Unfaithful complex hyperbolic triangle groups I: Involutions

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Abstract

A complex hyperbolic triangle group is the group of complex hyperbolic isometries generated by complex involutions fixing three complex lines in complex hyperbolic space. Such a group is called equilateral if there is an isometry of order three that cyclically permutes the three complex lines. We consider equilateral triangle groups for which the product of each pair of involutions and the product of all three involutions are all non-loxodromic. We classify all such groups that are discrete.

1 Introduction

A complex hyperbolic triangle group is a group generated by three complex reflections that fix complex lines in complex hyperbolic space. Unlike real reflections, complex reflections can be of arbitrary order. Much of the literature is confined to the case where the reflections have order two. In this paper we consider that case and in the subsequent paper [13] we consider the case where the generators have higher order.

The study of complex hyperbolic triangle groups was begun in [6] where ideal triangle groups were considered. Since then there have been many developments. There have been two strands to this work. First, following [6], discrete and faithful representations of triangle groups have been investigated; see [16] for example. On the other hand, there has been the study of discrete representations where certain group elements are required to be elliptic of finite order; see [3], [4], [12], [18] for example. These representations are necessarily unfaithful. Schwartz has given an excellent survey [17] that outlines progress and gives a conjectural overview of what we might expect. A more recent survey is also contained in Pratoussevitch's paper [14]. Pratoussevitch also considers the case where the generators are complex reflections of higher order, which is related to earlier work of Mostow; see [10] for example. We will treat groups with generators of higher order in the sequel to this paper [13].

Three complex lines L_1, L_2 and L_3 in complex hyperbolic space form an *equilateral triangle* if each pair intersect and if there is a symmetry map J of order 3 in $\mathrm{SU}(2,1)$ so that $J(L_j) = L_{j+1}$ (with indices taken cyclically). For $j = 1, 2, 3$ let I_j be the complex reflection of order 2 fixing L_j. Then $I_2 = JI_1J^{-1}$ and $I_3 = JI_2J^{-1} = J^{-1}I_1J$. We call the group $\Delta = \langle I_1, I_2, I_3\rangle$ an *equilateral triangle group*. In this case Δ is a normal subgroup of $\Gamma = \langle I_1, J\rangle$. At first sight, it might appear that there is a full S_3 symmetry group inside $\mathrm{SU}(2,1)$ operating here, but this is only the case when Δ preserves a Lagrangian plane. The symmetry preserving L_1 but interchanging L_2 and L_3 is antiholomorphic. We shall have more to say about this in Section 2.2.

Our starting point is the following theorem, proved by Schwartz in Section 3.3 of [16]:

Theorem 1.1 (Schwartz [16]) *Let $\Delta = \langle I_1,\, I_2,\, I_3\rangle$ be the group of complex hyperbolic isometries generated by complex involutions I_j each fixing a complex line. Suppose that there is a symmetry map J of order 3 so that $I_2 = JI_1J^{-1}$ and $I_3 = J^{-1}I_1J$. If I_1I_2 is parabolic and $I_1I_2I_3$ is elliptic then Δ is not discrete.*

The main theorem of this paper is to consider equilateral triangle groups $\Delta = \langle I_1,\, I_2,\, I_3\rangle$ where I_1I_2 and $I_1I_2I_3$ are both elliptic. Clearly $I_2I_3I_1$ and $I_3I_1I_2$ are conjugate to $I_1I_2I_3$ and so are elliptic; the fact that Δ is equilateral means that both I_2I_3 and I_3I_1 are conjugate to I_1I_2 and hence are elliptic; the fact that each I_j is an involution means that $I_2I_1 = (I_1I_2)^{-1}$ and $I_3I_2I_1 = (I_1I_2I_3)^{-1}$, and so both of these maps are elliptic as well. We classify all such Δ that are discrete and we find that there are remarkably few of them. The point is that I_1I_2 and $I_1I_2I_3$ should simultaneously have finite order. We now give a rough statement of our main theorem. For a more precise statement see Theorem 3.7 and Proposition 4.5.

Theorem 1.2 *Let $\Delta = \langle I_1,\, I_2,\, I_3\rangle$ be the group of complex hyperbolic isometries generated by complex involutions I_j each fixing a complex line. Suppose that there is a symmetry map J of order 3 so that $I_2 = JI_1J^{-1}$ and $I_3 = J^{-1}I_1J$. Suppose that I_1I_2 and $I_1I_2I_3$ are both elliptic. Then Δ is discrete if and only if one of the following is true:*

(i) Δ is finite;

(ii) Δ is a normal subgroup of one of Livné's lattices;

(iii) Δ is Deraux's lattice, with I_1I_2 of order 4 and $I_1I_2I_3$ of order 10;

(iv) Δ is the group described in Section 4.2, with I_1I_2 of order 14 and $I_1I_2I_3$ of order 14.

There are other possible theorems along these lines. According to Schwartz's conjectural picture [17] discreteness of $\Delta = \langle I_1, I_2, I_3\rangle$ is controlled by whether $I_1I_2I_3$ and $I_1I_2I_1I_3$ are non-elliptic (for equilateral triangle groups, by symmetry $I_1I_2I_1I_3$ is elliptic if and only if each of $I_2I_3I_2I_1$ and $I_3I_1I_3I_2$ are elliptic). We could have considered the case of equilateral triangle groups where I_1I_2 and $I_1I_2I_1I_3$ are elliptic of finite order. Either using the formulae of Pratoussevitch [14] or using the formulae of Section 2.3 we find that $\mathrm{tr}(I_1I_2) = |\tau|^2 - 1$ and $\mathrm{tr}(I_1I_2I_1I_3) = |\tau^2 - \overline{\tau}|^2 - 1$. Choosing τ so that $|\tau| = 2\cos(\pi/n)$ and $|\tau^2 - \overline{\tau}| = 2\cos(\pi/m)$ yields groups for which I_1I_2 has order n and $I_1I_2I_1I_3$ has order m. From this it is easy to see that if $I_1I_2I_3$ is loxodromic and $I_1I_2I_1I_3$ is elliptic then $1 < |\tau|^2 < 11/3$, which is equivalent to $4 \leq n \leq 10$. Conjecturally, for a given n in this range these groups are discrete for all sufficiently large values of m; compare Schwartz's remark just after Theorem 4.7 of [17]. (See also the second remark on page 8 of [19].) In [18] Schwartz proves the discreteness of the group of this type with $n = 4$ and $m = 7$. A similar proof should work for $n = 4$ and $m \geq 8$. Note that when $n = 4$ and $m = 5$ we obtain Deraux's lattice, in which case $I_1I_2I_3$ is elliptic and has order 10.

After finishing this paper, Julien Paupert and I were discussing complex hyperbolic equilateral triangle groups where the generators have higher order. It turns out that we can use the same equations to discuss discreteness of these groups. Certain values of τ yield some of Mostow's groups from [10] and other values of τ give normal subgroups of Mostow's groups. This simultaneously generalises Theorem 1.2 (ii), since Livné's lattices are examples of Mostow's groups, and also work of Sauter [15] on commensurability between Mostow's groups. There are also sporadic groups when the generators have higher order. The details may be found in [13].

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2 Parameters and traces

In this section we show how to parametrise equilateral complex hyperbolic triangle groups with a single complex parameter τ and we find the values of τ that correspond to such a group. We then describe how the properties of the group (for example the type of I_1I_2 and $I_1I_2I_3$) vary with τ. We will try to keep this account as self contained as possible. However, we shall assume a certain amount of background knowledge of complex hyperbolic geometry. For such background material on complex hyperbolic space see [5] and for material on complex hyperbolic triangle groups see [17] or [14].

2.1 Complex reflections

Let L_1 be a complex line in complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ and write I_1 for the complex reflection of order 2 fixing L_1. We may lift I_1 to a matrix in $\mathrm{SU}(2,1)$. If the polar vector of L_1 is $\mathbf{n}_1$ then I_1 is given by:

$$I_1(\mathbf{z}) = -\mathbf{z} + 2\frac{\langle \mathbf{z}, \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}\,\mathbf{n}_1. \tag{1}$$

In Section 3.3.2 of [5], Goldman uses the polar vectors of two complex lines to determine the geometry of their relative position.

Proposition 2.1 (Goldman [5]) *Suppose that L_1 and L_2 are complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with polar vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Let*

$$\mathcal{N}(L_1, L_2) = \frac{\langle \mathbf{n}_1, \mathbf{n}_2\rangle\langle \mathbf{n}_2, \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle\langle \mathbf{n}_2, \mathbf{n}_2\rangle}.$$

(i) If $\mathcal{N}(L_1, L_2) > 1$ then L_1 and L_2 are ultraparallel;

(ii) If $\mathcal{N}(L_1, L_2) = 1$ then either L_1 and L_2 are asymptotic or $L_1 = L_2$;

(iii) If $\mathcal{N}(L_1, L_2) < 1$ then L_1 and L_2 intersect with angle θ where $\mathcal{N}(L_1, L_2) = \cos^2(\theta)$.

