact that $\{Bo, B\infty\} \cap \{o, \infty\} = \emptyset$. Since A has precisely two fixed points, if $B_{n+1} = B_n A {B_n}^{-1}$ fixes both o and ∞ then B_n either fixes both o and ∞ or interchanges them. Without loss of generality, suppose that N is the smallest index for which $p_{N+1} = o$ and $q_{N+1} = \infty$. Since $\{B_0 o, B_0 \infty\} \cap \{o, \infty\} = \emptyset$, we may assume that $N \geq 1$. Then $B_N o = \infty$ and $B_N \infty = o$ and we see that B_N has an orbit of size 2. Thus B_N^2 fixes points that B_N does not. Since B_N is conjugate to A, this is a contradiction. This proves the theorem.

3 The octonionic hyperbolic plane

3.1 The octonions

The material in this section is well known, see for example Section 2 of Allcock [2] or Section 2 of Baez [3]. The octonions $\mathbb{O}$ comprise the real vector space spanned by $\mathbf{e}_j$ for $j = 0, \ldots, 7$ where $\mathbf{e}_0 = 1$ together with a non-associative multiplication defined on the basis vectors as follows and then extended to the whole of $\mathbb{O}$ by linearity. First for $j = 1, \ldots, 7$

$$\mathbf{e}_0\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_0 = \mathbf{e}_j, \quad \mathbf{e}_j^2 = -1.$$

Secondly for $j,\, k = 1, \ldots, 7$ and $j \neq k$

$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j.$$

Finally

$$\mathbf{e}_j\mathbf{e}_k = \mathbf{e}_l$$

precisely when (j,k,l) is a cyclic permutation of one of the triples:

$$(1,2,4), \quad (1,3,7), \quad (1,5,6), \quad (2,3,5), \quad (2,6,7), \quad (3,4,6), \quad (4,5,7).$$

A multiplication table is given on page 150 of [3].

We write an octonion z as

$$z = z_0 + \sum_{j=1}^{7} z_j\mathbf{e}_j.$$

Define the *conjugate* $\overline{z}$ of z to be

$$\overline{z} = z_0 - \sum_{j=1}^{7} z_j\mathbf{e}_j.$$

Conjugation is an anti-automorphism, that is for all octonions z and w

$$\overline{(zw)} = \overline{w}\,\overline{z}.$$

The *real part* of z is $\Re(z) = \frac{1}{2}(z+\overline{z})$ and the *imaginary part* of z is $\Im(z) = \frac{1}{2}(z-\overline{z})$. The *modulus* $|z|$ of an octonion is the non-negative real number defined by

$$|z|^2 = \overline{z}z = z\overline{z} = {z_0}^2 + {z_1}^2 + \cdots + {z_7}^2.$$

The modulus is multiplicative, that is $|zw| = |z|\,|w|$. Clearly $|z| > 0$ unless $z = 0$ (that is $z_0 = z_1 = \cdots = z_7 = 0$). An octonion z is a *unit* if $|z| = 1$.

The following result is due to Artin (see page 471 of [2] or pages 149–150 of [3]).

Proposition 3.1 *For any octonions x and y the subalgebra with a unit generated by x and y is associative. In particular, any product of octonions that may be written in terms of just two octonions is associative.*

The following identities are due to Ruth Moufang (see page 471 of [2]).

Proposition 3.2 *Suppose that x, y, z are octonions and μ is an imaginary unit (that is $|\mu| = 1$ and $\Re(\mu) = 0$ so $1 = |\mu|^2 = -\mu^2$). Then*

$$\begin{aligned} z(xy)z &= (zx)(yz), && (13)\\ \Re\big((xy)z\big) &= \Re\big(x(yz)\big) = \Re\big((yz)x\big), && (14)\\ (\mu x\overline{\mu})(\mu y) &= \mu(xy) && (15)\\ (x\mu)(\overline{\mu}y\mu) &= (xy)\mu && (16)\\ xy + yx &= (x\overline{\mu})(\mu y) + (y\overline{\mu})(\mu x) && (17) \end{aligned}$$

We write $\Re(xyz)$ for any of the 3 expressions in (14).