We shall be interested in equilateral triangle groups $\Delta = \langle I_1, I_2, I_3\rangle$ generated by complex involutions fixing complex lines L_1, L_2 and L_3. The hypothesis that the triangle is equilateral means there is a symmetry map J of order 3 in $\mathrm{SU}(2,1)$ so that $J(L_j) = L_{j+1}$, where the indices are taken mod 3. This implies that $J(\mathbf{n}_j) = \mathbf{n}_{j+1}$ and so

$$\mathcal{N}(L_j, L_{j+1}) = \frac{\langle \mathbf{n}_j, \mathbf{n}_{j+1}\rangle\langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle\langle \mathbf{n}_{j+1}, \mathbf{n}_{j+1}\rangle} = \frac{\left|\langle J(\mathbf{n}_j), \mathbf{n}_j\rangle\right|^2}{\langle \mathbf{n}_j, \mathbf{n}_j\rangle^2}.$$

2.2 The parameter space

Suppose we are given an equilateral triangle of complex lines L_1, L_2 and L_3 with polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ satisfying $J(\mathbf{n}_j) = \mathbf{n}_{j+1}$ where $j = 1, 2, 3$ taken mod 3. Because J preserves the Hermitian form, $\langle \mathbf{n}_j, \mathbf{n}_j\rangle$ is the same positive real number for each j. We normalise $\mathbf{n}_j$ so that this number is 2. Likewise $\langle \mathbf{n}_{j+1}, \mathbf{n}_j \rangle = \langle J(\mathbf{n}_j), \mathbf{n}_j \rangle$ is the same complex number for each j which we define to be τ. That is

$$\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \langle \mathbf{n}_3, \mathbf{n}_3 \rangle = 2, \qquad \langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \langle \mathbf{n}_3, \mathbf{n}_2 \rangle = \langle \mathbf{n}_1, \mathbf{n}_3 \rangle = \tau. \tag{2}$$

Let H be the matrix of the Hermitian form, that is $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}$. We define N to be the matrix whose columns are $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Then the ijth entry of $N^* H N$ is $\langle \mathbf{n}_j, \mathbf{n}_i \rangle$ and so

$$N^* H N = \begin{bmatrix} 2 & \tau & \overline{\tau} \\ \overline{\tau} & 2 & \tau \\ \tau & \overline{\tau} & 2 \end{bmatrix}. \tag{3}$$

Lemma 2.2 *Let L_1, L_2 and L_3 be complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Suppose that the Hermitian products of these vectors satisfy (2). Then the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are linearly independent if and only if* $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 \neq 0$.

Proof: We have

$$8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = \det \begin{bmatrix} 2 & \tau & \overline{\tau} \\ \overline{\tau} & 2 & \tau \\ \tau & \overline{\tau} & 2 \end{bmatrix} = \det(N^* H N) = \det(H) \big| \det(N) \big|^2.$$

Since $\det(H) \neq 0$ we see that $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 \neq 0$ if and only if N is non-singular. This proves the result. □

Let us now consider the case where the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are linearly independent. We consider the other case at the end of the section. Following Mostow, page 214 of [10], we choose coordinates so that

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In this case, the matrix N (whose columns are the vectors $\mathbf{n}_j$) is the identity. Hence, using equation (3), we see that the Hermitian form must be $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix} 2 & \tau & \overline{\tau} \\ \overline{\tau} & 2 & \tau \\ \tau & \overline{\tau} & 2 \end{bmatrix}. \tag{4}$$

We can immediately write down J and, using (1), the involutions I_j. They are

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad I_1 = \begin{bmatrix} 1 & \tau & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \overline{\tau} & 1 & \tau \\ 0 & 0 & -1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \tau & \overline{\tau} & 1 \end{bmatrix}.$$

From this it is clear that the groups $\Gamma = \langle I_1,\, J \rangle$ and $\Delta = \langle I_1,\, I_2,\, I_3 \rangle$ are completely determined, up to conjugation, by the parameter τ. However, not all values of τ correspond to complex hyperbolic triangle groups: it may be that the Hermitian matrix H_τ does not have signature $(2,1)$. We now determine this by finding the eigenvalues of H_τ. In this lemma and throughout the paper we write $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$.

Lemma 2.3 *Let H_τ be given by (4) and write $\tau = t + is$. The eigenvalues of H_τ are*

$$2 + \tau + \overline{\tau} = 2 + 2t, \qquad 2 + \tau\overline{\omega} + \overline{\tau}\omega = 2 - t + \sqrt{3}s, \qquad 2 + \tau\omega + \overline{\tau}\,\overline{\omega} = 2 - t - \sqrt{3}s.$$

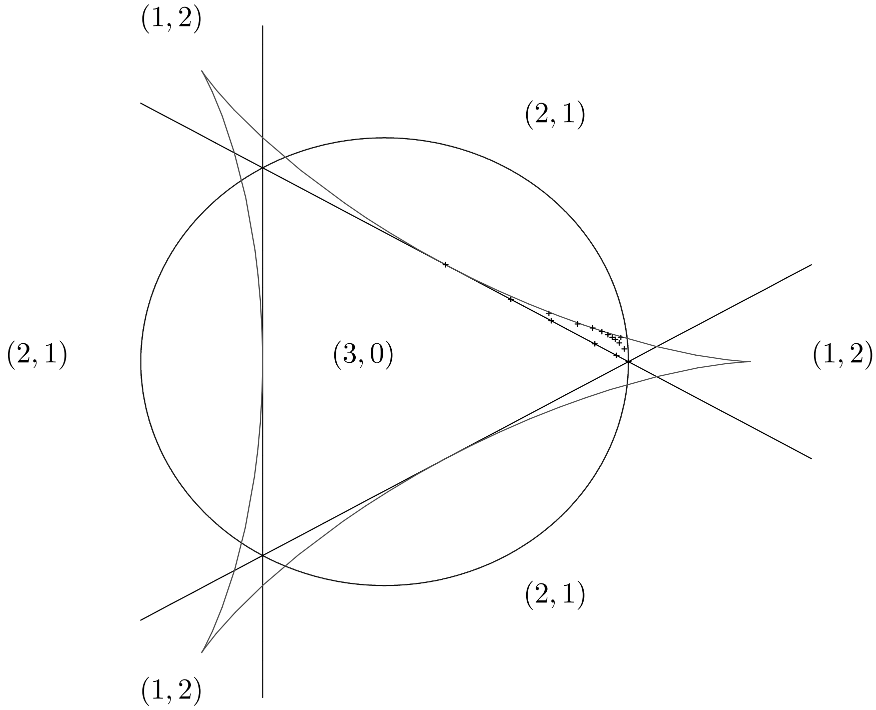

Figure 1: *The parameter space. We have drawn the points corresponding to the groups listed in Theorem 1.2. For more details of this part of the picture, see Figure 2.*

Proof: We observe that eigenvectors for H_τ are

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \begin{bmatrix}1\\\overline{\omega}\\\omega\end{bmatrix}, \qquad \begin{bmatrix}1\\\omega\\\overline{\omega}\end{bmatrix}.$$

Their eigenvalues are $2+\tau+\overline{\tau}$, $2+\tau\overline{\omega}+\overline{\tau}\omega$ and $2+\tau\omega+\overline{\tau}\,\overline{\omega}$ respectively. □

Corollary 2.4 *The matrix H_τ has signature $(2,1)$ if and only if*

$$6|\tau|^2-\tau^3-\overline{\tau}^3-8>0. \tag{5}$$

Proof: It is easy to check (for example by adding them) that all three eigenvalues cannot be negative. Thus H_τ has signature $(2,1)$ if and only if its determinant is negative. That is

$$0>(2+\tau+\overline{\tau})(2+\tau\overline{\omega}+\overline{\tau}\omega)(2+\tau\omega+\overline{\tau}\,\overline{\omega})=8+\tau^3+\overline{\tau}^3-6|\tau|^2.$$

□

We now describe how the signature of H_τ varies as τ varies in $\mathbb{C}$. There are three lines each of which is the locus where one of the eigenvalues vanishes. In Figure 1 we have drawn these three lines. These lines have seven complementary regions in $\mathbb{C}$, which fall into three types:

- The central triangle where all three eigenvalues are positive and so H_τ has signature $(3,0)$, that is it is positive definite.

- Three infinite components each sharing a common edge with the central triangle. In these regions two eigenvalues are positive and one negative and so H_τ has signature $(2,1)$. This is our parameter space.

- Three infinite components that each only abut the central triangle in a point. Here one eigenvalue is positive and two are negative and so H_τ has signature $(1,2)$. These correspond to groups of complex hyperbolic isometries generated by three complex involutions that each fix a point.

The values of τ satisfying (5) make up our parameter space. This space has three components related by multiplication by powers of $\omega = e^{2\pi i/3}$. This ambiguity corresponds to the choice we have made when lifting symmetry in $\mathrm{PU}(2,1)$ to the corresponding matrix J in $\mathrm{SU}(2,1)$, the triple cover of $\mathrm{PU}(2,1)$. In other words, the ordered triples $\{\mathbf{n}_1, \omega\mathbf{n}_2, \overline{\omega}\mathbf{n}_3\}$ and $\{\mathbf{n}_1, \overline{\omega}\mathbf{n}_2, \omega\mathbf{n}_3\}$ correspond to the same group as $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$. Hence τ is only defined up to a cube root of unity. Factoring out by this equivalence, our parameter space is in bijection with one of the three components where H_τ has signature $(2,1)$, which we described above. There is a further symmetry of our setup, namely complex conjugating τ. This corresponds to sending $\tau = \langle \mathbf{n}_{j+1}, \mathbf{n}_j\rangle$ to $\overline{\tau} = \langle \mathbf{n}_j, \mathbf{n}_{j+1}\rangle$. Up to conjugation, this preserves I_1 and sends J to J^{-1}. In particular, it swaps the roles of I_2 and I_3. Thus all the symmetries in S_3 (acting on L_1, L_2 and L_3) either preserve τ or send it to $\overline{\tau}$. Our parameter space also respects this symmetry. Hence we may restrict our attention to those τ whose argument lies in $[0, \pi/3]$. In fact, this description of the parameter space can be shown to be a reformulation in terms of τ of Pratoussevitch's result, Proposition 1 of [14], for our special case. We will give details of how to pass from τ to Pratoussevitch's parameters in the next section.

We conclude this section by considering the case where the vectors $\mathbf{n}_j$ are linearly dependent, that is when N is singular. Using Lemma 2.2, this implies that

$$0 = 8 + \tau^3 + \overline{\tau}^3 - 6|\tau|^2 = (2 + \tau + \overline{\tau})(2 + \tau\overline{\omega} + \overline{\tau}\omega)(2 + \tau\omega + \overline{\tau}\,\overline{\omega}).$$

It will be convenient to make a choice of which one of these linear factors is zero. In Section 4.1 below it will be useful to suppose that $\tau\omega + \overline{\tau}\,\overline{\omega} = -2$ and so we focus on that case here. We shall also explain how to obtain the formulae in the other two cases. We begin with a geometrical description of the complex lines L_1, L_2 and L_3.

Proposition 2.5 *Let L_1, L_2 and L_3 be complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Suppose that the Hermitian products of these vectors satisfy (2). Then*

(i) $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 0$ *and* $|\tau| < 2$ *if and only if L_1, L_2 and L_3 have a unique point of intersection in $\mathbf{H}^2_{\mathbb{C}}$;*

(ii) $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 0$ *and* $|\tau| = 2$ *if and only if L_1, L_2 and L_3 coincide*

(iii) $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 0$ *and* $|\tau| > 2$ *if and only if there is a complex line $L^\perp$ in $\mathbf{H}^2_{\mathbb{C}}$ that is orthogonal to each of L_1, L_2 and L_3.*

Proof: From Lemma 2.2 we see that $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 0$ if and only the matrix N defined above is singular. This is true if and only if there exists a non-zero vector $\mathbf{z}_0$ so that

$$\mathbf{0} = N^* H \mathbf{z}_0 = \begin{bmatrix} \langle \mathbf{z}_0, \mathbf{n}_1\rangle \\ \langle \mathbf{z}_0, \mathbf{n}_2\rangle \\ \langle \mathbf{z}_0, \mathbf{n}_3\rangle \end{bmatrix}.$$

Therefore $\mathbf{z}_0$ satisfies

$$\langle \mathbf{z}_0, \mathbf{n}_1\rangle = \langle \mathbf{z}_0, \mathbf{n}_2\rangle = \langle \mathbf{z}_0, \mathbf{n}_3\rangle = 0. \qquad (6)$$

Hence $\mathbf{z}_0$ either corresponds to a common intersection point of L_1, L_2 and L_3 or to the polar vector of a common orthogonal complex line. Which of these possibilities occurs depends on whether $\mathcal{N}(L_1, L_2) = |\tau|^2/4$ is greater than, equal to or less than 1, using Proposition 2.1. This proves (i) and (iii).

In order to prove the result, all that remains is to consider the case when $|\tau| = 2$ and to decide whether the complex lines are asymptotic or coincide. In this case

$$0 = 8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 2\mathrm{Re}\,(\tau^3) - 16$$

and so $\tau^3 = 8$ and so $\tau/2$ is a cube root of unity. In this case the matrix N has rank 1 and so N^*H has a two dimensional kernel. The projection of this kernel is $L_1 = L_2 = L_3$, proving (ii). $\square$

We can reinterpret Proposition 2.5 in terms of the group $\Delta = \langle I_1, I_2, I_3 \rangle$. This group fixes a point of $\mathbf{H}^2_{\mathbb{C}}$, that is it is elementary, if and only if $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 0$ and $|\tau| \leq 2$. If such a group is discrete then it must be finite. In particular, the case $|\tau| = 2$ corresponds to the order 2 group where $I_1 = I_2 = I_3$. On the other hand, Δ preserves a complex line if and only if $8 + 2\mathrm{Re}\,(\tau^3) - 6|\tau|^2 = 0$ and $|\tau| > 2$. If such a group is discrete then it is Fuchsian.

When $|\tau| \neq 2$, that is in the cases given by Proposition 2.5 (i) and (iii), we can once again choose a basis and use this to write down the Hermitian form and matrix representatives for I_1, I_2, I_3 and J. Suppose that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ satisfy (2). As in the proof of Proposition 2.5, there exists $\mathbf{z}_0$ satisfying (6). As was indicated in the proof of Proposition 2.5, this vector is negative when $|\tau| < 2$ and positive when $|\tau| > 2$ (recall we have excluded the case of $|\tau| = 2$). Thus, without loss of generality, we suppose that

$$\langle \mathbf{z}_0, \mathbf{z}_0 \rangle = |\tau| - 2.$$

Let N' be the matrix whose columns are $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{z}_0$. As our Hermitian form is non-degenerate, it is clear that $\mathbf{z}_0$ is not in the span of $\mathbf{n}_1$ and $\mathbf{n}_2$. Therefore $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{z}_0\}$ is a basis of $\mathbb{C}^3$ and so N' is non-degenerate. Moreover, if H is the matrix of the Hermitian form then

$$N'^* H N' = \begin{bmatrix} 2 & \tau & 0 \\ \overline{\tau} & 2 & 0 \\ 0 & 0 & |\tau| - 2 \end{bmatrix}.$$

We choose coordinates so that

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the matrix N' is the identity, and our Hermitian form is given by the following matrix H'_τ, which only depends on τ:

$$H'_\tau = \begin{bmatrix} 2 & \tau & 0 \\ \overline{\tau} & 2 & 0 \\ 0 & 0 & |\tau| - 2 \end{bmatrix}.$$

Notice that H'_τ has eigenvalues $2 + |\tau|$, $2 - |\tau|$ and $|\tau| - 2$ and so it has signature $(2, 1)$ whenever $|\tau| \neq 2$. Once more, we can use (1) to find I_1 and I_2. They are

$$I_1 = \begin{bmatrix} 1 & \tau & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \overline{\tau} & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Using $\tau\omega + \overline{\tau}\,\overline{\omega} = -2$, one may easily find $\mathbf{n}_3$ and this allows us to write down I_3 and J:

$$\mathbf{n}_3 = \begin{bmatrix} -\omega \\ -\overline{\omega} \\ 0 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1+\overline{\tau}\,\overline{\omega} & -\overline{\tau}\omega & 0 \\ -\tau\overline{\omega} & 1+\tau\omega & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -\omega & 0 \\ 1 & -\overline{\omega} & 0 \\ 0 & 0 & \overline{\omega} \end{bmatrix}, \tag{7}$$

Once again, it is clear that the groups $\Gamma = \langle I_1,\, J\rangle$ and $\Delta = \langle I_1,\, I_2,\, I_3\rangle$ are completely determined up to conjugation by the parameter τ. In the remaining two cases. namely $\tau\overline{\omega} + \overline{\tau}\omega = -2$ and $\tau + \overline{\tau} = -2$ the formulae for $\mathbf{n}_3$, I_3 and J are obtained from (7) by swapping ω and $\overline{\omega}$ or by replacing both ω and $\overline{\omega}$ by 1, respectively.

2.3 Traces

We have seen that if τ satisfies (5) then τ corresponds to a group $\Gamma = \langle I_1,\, J\rangle$, in $\mathrm{SU}(2,1)$, with a equilateral triangle group $\Delta = \langle I_1,\, I_2,\, I_3\rangle$ as an index three normal subgroup. In this section we write down the traces of certain elements of Γ in terms of τ. This should be compared to Theorem 9 of [14] where Pratoussevitch gives (in our language) formulae for the traces of elements in Δ that are integer polynomials in $|\tau|^2$, τ^3 and $\overline{\tau}^3$. We could write down the traces directly from our expressions for J, I_1, I_2 and I_3. We choose to give a more general argument as this is more illuminating and is independent of our choice of Hermitian form.

Lemma 2.6 *Let A be any element of* $\mathrm{SU}(2,1)$. *Then*

$$\mathrm{tr}(I_1A) = -\mathrm{tr}\,(A) + 2\frac{\langle A(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}.$$

PROOF: Using (1) we see that

$$I_1A(\mathbf{z}) = -A(\mathbf{z}) + 2\frac{\langle A(\mathbf{z}), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}\mathbf{n}_1.$$

We must find the trace of the matrix corresponding to the linear map $T : \mathbf{z} \longmapsto \langle A(\mathbf{z}), \mathbf{n}_1\rangle \mathbf{n}_1$. Writing $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^*H\mathbf{z}$ and using the fact that this is a complex scalar, we have

$$\langle A(\mathbf{z}), \mathbf{n}_1\rangle \mathbf{n}_1 = \mathbf{n}_1\mathbf{n}_1^*HA\mathbf{z} = \big(\mathbf{n}_1(A^*H\mathbf{n}_1)^*\big)\mathbf{z}.$$

Hence the matrix of T is $\mathbf{n}_1(A^*H\mathbf{n}_1)^*$. Now if a matrix can be written in the form $\mathbf{u}\mathbf{v}^*$ for column vectors $\mathbf{u}$ and $\mathbf{v}$, then its trace is just $\mathbf{v}^*\mathbf{u}$. Thus

$$\mathrm{tr}\big(\mathbf{n}_1(A^*H\mathbf{n}_1)^*\big) = (A^*H\mathbf{n}_1)^*\mathbf{n}_1 = \mathbf{n}_1^*HA\mathbf{n}_1 = \langle A(\mathbf{n}_1), \mathbf{n}_1\rangle.$$