3.2 Jordan algebras and the octonionic hyperbolic plane

In this section we construct the octonionic hyperbolic plane. This should be compared to the analogous construction of the octonionic projective plane given by Baez [3]. Our construction follows Allcock [2] but we use notation that is closer to that used for complex and quaternionic hyperbolic geometries in [6], [10], [11]. Let

$$\Psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $\mathrm{M}(3, \mathbb{O})$ denote the real vector space of 3×3 matrices with octonionic entries. Let X^* denote the conjugate transpose of a matrix X in $\mathrm{M}(3, \mathbb{O})$. Define

$$J = \Big\{ X \in \mathrm{M}(3, \mathbb{O}) \; : \; \Psi X = X^* \Psi \Big\}.$$

Then J is closed under the Jordan multiplication

$$X * Y = \frac{1}{2}(XY + YX),$$

and so we call J the *Jordan algebra* associated to Ψ. Real numbers act on $\mathrm{M}(3, \mathbb{O})$ by multiplication of each entry of X. We define an equivalence relation on J by $X \sim Y$ if and only if $Y = kX$ for some non-zero real number k. Then $\mathbb{P}J$ is defined to be the set of equivalence classes $[X]$.

Following Allcock we define

$$\mathbb{O}_0^3 = \left\{ \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; : \; x, \, y, \, z \text{ all lie in some associative subalgebra of } \mathbb{O} \right\}.$$

We define an equivalence relation on $\mathbb{O}_0^3$ by $\mathbf{v} \sim \mathbf{w}$ if $\mathbf{w} = \mathbf{v}\alpha$ for some α in an associative subalgebra of $\mathbb{O}$ containing the entries x, y, z of $\mathbf{v}$. Let $\mathbb{P}\mathbb{O}_0^3$ be the set of equivalence classes $[\mathbf{v}]$. Define a map $\pi_\Psi : \mathbb{O}_0^3 \longrightarrow J$ by

$$\pi_\Psi(\mathbf{v}) = \mathbf{v}\mathbf{v}^*\Psi = \begin{pmatrix} x\overline{z} & x\overline{y} & |x|^2 \\ y\overline{z} & |y|^2 & y\overline{x} \\ |z|^2 & z\overline{y} & z\overline{x} \end{pmatrix}.$$

It is easy to check that if x, y, z, α all lie in an associative subalgebra of $\mathbb{O}$ then

$$\pi_\Psi \begin{pmatrix} x\alpha \\ y\alpha \\ z\alpha \end{pmatrix} = |\alpha|^2 \pi_\Psi \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Hence π_Ψ descends to a map $\overline{\pi}_\Psi : \mathbb{P}\mathbb{O}_0^3 \longrightarrow \mathbb{P}J$ defined by $\overline{\pi}_\Psi[\mathbf{v}] = \big[\pi_\Psi(\mathbf{v})\big]$.

We define an (indefinite) norm $|\mathbf{v}|_\Psi$ on $\mathbb{O}_0^3$ by

$$|\mathbf{v}|_\Psi = \mathbf{v}^*\Psi\mathbf{v} = \mathrm{tr}\big(\pi_\Psi(\mathbf{v})\big).$$

Observe that $|\mathbf{v}\alpha|_\Psi = |\alpha|^2|\mathbf{v}|_\Psi$ and so the sign of $|\mathbf{v}|_\Psi$ is defined consistently within classes of $\mathbb{P}\mathbb{O}_0^3$. Following Allcock, a model of the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$ is the subset of $\mathbb{P}J$ comprising $\big[\pi_\Psi(\mathbf{v})\big] = \overline{\pi}_\Psi[\mathbf{v}]$ for $\mathbf{v} \in \mathbb{O}_0^3$ with $|\mathbf{v}|_\Psi < 0$. Likewise its boundary is subset of $\mathbb{P}J$ comprising $\big[\pi_\Psi(\mathbf{v})\big]$ for $\mathbf{v} \in \mathbb{O}_0^3$ with $|\mathbf{v}|_\Psi = 0$.

Let $\mathcal{H}^{15}$ be the 15 dimensional variety

$$\mathcal{H}^{15} = \Big\{(x,y)\ :\ x,\, y \in \mathbb{O},\ x + \overline{x} + |y|^2 = 0\Big\}.$$

The following (associative) product gives $\mathcal{H}^{15}$ the structure of a nilpotent Lie group

$$(x,y) * (w,z) = (x + w - \overline{y}z, y + z).$$

(These coordinates on $\mathcal{H}^{15}$ are related to those given by Allcock, page 479 of [2], by $(x,y) = (-\frac{1}{2}|\xi|^2 + \eta, \xi)$.) Consider the map $\psi : \mathcal{H}^{15} \cup \{\infty\} \longrightarrow J$ given by

$$\begin{aligned} \psi(x,y) &= \pi_\Psi \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & x\overline{y} & |x|^2 \\ y & |y|^2 & y\overline{x} \\ 1 & \overline{y} & \overline{x} \end{pmatrix} \\ \psi(\infty) &= \pi_\Psi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The map $\mathbb{P}\psi : \mathcal{H}^{15} \cup \{\infty\} \longrightarrow \partial\mathbf{H}^2_{\mathbb{O}} \subset \mathbb{P}J$ defines a bijection between $\mathcal{H}^{15} \cup \{\infty\}$ and Allcock's model of the boundary of the octonionic hyperbolic plane (compare with the penultimate sentence on page 483 of [2]). Henceforth, unless we specify otherwise, we identify $\mathcal{H}^{15} \cup \{\infty\}$ with $\partial\mathbf{H}^2_{\mathbb{O}}$.

We define certain transformations from $\mathcal{H}^{15} \cup \{\infty\}$ to itself. First $\mathcal{H}^{15}$ acts transitively on itself by left translation. For $(t,s) \in \mathcal{H}^{15}$ define (see page 479 of [2])

$$T_{(t,s)}(x,y) = (t,s) * (x,y) = (t + x - \overline{s}y, s + y), \quad T_{(t,s)}(\infty) = \infty. \tag{18}$$

For any imaginary unit octonion μ we define (see page 476 or page 479 of [2])

$$S_\mu(x,y) = (\mu x \overline{\mu}, y\overline{\mu}), \quad S_\mu(\infty) = \infty. \tag{19}$$

We remark that, in general, $S_\mu S_\nu \neq S_{\mu\nu}$. The S_μ generate the compact group $\mathrm{Spin}_7(\mathbb{R})$ (see page 476 of [2]). For any positive real number d define

$$D_d(x,y) = (d^4 x, d^2 y), \quad D_d(\infty) = \infty. \tag{20}$$

Finally, if $x \neq 0$ then define (see page 476 of [2])

$$R(x, y) = (\overline{x}/|x|^2, -y\overline{x}/|x|^2), \quad R(0,0) = \infty, \quad R(\infty) = (0,0). \tag{21}$$

Let G be the group generated by R, S_μ, $T_{(t,s)}$ for all imaginary unit octonions μ and all $(t, s) \in \mathcal{H}^{15}$. Allcock considers this group acting not on $\mathcal{H}^{15} \cup \{\infty\}$ but on J (we have kept Allcock's names for these transformations). From this point of view, Allcock shows (Theorem 4.4 (iv) of [2]) that G is precisely the automorphism group $\mathrm{Aut}(J)$ of J, which is known to be $F_{4(-20)}$, an exceptional connected, simply connected Lie group of 52 dimensions (see for example Theorem 4.5 of [2]).

However, in [2] Allcock does not mention the dilations D_d. In fact the D_d is in G for all $d > 0$. For example, one may easily check that if d is a positive real number then

$$D_d(x, y) = RT_{(-2/d^2, -2/d)}RT_{(-(d-1)^2/2, d-1)}RT_{(-2,2)}RT_{(-(1-d)^2/2d^2, (1-d)/d)}(x, y).$$