Hence

$$\mathrm{tr}(I_1A) = -\mathrm{tr}\,(A) + 2\frac{\mathrm{tr}\big(\mathbf{n}_1(A^*H\mathbf{n}_1)^*\big)}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle} = -\mathrm{tr}\,(A) + 2\frac{\langle A(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}.$$

□

Corollary 2.7 *Let I_1 be a complex involution fixing a complex line L_1 with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ be a regular elliptic map of order 3. Then*

$$\mathrm{tr}(I_1J) = 2\frac{\langle J(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}.$$

PROOF: This follows directly from the previous lemmas using the fact that, since J is regular elliptic of order three, its trace is zero. $\square$

Using equation (2) and the fact that $\mathbf{n}_{j+1} = J(\mathbf{n}_j)$, an immediate consequence of Corollary 2.7 is that $\mathrm{tr}(I_j J) = \tau$. This fact is our justification for naming our parameter τ. The following theorem is just a restatement of Theorem 6.2.4 of [5]

Proposition 2.8 *Let τ be given by (2). Then $I_1 J$ is regular elliptic if and only if*

$$|\tau|^4 - 4\tau^3 - 4\overline{\tau}^3 + 18|\tau|^2 - 27 < 0.$$

The curve given by the equality in Proposition 2.8 is a deltoid; see Figure 1. Groups for which $I_1 J$ is regular elliptic correspond to points in the interior of this deltoid. Points on the deltoid correspond to points where $I_1 J$ is either a complex reflection or is parabolic. Since $I_1 I_2 I_3 = (I_1 J)^3$ we can determine the type of $I_1 I_2 I_3$ from $I_1 J$. (Note that it may be that $I_1 J$ is regular elliptic and that $I_1 I_2 I_3$ is a complex reflection.)

We now consider $I_1 I_2$, and hence by symmetry $I_2 I_3$ and $I_3 I_1$ as well.

Proposition 2.9 *Let L_1 and L_2 be complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with polar vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively. Suppose that $\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = 2$ and $\langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \tau$. Let I_1 and I_2 denote the complex involutions fixing L_1 and L_2. Then*

$$\mathrm{tr}(I_1 I_2) = |\tau|^2 - 1.$$

PROOF: Let $\mathbf{z}_2$ and $\mathbf{z}_3$ be any distinct vectors on L_1 (for example, if the fixed point of J is not on L_1, we could choose $\mathbf{z}_2 = L_1 \cap L_3$ and $\mathbf{z}_3 = L_1 \cap L_2$). Then $\{\mathbf{n}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is a basis for $\mathbb{C}^{2,1}$. We write $\mathbf{n}_2$ in terms of this basis as:

$$\mathbf{n}_2 = \alpha \mathbf{n}_1 + \beta \mathbf{z}_2 + \gamma \mathbf{z}_3.$$

Since $\mathbf{n}_1$ is orthogonal to $\mathbf{z}_2$ and $\mathbf{z}_3$ then $\langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \alpha \langle \mathbf{n}_1, \mathbf{n}_1 \rangle$ and so $\alpha = \tau/2$. Since $\mathbf{n}_1$ is a 1-eigenvector for I_1 and $\mathbf{z}_2$, $\mathbf{z}_3$ are both (-1)-eigenvectors we have

$$\mathbf{n}_2 = \frac{\tau}{2}\mathbf{n}_1 + \beta \mathbf{z}_2 + \gamma \mathbf{z}_3, \qquad I_1(\mathbf{n}_2) = \frac{\tau}{2}\mathbf{n}_1 - \beta \mathbf{z}_2 - \gamma \mathbf{z}_3.$$

Adding these two expressions we see that $I_1(\mathbf{n}_2) = \tau \mathbf{n}_1 - \mathbf{n}_2$. Therefore

$$\langle I_1(\mathbf{n}_2), \mathbf{n}_2 \rangle = \tau \langle \mathbf{n}_1, \mathbf{n}_2 \rangle - \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \frac{|\tau|^2}{2} \langle \mathbf{n}_2, \mathbf{n}_2 \rangle - \langle \mathbf{n}_2, \mathbf{n}_2 \rangle.$$

Hence using Lemma 2.6 we have

$$\mathrm{tr}(I_2 I_1) = -\mathrm{tr}(I_1) + 2\frac{\langle I_1(\mathbf{n}_2), \mathbf{n}_2 \rangle}{\langle \mathbf{n}_2, \mathbf{n}_2 \rangle} = 1 + |\tau|^2 - 2 = |\tau|^2 - 1.$$

$\square$

If an element of $\mathrm{SU}(2,1)$ with real trace is elliptic then its trace lies in $[-1,3)$ and conversely any element of $\mathrm{SU}(2,1)$ whose trace lies in the real interval $(-1,3)$ is elliptic. If the trace is -1 it is either elliptic or parabolic. Hence we see that $I_1 I_2$ is elliptic if and only if $|\tau| < 2$. In Figure 1 we show how the circle $|\tau| = 2$ compares to the deltoid of Proposition 2.8. Furthermore, we can now see that Pratoussevitch's parameters [14] may be written in terms of τ as:

$$r_1 = r_2 = r_3 = |\tau|/2, \qquad \alpha = \arg(\tau^3).$$

Using Proposition 2.1 we can, in fact, relate the trace of $I_1 I_2$ to the relative position of I_1 and I_2.

Corollary 2.10 *Let L_1 be a complex line in $\mathbf{H}^2_{\mathbb{C}}$ with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ and write $L_2 = J(L_1)$. Let I_1 and I_2 denote the complex involutions fixing L_1 and L_2. Then:*

(i) If L_1 and L_2 are ultraparallel then $\mathrm{tr}(I_1I_2) > 3$*;*

(ii) If L_1 and L_2 are asymptotic then $\mathrm{tr}(I_1I_2) = 3$*;*

(iii) If L_1 and L_2 intersect with angle θ then $\mathrm{tr}(I_1I_2) = 2\cos(2\theta) + 1$.

3 When $I_1I_2I_3$ is elliptic and I_1I_2 is non-loxodromic

This section is the heart of the paper. We restrict our attention to those groups for which $I_1I_2I_3$ is elliptic of finite order and I_1I_2 is either elliptic of finite order or else parabolic. These are groups for which τ lies inside or on the deltoid and inside or on the circle in Figure 1. Since they have finite order, the eigenvalues of $I_1I_2I_3$ and I_1I_2 are all roots of unity (in the case where I_1I_2 is parabolic then its eigenvalues are all 1). This fact leads to a linear equation in certain cosines of rational multiples of π. We find all solutions to this equation using a theorem of Conway and Jones [1]. We then go on to find which of these solutions lie in parameter space, that is, which of the solutions lie outside the central triangle in Figure 1. As we have already indicated, it suffices to consider those τ whose argument lies in $[0, \pi/3]$. Such values of τ lying outside the central triangle and yet inside both the deltoid and circle are shown in Figure 2.

3.1 The eigenvalue equation

We now investigate when both I_1I_2 and $I_1I_2I_3$ are elliptic of finite order. In fact our proof will be valid when I_1I_2 is parabolic and yields a new proof of Theorem 1.1. We know that, I_1J (and hence $I_1I_2I_3$) is elliptic of finite order if and only if

$$\tau = \mathrm{tr}(I_1J) = e^{i\alpha} + e^{i\beta} + e^{-i\alpha-i\beta}, \tag{8}$$

where α and β are rational multiples of π. Likewise for I_1I_2. In fact we know slightly more. Since the intersection of L_1 and L_2 is a (-1)-eigenvector for each of I_1 and I_2 it must be a $(+1)$-eigenvector for I_1I_2. Hence the eigenvalues of I_1I_2 are 1, $e^{2i\theta}$ and $e^{-2i\theta}$. That is

$$|\tau|^2 - 1 = \mathrm{tr}(I_1I_2) = 2\cos(2\theta) + 1, \tag{9}$$

where θ is a rational multiple of π. From Corollary 2.10 (iii) we see that, geometrically, θ is just the angle between L_1 and L_2. If $\theta = 0$ then I_1I_2 is parabolic (or the identity) and we shall include this case in our analysis.

We solve equations (8) and (9) by eliminating τ. That is, we seek θ, α, β rational multiples of π so that

$$2\cos(2\theta) + 2 = |\tau|^2 = 3 + 2\cos(\alpha-\beta) + 2\cos(\alpha+2\beta) + 2\cos(-2\alpha-\beta).$$

Rearranging, this becomes

$$\frac{1}{2} = \cos(2\theta) - \cos(\alpha-\beta) - \cos(\alpha+2\beta) - \cos(-2\alpha-\beta). \tag{10}$$

Notice that there is a certain amount of ambiguity in the solutions of this equation. Given one solution $\alpha-\beta$, $\alpha+2\beta$ and $2\alpha-\beta$ we obtain other solutions by a sequence of the following operations:

(i) permuting $\alpha-\beta$, $\alpha+2\beta$ and $-2\alpha-\beta$,

(ii) changing the sign of all three of them;

(iii) adding a multiple of 2π to one of them and subtracting the same multiple of 2π from another.