Passing between our action and Allcock's is slightly subtle. Pushing our action forward under $\mathbb{P}\psi$ and Allcock's under $\mathbb{P}$ gives the same action on $\partial\mathbf{H}^2_{\mathbb{O}} \subset \mathbb{P}J$. The preimage of this action on J used by Allcock preserves the trace (Lemma 4.1 of [2]). On the other hand, we choose the preimage under $\mathbb{P}$ so that $\psi(\mathcal{H}^{15} \cup \{\infty\})$ is preserved. Thus in our notation $\psi\big(R(x, y)\big)$ differs from the Allcock's by multiplication by $1/|x|^2$. Likewise, the action of a dilation on J is given by $d^{-4}\psi\big(D_d(x, y)\big)$ and $d^4\psi\big(D_d(\infty)\big)$. Thus, from Allcock's viewpoint, $\psi(o)$ and $\psi(\infty)$ are eigenvectors of D_d whose eigenvalues are d^{-4} and d^4 respectively. Since $d \neq 1$ these are not fixed by D_d. Hence Allcock does not include them in the stabilisers of $\psi(o)$ and $\psi(\infty)$ in Theorem 4.4 (v), (vi) of [2]. However, considering the action on $\mathbb{P}J$, we see that D_d fixes $\big[\psi(o)\big]$ and $\big[\psi(\infty)\big]$. So we may adapt Allcock's theorem to show that, in terms of the action of G on $\mathcal{H}^{15} \cup \{\infty\}$, the stabiliser of o and ∞ is generated by the S_μ and D_d, and so is isomorphic to $\mathrm{Spin}_7(\mathbb{R}) \times \mathbb{R}_+$. Likewise, in terms of its action on $\mathcal{H}^{15} \cup \{\infty\}$, the stabiliser in G of ∞ is $\mathcal{H}^{15} \rtimes (\mathrm{Spin}_7(\mathbb{R}) \times \mathbb{R}_+)$.

3.3 The Cygan metric

Define a bilinear form on $\mathrm{M}(3, \mathbb{O})$ (see page 474 of [2]) by

$$\langle X|Y\rangle = \Re\, \mathrm{tr}(X * Y).$$

We claim that following map $\rho_0 : \mathcal{H}^{15} \times \mathcal{H}^{15} \longrightarrow \mathbb{R}$ gives a metric, called the *Cygan metric*, on $\mathcal{H}^{15}$:

$$\rho_0\big((x, y), (w, z)\big) = \big\langle \psi(x, y)|\psi(w, z)\big\rangle^{1/4}. \tag{22}$$

This metric generalises the standard Cygan metric on the usual Heisenberg group $\mathcal{H}^3$. Indeed, any pair of points in $\mathcal{H}^{15}$ lie on a copy of $\mathcal{H}^3$ inside $\mathcal{H}^{15}$ and ρ_0 gives their (standard) Cygan distance in this copy of $\mathcal{H}^3$. Before we prove that ρ_0 is a metric we investigate some of its other properties.

First, observe that, using the Moufang identity (14), we have

$$\begin{aligned}
\rho_0\big((x,y),(w,z)\big)^4 &= \big\langle \psi(x,y)|\psi(w,z)\big\rangle \\
&= |x|^2+|w|^2+|y|^2|z|^2+\Re(xw+wx) \\
&\quad +\Re\big((x\overline{y})z+y(w\overline{z})+(y\overline{x})\overline{z}+\overline{y}(z\overline{w})\big) \\
&= |x|^2+|w|^2+|y|^2|z|^2+2\Re(xw)+2\Re(x\overline{y}z+\overline{y}z\overline{w}) \\
&= |x+\overline{w}|^2+|\overline{z}y|^2+2\Re\big((x+\overline{w})\overline{y}z\big) \\
&= |x+\overline{w}+\overline{z}y|^2.
\end{aligned}$$

We now show that $T_{(t,s)}$ and S_μ preserve ρ_0 and that ρ_0 scales by d^2 under D_d.

Proposition 3.3 *Let $(t,s)\in\mathcal{H}^{15}$, μ be an imaginary unit and d be a positive real number. Then for all (x,y), $(z,w)\in\mathcal{H}$ we have*

$$\begin{aligned}
\rho_0\big(T_{(t,s)}(x,y),T_{(t,s)}(w,z)\big) &= \rho_0\big((x,y),(w,z)\big), \\
\rho_0\big(S_\mu(x,y),S_\mu(w,z)\big) &= \rho_0\big((x,y),(w,z)\big), \\
\rho_0\big(D_d(x,y),D_d(w,z)\big) &= d^2\rho_0\big((x,y),(w,z)\big).
\end{aligned}$$

Proof: We have

$$\begin{aligned}
\rho_0\big(T_{(t,s)}(x,y),T_{(t,s)}(w,z)\big) &= \Big|t+x-\overline{s}y+\overline{t}+\overline{w}-\overline{z}s+(\overline{z}+\overline{s})(s+y)\Big|^{1/2} \\
&= \big|t+\overline{t}+|s|^2+x+\overline{w}+\overline{z}y\big|^{1/2} \\
&= \rho_0\big((x,y),(w,z)\big)
\end{aligned}$$

since $t+\overline{t}+|s|^2=0$.

Using the Moufang identity (13) and $\overline{\mu}=-\mu$ we have

$$\begin{aligned}
\rho_0\big(S_\mu(x,y),S_\mu(w,z)\big) &= \big|\mu x\overline{\mu}+\mu\overline{w}\,\overline{\mu}+(\mu\overline{z})(y\overline{\mu})\big|^{1/2} \\
&= \big|\mu x\overline{\mu}+\mu\overline{w}\,\overline{\mu}+\mu(\overline{z}y)\overline{\mu})\big|^{1/2} \\
&= \big|\mu(x+\overline{w}+\overline{z}y)\overline{\mu}\big|^{1/2} \\
&= \rho_0\big((x,y),(w,z)\big).
\end{aligned}$$

Finally

$$\rho_0\big(D_d(x,y),D_d(w,z)\big)=|d^4x+d^4\overline{w}+d^2\overline{z}d^2y|^{1/2}=d^2\rho_0\big((x,y),(w,z)\big).$$

□

We are now in a position to show that ρ_0 is a metric.

Proposition 3.4 *The map $\rho_0:\mathcal{H}^{15}\times\mathcal{H}^{15}\longrightarrow\mathbb{R}$ given by*

$$\rho_0\big((x,y),(w,z)\big)=\big\langle\psi(x,y)|\psi(w,z)\big\rangle^{1/4}=|x+\overline{w}+\overline{z}y|^{1/2}$$

is a metric.

Proof: Clearly $\rho_0\big((x,y),(x,y)\big) = \big|x+\overline{x}+|y|^2\big|^{1/2} = 0$ and for $(x,y)\neq(z,w)$ then $\rho_0\big((x,y),(w,z)\big) = \rho_0\big((w,z),(x,y)\big)$.

If $\rho_0\big((x,y),(w,z)\big)=0$ then $x+\overline{w}+\overline{z}y=0$. Hence

$$\begin{aligned} |y-z|^2 &= |y|^2-\overline{z}y-\overline{y}z+|z|^2 \\ &= -x-\overline{x}-\overline{z}y-\overline{y}z-w-\overline{w} \\ &= -2\Re(x+\overline{w}+\overline{z}y) \\ &= 0, \end{aligned}$$

where we have used $x+\overline{x}+|y|^2 = w+\overline{w}+|z|^2 = 0$. Thus $y=z$ and so $x=-\overline{w}-\overline{z}y = -\overline{w}-|z|^2 = w$. Hence $(x,y)=(w,z)$ as required.

It remains to prove the triangle inequality. Using the fact that $\mathcal{H}^{15}$ acts transitively on itself by left translation $T_{(t,s)}$, and that this action preserves ρ_0, it suffices to show that for all (x,y), $(w,z)\in\mathcal{H}^{15}$ we have

$$\rho_0\big((x,y),(w,z)\big) \leq \rho_0\big((x,y),o\big)+\rho_0\big(o,(w,z)\big)$$

where $o=(0,0)\in\mathcal{H}^{15}$. In order to see this, observe

$$\begin{aligned} \rho_0\big((x,y),(w,z)\big) &= |x+\overline{w}+\overline{z}y|^{1/2} \\ &\leq \big(|x|+|w|+|z|\,|y|\big)^{1/2} \\ &= \Big(|x|+|w|+\sqrt{4\Re(x)\Re(w)}\Big)^{1/2} \\ &\leq \big(|x|+2|x|^{1/2}|w|^{1/2}+|w|\big)^{1/2} \\ &= |x|^{1/2}+|w|^{1/2} \\ &= \rho_0\big((x,y),o\big)+\rho_0\big(o,(w,z)\big). \end{aligned}$$