Using the fact that all three angles sum to zero, the net result of these ambiguities is to possibly complex conjugate τ and/or to multiply τ by a power of $\omega = e^{2\pi i/3}$. These are precisely the ambiguities in τ we already know about. For example, adding 2π to $\alpha + 2\beta$ and subtracting 2π from $-2\alpha - \beta$ sends α to $\alpha + 2\pi/3$ and β to $\beta + 2\pi/3$. Hence it sends τ to $\tau\omega$. Likewise, swapping $\alpha + 2\beta$ and $-2\alpha - \beta$ sends α to $-\beta$ and β to $-\alpha$. This has the effect of sending τ to $\overline{\tau}$.

The theorem that we use to find all solutions to (10) is the following wonderful theorem due to Conway and Jones, Theorem 7 of [1]:

Theorem 3.1 (Conway and Jones [1]) *Suppose that we are given at most four distinct rational multiples of π lying strictly between 0 and $\pi/2$ for which some rational linear combination of their cosines is rational, but no proper subsum has this property. Then this linear combination is proportional to one of the following:*

(a) $\frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$,

(b) $0 = -\cos(\phi) + \cos\left(\phi - \frac{\pi}{3}\right) + \cos\left(\phi + \frac{\pi}{3}\right)$ *where* $0 < \phi < \frac{\pi}{6}$,

(c) $\frac{1}{2} = \cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right)$,

(d) $\frac{1}{2} = \cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right)$,

(e) $\frac{1}{2} = \cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{15}\right) + \cos\left(\frac{4\pi}{15}\right)$,

(f) $\frac{1}{2} = -\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{15}\right) - \cos\left(\frac{7\pi}{15}\right)$,

(g) $\frac{1}{2} = \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) - \cos\left(\frac{\pi}{21}\right) + \cos\left(\frac{8\pi}{21}\right)$,

(h) $\frac{1}{2} = \cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{21}\right) - \cos\left(\frac{5\pi}{21}\right)$,

(i) $\frac{1}{2} = -\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{4\pi}{21}\right) + \cos\left(\frac{10\pi}{21}\right)$,

(j) $\frac{1}{2} = -\cos\left(\frac{\pi}{15}\right) + \cos\left(\frac{2\pi}{15}\right) + \cos\left(\frac{4\pi}{15}\right) - \cos\left(\frac{7\pi}{15}\right)$.

We claim that we may find all solutions to equation (10) by inspection from this theorem. In order to see this, observe that by sending ϕ to $\pi - \phi$ we send $\cos\phi$ to $-\cos\phi$. Thus, by allowing the angles to lie in $(0, \pi)$, for each equation in Theorem 3.1 we arrange for all the signs in front of the cosines to be the same. We then look for three of the angles that (after possibly changing their sign) add up to a multiple of 2π. Once again, working mod 2π, we adjust the angles so that they sum to zero. The resulting angles are $\alpha - \beta$, $\alpha + 2\beta$ and $-2\alpha - \beta$. The trickiest cases are those where there are fewer than four angles listed in Theorem 3.1. One then has to use one of the following identities to reconstruct equation (10):

(k) $1 = \cos(0)$,

(l) $0 = \cos\left(\frac{\pi}{2}\right)$,

(m) $0 = \cos(\phi) + \cos(\pi - \phi)$ for some angle ϕ.

Note the identity given in Theorem 3.1 (b) holds for all angles $\phi \in [0, 2\pi)$. The condition $0 < \phi < \pi/6$ was only there to ensure that the three angles are in $(0, \pi/2)$. Since we are adding multiples of π to our angles and changing their sign where necessary, when using Theorem 3.1 (b) we allow ϕ to be any angle.

Hence, by inspection, we find the only candidates for $\alpha - \beta$, $\alpha + 2\beta$, $-2\alpha - \beta$ solving equation (10) and we list them in the following table along with θ (we have ordered them so that $\alpha - \beta < \alpha + 2\beta < 2\alpha + \beta$). From this we find α, β and $\alpha + \beta$. In the last column we indicate which of the identities (a) to (m) we have used.

	2θ	$\alpha - \beta$	$\alpha + 2\beta$	$2\alpha + \beta$	α	β	$\alpha + \beta$	
(i)	$2\pi/3$	$\pi - \phi/2$	π	$2\pi - \phi/2$	$\pi - \phi/3$	$\phi/6$	$\pi - \phi/6$	$(a),\ (k),\ (m)$
(ii)	ϕ	$\pi/3 - \phi$	$\pi/3 + \phi$	$2\pi/3$	$\pi/3 - \phi/3$	$2\phi/3$	$\pi/3 + \phi/3$	$(a),\ (b)$
(iii)	$\pi/3$	$\pi/4$	$\pi/2$	$3\pi/4$	$\pi/3$	$\pi/12$	$5\pi/12$	$(a),\ (l),\ (m)$
(iv)	$\pi/5$	$3\pi/10$	$2\pi/5$	$7\pi/10$	$\pi/3$	$\pi/30$	$11\pi/30$	$(c),\ (m)$
(v)	$3\pi/5$	$\pi/10$	$4\pi/5$	$9\pi/10$	$\pi/3$	$7\pi/30$	$17\pi/30$	$(c),\ (m)$
(vi)	$\pi/2$	$2\pi/7$	$4\pi/7$	$6\pi/7$	$8\pi/21$	$2\pi/21$	$10\pi/21$	$(d),\ (l)$
(vii)	$\pi/2$	$\pi/15$	$11\pi/15$	$4\pi/5$	$13\pi/45$	$2\pi/9$	$23\pi/45$	$(e),\ (l)$
$(viii)$	$\pi/2$	$7\pi/15$	$17\pi/15$	$8\pi/5$	$31\pi/45$	$2\pi/9$	$41\pi/45$	$(f),\ (l)$
(ix)	$\pi/7$	$\pi/21$	$4\pi/7$	$13\pi/21$	$2\pi/9$	$11\pi/63$	$25\pi/63$	(g)
(x)	$5\pi/7$	$5\pi/21$	$19\pi/21$	$8\pi/7$	$29\pi/63$	$2\pi/9$	$43\pi/63$	(h)
(xi)	$3\pi/7$	$11\pi/21$	$25\pi/21$	$12\pi/7$	$47\pi/63$	$2\pi/9$	$61\pi/63$	(i)

We reiterate that each line in this table is a representative of several equivalent solutions. These are obtained by permutation, changing sign and adding a multiple of $2\pi/3$ to both α and β. For example, the solution (iii) also corresponds to the following pair of solutions (reordered so that $\beta < \alpha$):

$$\begin{array}{lll} \alpha = \pi/3 + 2\pi/3 = \pi, & \beta = \pi/12 + 2\pi/3 = 3\pi/4, & \alpha + \beta = 5\pi/12 + 4\pi/3 = 7\pi/4; \\ \alpha = 2\pi/3 - \pi/12 = 7\pi/12, & \beta = 2\pi/3 - \pi/3 = \pi/3, & \alpha + \beta = 4\pi/3 - 5\pi/12 = 11\pi/12. \end{array}$$

We then write down $\tau = \mathrm{tr}(I_1 J) = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$ and $\mathrm{tr}\,(I_1 I_2 I_3) = e^{3i\alpha} + e^{3i\beta} + e^{-3i\alpha - 3i\beta}$ using this table. As indicated earlier, the parameters $\tau\omega^j$ and $\overline{\tau}\omega^j$ correspond to the same group as τ. So in the case of (iii) where $\tau = e^{i\pi/3} + e^{-i\pi/6} 2\cos(\pi/4)$ the two equivalent solutions listed above yield, respectively:

$$\begin{aligned} \tau &= -1 + i2\cos(\pi/4) &= e^{2\pi i/3}\big(e^{i\pi/3} + e^{-i\pi/6} 2\cos(\pi/4)\big), \\ \tau &= e^{i\pi/3} - e^{-i\pi/6} 2\cos(\pi/4) &= e^{2\pi i/3}\big(e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/4)\big). \end{aligned}$$

Evaluating τ from each line in the table gives the following result. We have kept the same labelling (i) to (xi) as in the table.

Proposition 3.2 *Suppose that $I_1 I_2$ and $I_1 I_2 I_3$ are both elliptic of finite order (or possibly $I_1 I_2$ is parabolic). Up to complex conjugating τ and multiplying by a power of ω, then one of the following is true:*

(i) $\tau = -e^{-i\phi/3}$ *for some angle ϕ that is a rational multiple of π;*

(ii) $\tau = e^{2i\phi/3} + e^{-i\phi/3} = e^{i\phi/6}\, 2\cos(\phi/2)$ *for some angle ϕ that is a rational multiple of π;*

(iii) $\tau = e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/4)$;

(iv) $\tau = e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/5)$;

(v) $\tau = e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(2\pi/5)$;

(vi) $\tau = e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}$;

(vii) $\tau = e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/5)$;

(viii) $\tau = e^{2\pi i/9} + e^{-\pi i/9}\, 2\cos(4\pi/5)$;

(ix) $\tau = e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/7)$;

(x) $\tau = e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(4\pi/7)$;

(xi) $\tau = e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(6\pi/7)$.

Observe that the first two cases of the above theorem include various elementary groups:

- Putting $\phi = 0$ in (i) we obtain $\tau = -1$ which yields the elementary group of order 6 where $J = I_1 I_2$. Multiplying by $\overline{\omega}$, we see that this value of τ is equivalent to $\tau = e^{i\pi/3}$.
- Putting $\phi = 0$ in (ii) we obtain $\tau = 2$ which yields the elementary group of order 2 where $I_1 = I_2 = I_3$.
- Putting $\phi = \pi/2$ in (ii) gives $\tau = e^{\pi i/3}(1 - i)$.