□

We have seen in Proposition 3.3 that $T_{(t,s)}$ and S_μ are isometries of the metric ρ_0 and that D_d satisfies (5) with $d_{D_d}=d$. We will take $\mathrm{Aut}(\mathcal{H}^{15})$ to be the subgroup of G fixing ∞ and preserving ρ_0. If $A\in G$ fixes ∞ then, in Allcock's normalisation, the eigenvalue of $\psi(\infty)$ is ${d_A}^4$ where ${d_A}^2$ is the dilation factor of A. Thus, if A also preserves ρ_0, we see that $\psi(\infty)$ is fixed by A. Using Theorem 4.4 (vi) of [2], we see that $\mathrm{Aut}(\mathcal{H}^{15}) = \mathcal{H}^{15}\rtimes \mathrm{Spin}_7(\mathbb{R})$. Thus $\mathrm{Aut}(\mathcal{H}^{15})$ acts transitively on $\mathcal{H}^{15}$. Moreover, the stabiliser of $o=(0,0)$ in $\mathrm{Aut}(\mathcal{H}^{15})$ is the compact group $\mathrm{Spin}_7(\mathbb{R})$.

Finally, we show that R satisfies (6) and (7). This will imply that G, the group generated by the $T_{(t,s)}$, S_μ, D_d and R, is in fact $\mathrm{Möb}(\mathcal{H}^{15})$ in the sense of the previous section.

Proposition 3.5 *Writing $o=(0,0)\in\mathcal{H}^{15}$, for all (x,y), $(z,w)\in\mathcal{H}^{15}-\{o\}$ we have*

$$\begin{aligned} \rho_0\big(R(x,y),o\big) &= \frac{1}{\rho_0\big((x,y),o\big)}, \\ \rho_0\big(R(x,y),R(w,z)\big) &= \frac{\rho_0\big((x,y),(w,z)\big)}{\rho_0\big((x,y),o\big)\rho_0\big((w,z),o\big)}. \end{aligned}$$

Proof: Using (21) and the Moufang identity (14) we have

$$\begin{aligned}
\rho_0\big(R(x,y),R(w,z)\big)^4 &= \rho_0\big((\overline{x}/|x|^2,-y\overline{x}/|x|^2),(\overline{w}/|w|^2,-z\overline{w}/|w|^2)\big)^4 \\
&= \Big|\overline{x}/|x|^2+w/|w|^2+(-w\overline{z}/|w|^2)(-y\overline{x}/|x|^2)\Big|^2 \\
&= \frac{1}{|x|^2}+\frac{1}{|w|^2}+\frac{|z|^2}{|w|^2}\cdot\frac{|y|^2}{|x|^2}+2\Re\left(\frac{\overline{x}}{|x|^2}\cdot\frac{\overline{w}}{|w|^2}\right) \\
&\quad +2\Re\left(\frac{\overline{x}}{|x|^2}\cdot\frac{x\overline{y}}{|x|^2}\cdot\frac{z\overline{w}}{|w|^2}\right)+2\Re\left(\frac{x\overline{y}}{|x|^2}\cdot\frac{z\overline{w}}{|w|^2}\cdot\frac{w}{|w|^2}\right) \\
&= \frac{|x+\overline{w}+\overline{z}y|^2}{|x|^2|w|^2} \\
&= \frac{\rho_0\big((x,y),(w,z)\big)^4}{\rho_0\big((x,y),o\big)^4\rho_0\big((w,z),o\big)^4}.
\end{aligned}$$

A similar, but easier, argument gives

$$\rho_0\big(R(x,y),o\big)=\rho_0\big((\overline{x}/|x|^2,-y\overline{x}/|x|^2),o\big)=\frac{1}{|x|^{1/2}}=\frac{1}{\rho_0\big((x,y),o\big)}.$$

□

3.4 An octonionic Jørgensen's inequality

The main result (see Chapter 4 of Markham [11]) of this section is

Theorem 3.6 *Suppose that A is a loxodromic element of $\text{Möb}(\mathcal{H}^{15})$ conjugate to D_dS_μ for some $d>0$ and some imaginary unit μ. Denote the fixed points of A by p and q. Let Γ be a discrete subgroup of $\text{Möb}(\mathcal{H}^{15})$ containing A. Then for all $B\in\Gamma$ with $\{Bp,\,Bq\}\cap\{p,\,q\}=\emptyset$ we have*

$$\Big(|d^2\mu-1|+|d^{-2}\mu-1|\Big)\Big(\mathbb{X}(Bp,q;p,Bq)+1\Big)\geq 1.$$

This theorem will follow directly from Theorem 2.4. We have already verified some of the hypotheses of Theorem 2.4. We know that $\text{Aut}(\mathcal{H}^{15})=\mathcal{H}^{15}\rtimes\text{Spin}_7(\mathbb{R})$ acts transitively on $\mathcal{H}^{15}$ with compact stabilisers. Also, $\text{Möb}(\mathcal{H}^{15})$ is generated by $T_{(t,s)}$, S_μ, D_d and R which satisfy the hypotheses (5), (6) and (7). It remains to find a constant m_A satisfying (9). In fact we find such an m_A satisfying the equivalent condition (10).

Proposition 3.7 *Let d be a positive real number and let μ be an imaginary unit octonion. For all $(x,y)\in\mathcal{H}^{15}$ we have*

$$\rho_0\big(D_dS_\mu(x,y),(x,y)\big)\leq d\big(|d^2\mu-1|+|d^{-2}\mu-1|\big)^{1/2}\rho_0\big((x,y),o\big).$$

Proof: Using Proposition 3.1 and $|y|^2 = -x - \overline{x}$, we have

$$\begin{aligned} \rho_0\big(D_d S_\mu(x,y),(x,y)\big) &= \rho_0\big((d^4\mu x\overline{\mu}, d^2 y\overline{\mu}),(x,y)\big) \\ &= \big|d^4\mu x\overline{\mu} + \overline{x} + d^2\overline{y}(y\overline{\mu})\big|^{1/2} \\ &= \big|d^4\mu x\overline{\mu} + \overline{x} - d^2 x\overline{\mu} - d^2\overline{x}\,\overline{\mu}\big|^{1/2} \\ &= \big|(d^2\mu - 1)d^2 x\overline{\mu} + \overline{x} d^2\overline{\mu}(d^{-2}\mu - 1)\big|^{1/2} \\ &\leq d\big(|d^2\mu - 1| + |d^{-2}\mu - 1|\big)^{1/2}|x|^{1/2} \\ &= d\big(|d^2\mu - 1| + |d^{-2}\mu - 1|\big)^{1/2}\rho_0\big((x,y),o\big). \end{aligned}$$

□

Therefore if $A = D_d S_\mu$ we have $d_A = d$ and $m_A = \big(|d^2\mu - 1| + |d^{-2}\mu - 1|\big)^{1/2}$, satisfying (10). Hence all the hypotheses of Theorem 2.4 are satisfied. This proves Theorem 3.6.

Using a similar method, it is not hard to show that $m_{D_d} = |d^2 - d^{-2}|^{1/2}$ and so in this case we could have obtained an analogous result to Theorem 3.6. However, there is an obstacle to proving Theorem 2.4 for general loxodromic maps in Möb$(\mathcal{H}^{15})$. This can be seen by considering $A = D_d S_\mu S_\nu$. We have

$$\begin{aligned} \rho_0\big(D_d S_\mu S_\nu(x,y),(x,y)\big) &= \rho_0\Big(\big(d^4\mu(\nu x\overline{\nu})\overline{\mu}, d^2(y\overline{\nu})\overline{\mu}\big),(x,y)\Big) \\ &= \Big|d^4\mu(\nu x\overline{\nu})\overline{\mu} + \overline{x} + d^2\overline{y}\big((y\overline{\nu})\overline{\mu}\big)\Big|^{1/2}. \end{aligned}$$

In general, we cannot write $\overline{y}\big((y\overline{\nu})\overline{\mu}\big) = |y|^2\overline{\nu}\,\overline{\mu}$ and then substitute $-x - \overline{x} = |y|^2$.

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