These three groups will be important for our discussion of Theorem 3.7 (i) below. We shall discuss elementary groups in more detail in Section 4.1. Moreover, it is clear that the only solution with $\theta = 0$ involves setting $\phi = 0$ in part (ii). This gives a new proof of Schwartz's theorem, Theorem 1.1.

3.2 Solutions in parameter space

We now consider the values of τ found in Proposition 3.2 and we check which of them satisfy the conditions of Corollary 2.4. In other words, we find which of them lies in one of the regions in Figure 1 where the signature is $(2,1)$. Note that since $|\tau| \le 2$ the only possibilities are that H_τ has signature $(2,1)$ or $(3,0)$. We state our results in terms of the signature of H_τ. First, for τ given in part (i) H_τ has signature $(3,0)$ unless $\tau = -1$ (or $-\omega$ or $-\overline{\omega}$), when H_τ is degenerate. We now consider the other cases one by one.

Lemma 3.3 *If $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ then H_τ has signature $(2,1)$ if and only if $0 < \cos\phi < 1$ and signature $(3,0)$ if and only if $-1 \le \cos\phi < 0$. When $\cos\phi = 0$ or 1 then H_τ is degenerate.*

PROOF: We have $|\tau|^2 = 2 + 2\cos\phi$ and $\tau^3 = e^{2i\phi} + 3e^{i\phi} + 3 + e^{-i\phi}$. Therefore

$$6|\tau|^2 - \tau^3 - \overline{\tau}^3 - 8 = 6(2 + 2\cos\phi) - 2(2\cos^2\phi + 4\cos\phi + 2) - 8 = 4\cos\phi - 4\cos^2\phi.$$

This is positive when $0 < \cos\phi < 1$ and negative when $-1 \le \cos\phi < 0$. □

Lemma 3.4 *If $\tau = e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}$ then H_τ has signature $(3,0)$.*

PROOF: We can rewrite τ as $(-1 + i\sqrt{7})/2$. Thus from Lemma 2.3, the eigenvalues of H_τ are

$$2 + \tau + \overline{\tau} = 1, \qquad 2 + \tau\overline{\omega} + \overline{\omega}\tau = \frac{5 + \sqrt{21}}{2}, \qquad 2 + \tau\omega + \overline{\tau}\,\overline{\omega} = \frac{5 - \sqrt{21}}{2}.$$

These are all positive. □

Lemma 3.5 *Suppose that $\tau = e^{\pi i/3} + e^{-\pi i/6} 2\cos\phi$ for some ϕ. Then H_τ is degenerate, that is it has determinant zero.*

Proof: We have

$$\tau\omega + \overline{\tau}\,\overline{\omega} = (-1 + 2i\cos\phi) + (-1 - 2i\cos\phi) = -2.$$

Therefore, using Lemma 2.3 we see that H_τ has an eigenvalue of 0, and hence is degenerate. □

Lemma 3.6 *If $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos\phi$ for some ϕ then H_τ has signature $(2,1)$ if and only if $\cos(3\phi) < -1/2$ and signature $(3,0)$ if and only if $\cos(3\phi) > -1/2$. If $\cos(3\phi) = -1/2$ then H_τ is degenerate.*

Proof: We have

$$|\tau|^2 = 1 + 2\cos\phi + 4\cos^2\phi, \qquad \tau^3 = e^{2\pi i/3} + e^{\pi i/3} 6\cos\phi + 12\cos^2\phi + e^{-\pi i/3} 8\cos^3\phi.$$

Thus

$$\begin{aligned} 6|\tau|^2 - \tau^3 - \overline{\tau}^3 - 8 &= 6(1 + 2\cos\phi + 4\cos^2\phi) - (-1 + 6\cos\phi + 24\cos^2\phi + 8\cos^3\phi) - 8 \\ &= -1 + 6\cos\phi - 8\cos^3\phi \\ &= -1 - 2\cos(3\phi). \end{aligned}$$

This is positive if and only if $\cos(3\phi) < -1/2$ and negative if and only if $\cos(3\phi) > -1/2$. □

We have shown that the values of τ given in parts (vi), (viii), (x) and (xi) of Proposition 3.2 correspond to values of τ for which H_τ has signature $(3,0)$. Thus they do not correspond to groups in $\mathrm{SU}(2,1)$. There are six values of τ where H_τ is degenerate, each of which has the form $\tau = e^{\pi i/3} + e^{-\pi i/6} 2\cos(\phi)$. First, the three values with $\phi = \pi/4$, $\pi/5$, $2\pi/5$ come from parts (iii), (iv) and (v) of Proposition 3.2. Secondly, there are the three values $\tau = e^{i\pi/3}$, 2, $e^{i\pi/3}(1-i)$ listed at the end of Section 3.1 and which correspond to elementary groups. These values correspond to $\phi = \pi/2$, $\pi/6$, $\pi/3$.

We now summarise the results of this section.

Theorem 3.7 *Let I_1, $I_2 = JI_1J^{-1}$ and $I_3 = J^{-1}I_1J$ be involutions in $\mathrm{SU}(2,1)$ each fixing a complex line, where $J \in \mathrm{SU}(2,1)$ has order three. Suppose that I_1I_2 and $I_1I_2I_3 = (I_1J)^3$ both have finite order. If $\Delta = \langle I_1, I_2, I_3\rangle$ is discrete then, up to complex conjugating or multiplying by ω or $\overline{\omega}$, one of the following is true:*

(i) $\tau = e^{\pi i/3} + e^{-\pi i/6} 2\cos(2\pi/n) = e^{i\pi/3}(1 - 2i\cos(2\pi/n))$ *where* $n = 4$, 5, 6, 8, 10 *or* 12.

(ii) $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ *where ϕ is a rational multiple of π in $(0, \pi/2)$;*

(iii) $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(2\pi/5)$*;*

(iv) $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(2\pi/7)$*.*

Note that the cases $n = 6$ and $n = 12$ in (i) correspond to the endpoints of the open interval given in (ii).

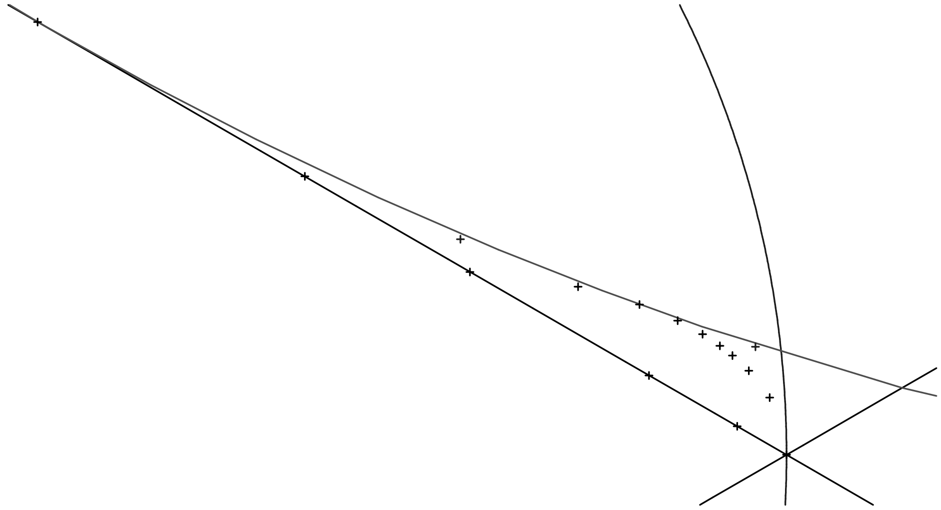

Figure 2: *An enlarged view of part of Figure 1 showing the values of* τ *in Theorem 3.7. Of the groups from Theorem 3.7 (ii) we have only plotted those that are discrete, as enumerated in Proposition 4.5.*

4 The discrete groups

In this section we analyse all the groups from Theorem 3.7. We show that those groups listed in parts (i), (iii) and (iv) are all discrete and that finitely many of those given in (ii) are discrete. The parameter values corresponding to discrete groups are all plotted in Figure 2.

4.1 The elementary groups

In this section we consider what happens when $\tau = e^{\pi i/3} + e^{-\pi i/6} 2\cos\phi = e^{i\pi/3}(1 - 2i\cos\phi)$ where $\phi = 2\pi/n$ for $n = 4$, 5, 6, 8, 10 or 12. These are the groups from Proposition 3.2 (iii), (iv), (v) together with the cases where $\tau = e^{i\pi/3}$, $\tau = 2$ and $\tau = e^{\pi i/3}(1-i)$. These are the six parameter values listed in Theorem 3.7 (i).

When $\tau = 2$ all three complex lines coincide and so J maps L_1 to itself, fixing a single point of L_1. Moreover, I_1 and J commute and $\langle I_1, J\rangle$ is a cyclic group of order six, generated by I_1J.

In each case τ satisfies $\tau\omega + \overline{\tau}\,\overline{\omega} = -2$ which is one of the three linear factors of the equation $6|\tau|^2 - \tau^3 - \overline{\tau}^3 - 8 = 0$. Thus, by Lemma 2.2 the polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are linearly dependent. In all cases, except for $\tau = 2$, we can write down matrix representatives for I_1, I_2, I_3 and J as was done at the very end of Section 2.2. These matrices are block diagonal, the upper left hand block lying in a copy of U(2) (preserving the Hermitian form given by the upper left hand 2×2 block of H'_τ). We can multiply them by scalars so that the upper left hand block has determinant 1. This yields

$$iI_1 = \begin{bmatrix} i & i\tau & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}, \qquad \omega J = \begin{bmatrix} 0 & -\overline{\omega} & 0 \\ \omega & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad i\omega I_1 J = \begin{bmatrix} -2\cos\phi - i & 2\overline{\omega}\cos\phi & 0 \\ -i\omega & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

where we have used $\tau = e^{\pi i/3}(1 - 2i\cos\phi) = -\overline{\omega} + 2i\overline{\omega}\cos\phi$. By examining the traces of the upper left hand blocks of iI_1, ωJ and $i\omega I_1 J$ we see that they form a dihedral ($n = 4$), tetrahedral ($n = 6$), octahedral ($n = 8$) or icosahedral ($n = 5$, 10) group. Therefore $\langle I_1, J\rangle$ is a finite central extension of such a group and hence, in each case, is finite

We remark that in each case $\mathrm{tr}(I_j I_{j+1}) = 4\cos^2\phi = 1 + 2\cos(2\theta)$. We could read off the values of θ from the table given earlier or we can calculate them directly. They are:

ϕ	$4\cos^2\phi$	$2\cos(2\theta)$	2θ
$\pi/2$	0	-1	$2\pi/3$
$2\pi/5$	$(3-\sqrt{5})/2$	$(1-\sqrt{5})/2$	$3\pi/5$
$\pi/3$	1	0	$\pi/2$
$\pi/4$	2	1	$\pi/3$
$\pi/5$	$(3+\sqrt{5})/2$	$(1+\sqrt{5})/2$	$\pi/5$
$\pi/6$	3	2	0

4.2 The groups with $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(\phi)$.

In this section we consider the equilateral triangle groups corresponding to $\tau = e^{2\pi i/9}\big(1 - 2\omega\cos\phi\big)$ where $\omega = e^{2\pi i/3}$ is a cube root of unity and $\phi = 2\pi/5$ or $2\pi/7$. We remark that when $\phi = 2\pi/6$ we obtain $\tau = e^{2\pi i/9}\big(1 - 2\omega\cos\phi\big) = e^{2\pi i/9} + e^{-\pi i/3}$ which is one of the groups from Theorem 3.7 (ii) and is treated in Section 4.3. We want to eliminate the factor of $e^{2\pi i/9}$ from our matrix entries. Therefore we apply the matrix $C = \mathrm{diag}(e^{-2\pi i/9}, 1, e^{2\pi i/9})$ to the $\mathbf{n}_j$, and hence to the whole set-up. The images of the polar vectors under C are:

$$\mathbf{n}_1 = \begin{bmatrix} e^{-2\pi i/9} \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ e^{2\pi i/9} \end{bmatrix}.$$

The matrix N whose columns are $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ is simply C. Therefore, the new Hermitian form, which we call H_ϕ, satisfies $C^* H_\phi C = H_\tau$. Clearly $C^* = C^{-1}$ and so

$$H_\phi = C H_\tau C^{-1} = \begin{bmatrix} 2 & 1 - 2\omega\cos\phi & \overline{\omega} - 2\omega\cos\phi \\ 1 - 2\overline{\omega}\cos\phi & 2 & 1 - 2\omega\cos\phi \\ \omega - 2\overline{\omega}\cos\phi & 1 - 2\overline{\omega}\cos\phi & 2 \end{bmatrix}.$$

Obviously, since H_τ and H_ϕ are conjugate, they have the same eigenvalues. Using Lemma 3.6 we immediately have

Lemma 4.1 *When $\phi = 2\pi/5$ or $\phi = 2\pi/7$ the matrix H_ϕ has signature* $(2,1)$. *When $\phi = 4\pi/5$, $4\pi/7$ or $6\pi/7$ the matrix H_ϕ has signature* $(3,0)$.

After conjugating by C, the complex involutions are given by

$$I_1 = \begin{bmatrix} 1 & 1 - 2\omega\cos\phi & \overline{\omega} - 2\omega\cos\phi \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad (11)$$

$$I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 - 2\overline{\omega}\cos\phi & 1 & 1 - 2\omega\cos\phi \\ 0 & 0 & -1 \end{bmatrix}, \qquad (12)$$

$$I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \omega - 2\overline{\omega}\cos\phi & 1 - 2\overline{\omega}\cos\phi & 1 \end{bmatrix}. \qquad (13)$$

The entries of I_1, I_2 and I_3 all have determinant one and lie in the ring $\mathbb{Z}[\omega, 2\cos\phi]$. It is standard to write $\mathrm{SU}(H;\mathcal{O})$ for the group of unimodular matrices preserving the Hermitian form H whose entries lie in the ring $\mathcal{O}$. Thus:

Lemma 4.2 *The group* $\Delta = \langle I_1, I_2, I_3 \rangle$ *generated by the matrices given in (11), (12) and (13) is a subgroup of* $\mathrm{SU}\big(H_\phi; \mathbb{Z}[\omega, 2\cos\phi]\big)$.

Since ω and $2\cos(2\pi/n)$ are both algebraic integers, we see that every element of the ring $\mathbb{Z}\big[\omega,\, 2\cos(2\pi/n)\big]$ is an algebraic integer in $\mathbb{Q}\big(\omega,\, 2\cos(2\pi/n)\big)$. The field $\mathbb{Q}\big(\omega,\, 2\cos(2\pi/n)\big)$ is a totally imaginary quadratic extension of the totally real number field $\mathbb{Q}\big(2\cos(2\pi/n)\big)$. The following result (in the case $\phi = 2\pi/5$) is essentially identical to that given by Deraux in Corollary 2.6 of [3] and is similar to the proof of Corollary 1.4 of [18].

Proposition 4.3 *For* $\phi = 2\pi/5$ *or* $\phi = 2\pi/7$ *the group* $\mathrm{SU}\big(H_\phi; \mathbb{Z}[\omega, 2\cos\phi]\big)$ *is arithmetic and hence discrete. In particular,* Δ *is discrete.*

Proof: We give the proof in the case of $n = 7$. The proof for $n = 5$ is almost the same; see also Corollary 2.6 of [3].

The field $\mathbb{Q}\big(\omega, 2\cos(2\pi/7)\big)$ is a totally imaginary quadratic extension of the totally real number field $\mathbb{Q}\big(2\cos(2\pi/7)\big)$. Let $\mathbb{Q}(c')$ be the totally real number field obtained from any non-trivial Galois conjugate of c' of $2\cos(2\pi/7)$ and let $\mathbb{Q}(\omega, c')$ be a compatible quadratic extension corresponding to $\mathbb{Q}(\omega, c)$. The only Galois conjugates of $2\cos(2\pi/7)$ are $2\cos(4\pi/7)$ and $2\cos(6\pi/7)$.

Let $H_{4\pi/7}$ and $H_{6\pi/7}$ be the Hermitian forms obtained by applying these Galois automorphisms to $H_{2\pi/7}$. From Lemma 4.1 we see that $H_{4\pi/7}$ and $H_{6\pi/7}$ are positive definite. Therefore the corresponding groups $\mathrm{SU}(H_{4\pi/7})$ and $\mathrm{SU}(H_{6\pi/7})$ are compact.

Let $x \in \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]$. Then x is an algebraic integer in $\mathbb{Q}\big(\omega, 2\cos(2\pi/7)\big)$. Let x' and x'' be its Galois conjugates in $\mathbb{Q}\big(\omega, 2\cos(4\pi/7)\big)$ and $\mathbb{Q}\big(\omega, 2\cos(6\pi/7)\big)$. The map $x \longmapsto (x, x', x'')$ maps $\mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]$ to a discrete subset of $\mathbb{C}^3$. Hence

$$\mathrm{SU}\big(H_{2\pi/7}; \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]\big) \times \mathrm{SU}\big(H_{4\pi/7}; \mathbb{Z}\big[\omega, 2\cos(4\pi/7)\big]\big) \times \mathrm{SU}\big(H_{6\pi/7}; \mathbb{Z}\big[\omega, 2\cos(6\pi/7)\big]\big)$$

is discrete. Since $\mathrm{SU}(H_{4\pi/7})$ and $\mathrm{SU}(H_{6\pi/7})$ are compact, the image of projection onto the first factor, namely $\mathrm{SU}\big(H_{2\pi/7}; \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]\big)$ is also discrete. □

Note that the groups we eliminated from Proposition 3.2 (viii), (x) and (xi) using Lemma 3.6 are just the Galois conjugates of the two groups we are considering.

Proposition 4.4 *The group with* $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(2\pi/5)$ *is Deraux's lattice.*

Proof: We calculate that $|\tau|^2 = 1 + 2\cos(2\pi/5) + 4\cos^2(2\pi/5) = 2$. Hence $I_1 I_2$ has order 4. The eigenvalues of $I_1 I_2 I_3$ are $(e^{2\pi i/9})^3 = \omega$, $(e^{-\pi i/9 \pm 2\pi i/5})^3 = \omega e^{\pm i\pi/5}$. These are the same as the eigenvalues of $I_1 I_2 I_3$ found by Deraux in equation (2.15) of [3]. Thus the groups are the same. □

We now briefly discuss the group with $\tau = e^{2\pi i/9} + e^{-\pi i/9} 2\cos(2\pi/7)$. This does not seem to have previously appeared in the literature. It is easy to show that $\mathrm{tr}(I_1 I_2) = 1 + 2\cos(\pi/7)$ and so $I_1 I_2$ has order 14. Furthermore, $\mathrm{tr}(I_1 I_2 I_1 I_3) = 2\cos(\pi/7) + 2 > 3$ and so $I_1 I_2 I_1 I_3$ is loxodromic. Since $I_1 I_2 I_3$ is elliptic, this means that, in the language of [17], Δ is of Type B; see also Proposition 7.5 of [12] for other discrete, unfaithful groups of Type B. It is not clear whether or not Δ is the whole of the lattice $\mathrm{SU}\big(H_7; \mathbb{Z}\big[\omega, 2\cos(2\pi/7)\big]\big)$. The fact that $I_1 I_2 I_1 I_3$ is loxodromic would indicate that, in fact, Δ may not be a lattice. This group merits further investigation.

4.3 Subgroups of Livné's lattices

We consider the groups for which $\tau = e^{2i\phi/3} + e^{-i\phi/3} = e^{i\phi/6} 2\cos(\phi/2)$ for some angle $\phi \in (0, \pi/2)$. Note that we have already treated the cases of $\phi = 0$ and $\phi = \pi/2$ in Section 4.1. The main result of this section is

Proposition 4.5 *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ for some angle $\phi \in (0, \pi/2)$. Then the group $\langle I_1, I_1, I_3 \rangle$ is discrete if and only if $\phi = 2\pi/p$ where $p = 5, 6, 7, 8, 9, 10, 12$ or 18.*

We remark that if $p = 6$ then $I_1 I_2 I_3$ is parabolic and so we are not in the case considered in Theorem 3.7. In this case it is particularly easy to prove discreteness and we include it for completeness. As we remarked in Section 4.2, when $p = 6$ we can write $\tau = e^{2\pi i/9}(1 - 2\omega \cos\phi) = e^{2\pi i/9}(1 - \omega)$. We can then conjugate I_1, I_2 and I_3 into the forms (11), (12) and (13) respectively. These matrices all have entries in the ring of Eisenstein integers $\mathbb{Z}[\omega]$, as does every matrix in $\Delta = \langle I_1, I_2, I_3 \rangle$. Since $\mathbb{Z}[\omega]$ is a discrete subring of $\mathbb{C}$ we see that Δ is discrete; see [4] for a more detailed discussion of this group.

We begin by showing that if ϕ does not take one of the values listed in Proposition 4.5 then $\Delta = \langle I_1, I_2, I_3 \rangle$ cannot be discrete. We do this by showing that the subgroup $\langle I_1, I_1 I_2 I_3 \rangle$ of Δ is not discrete for these values of τ. In this discussion we exclude the case of $\phi = \pi/3$ (that is $p = 6$) which we have already discussed.

We can see from line (ii) of the table given given in Section 3.1 that the eigenvalues of $I_1 J$ are $e^{i\alpha} = e^{i(\pi - \phi)/3}$, $e^{i\beta} = e^{2i\phi/3}$ and $e^{-i(\alpha+\beta)} = e^{i(-\pi-\phi)/3}$. Hence the eigenvalues of $I_1 I_2 I_3 = (I_1 J)^3$ are $e^{3i\alpha} = -e^{-i\phi}$, $e^{3i\beta} = e^{2i\phi}$ and $e^{-3i(\alpha+\beta)} = -e^{-i\phi}$. Therefore in this case $I_1 I_2 I_3$ has a repeated eigenvalue and so is a complex reflection with rotation angle $\pi - 3\phi$; it cannot be parabolic as it is the cube of an elliptic map. (Note that when $\phi = 2\pi/6$ then $I_1 J$ has a repeated eigenvalue and is parabolic, as is $I_1 I_2 I_3$.) By examining the eigenvectors, one can show that when $\phi < 2\pi/6$ then $I_1 I_2 I_3$ is complex reflection in a complex line and when $\phi > 2\pi/6$ then $I_1 I_2 I_3$ is complex reflection in a point. We will give the details in the former case. The latter case is almost identical.

Lemma 4.6 *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ with $0 < \phi < \pi/3$. Let L_1 and L_{123} be the complex lines fixed by I_1 and $I_1 I_2 I_3$ respectively. These complex lines are ultraparallel and their common orthogonal $L^\perp$ is preserved by the group $\langle I_1, I_1 I_2 I_3 \rangle$. This group acts on $L^\perp$ as the index 2 holomorphic subgroup of the group generated by reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\phi/2$, $(\pi - 3\phi)/2$.*

Proof: Since I_1 and $I_1 I_2 I_3$ are complex reflections, they preserve all complex lines orthogonal to L_1 and L_{123}, respectively. Suppose that L_1 and L_{123} are not ultraparallel. Let their intersection be $\mathbf{z}$. Then $\mathbf{z}$ is fixed by I_1 and by $I_1 I_2 I_3$. Hence it is also fixed by $I_2 I_3$ and so must be $L_2 \cap L_3$. In other words, $\mathbf{z}$ is fixed by I_1, I_2 and I_3 and hence the group must be elementary. This is a contradiction to Lemmas 2.2 and 3.3.

Hence L_1 and L_{123} are ultraparallel. Their common orthogonal is preserved by I_1 and $I_1 I_2 I_3$ which act as rotations through angles π and $\pi - 3\phi$ respectively. Moreover $L^\perp$ is preserved by $I_2 I_3$ and so its polar vector must be an eigenvector of $I_2 I_3$. The eigenvalues of $I_2 I_3$ corresponding to positive vectors are $e^{i\phi}$ and $e^{-i\phi}$. Hence $I_2 I_3$ acts on $L^\perp$ as a rotation through ϕ. □

In the case where $\pi/3 < \phi < \pi/2$ then $I_1 I_2 I_3$ is complex reflection in a point. The complex line $L^\perp$ is now the complex line through this point orthogonal to L_1. A similar argument shows that $\langle I_1, I_1 I_2 I_3 \rangle$ acts on $L^\perp$ as the index 2 holomorphic subgroup of the group generated by reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\phi/2$, $(3\phi - \pi)/2$. We can now use plane hyperbolic geometry to complete the proof of Proposition 4.5.

Proposition 4.7 *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ with $0 < \phi < \pi/2$ and $\phi \neq \pi/3$. The group $\langle I_1, I_1I_2I_3\rangle$ is discrete if and only if $\phi = 2\pi/p$ for $p = 5, 7, 8, 9, 10, 12$ or 18.*

PROOF: We consider the subgroup $\langle I_1, I_1I_2I_3\rangle$ and its action on $L^\perp$. This group is generated by two elliptic maps whose product is also elliptic. In Theorem 2.3 of [8], Knapp has characterised when such a group is discrete. Our case is particularly easy because I_1 has order 2 and so one of the angles in our triangle is a right angle. Hence if $\langle I_1, I_1I_2I_3\rangle$ is discrete then we are in Case I or Case IV of Knapp's theorem. In other words either $\phi/2 = \pi/p$ and $|\pi - 3\phi|/2 = \pi/d$ or else one of $\phi/2$ and $|\pi - 3\phi|/2$ equals π/m and the other equals $2\pi/m$ for some odd integer m. In fact, if we solve for m in the last case we see that either $m = 10$ or $m = 14$, neither of which is odd. Therefore we must have $\phi = 2\pi/p$ and $|\pi - 3\phi| = \pi|p - 6|/p = 2\pi/d$. The values of p in the proposition are precisely those (with $2\pi/p < \pi/2$) for which $2p/|p - 6|$ is an integer d. □

This shows that when ϕ is not one of the given values then the group Δ is not discrete. It remains to show that the values of ϕ listed in Proposition 4.5 do indeed correspond to discrete groups. This follows immediately from the following result and will complete the proof of Proposition 4.5.

Proposition 4.8 (Corollary 7.4 of [12]) *Let $\tau = e^{2i\phi/3} + e^{-i\phi/3}$. When $\phi = 2\pi/p$ for $p = 5, 6, 7, 8, 9, 10, 12$ or 18 the group $\Delta = \langle I_1, I_2, I_3\rangle$ is discrete. Moreover, when $p = 5$ this group is a cocompact lattice in $\mathrm{SU}(2,1)$, when $p = 6, 7, 8, 9, 10, 12$ or 18 it is geometrically infinite.*

We now give a brief discussion of how one might prove Proposition 4.8. In [12], this result is proved by demonstrating that, for such ϕ, the group Δ is a normal subgroup of one of the lattices first described by Livné [9]. There are several ways to show that Livné's groups are discrete. For example, fundamental domains for these groups were constructed in [12], and presentations were given using Poincaré's polyhedron theorem. Alternatively, one could relate such a group to one of Mostow's ball 5-tuples and then use his discreteness criterion ΣINT [11]; see also Theorem 16.1 of [2]. The fact that this criterion precisely characterises discreteness is due to Sauter [15], who analysed the few remaining cases not treated by Mostow.

Alternatively, for $p = 5, 6, 7, 8, 10, 12, 18$ we could show that Δ is arithmetic and hence discrete using a similar argument to Proposition 4.3. In doing this we use Lemma 3.3 to show that H_τ has signature $(2,1)$ if and only if $0 < \cos\phi < 1$. This argument does not work when $p = 9$. The Galois conjugates of $2\cos(2\pi/9)$ are $2\cos(4\pi/9)$ and $2\cos(6\pi/9)$. Because $\cos(4\pi/9) > 0$, Lemma 3.3 implies that the corresponding Hermitian form has signature $(2,1)$. In fact when $p = 9$ the group Δ is non-arithmetic and, in this case, one must use a geometrical argument.

Furthermore, there are other ways of showing that values of τ listed in Proposition 4.5 are the only ones of this form that correspond to discrete groups. We could again use Mostow's ΣINT condition. A more direct approach would be to use the complex hyperbolic Jørgensen's inequality [7] to show that when ϕ is not one of the angles listed above then Δ is not discrete.

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