Review of:

COMPLEX HYPERBOLIC GEOMETRY

William M. Goldman

The unit ball in $\mathbb{C}^n$, and complex manifolds for which this is the universal cover, have been studied from many points of view ranging from complex analysis to algebraic geometry. Until recently the literature contained comparatively little about these subjects from a purely geometrical viewpoint. After an auspicious beginning with major work of Picard, Giraud and Élie Cartan the geometrical side of the subject fell into decline. A revival of interest began about 25 years ago with major contributions by Chen, Greenberg, Mostow and others. This resurgence has been intensified over the last ten years, largely inspired by Goldman's interest. This makes the publication of the book under review very timely as well as an invaluable guide to the recent developments.

The book is a monumental result of an investigation of complex hyperbolic geometry conducted over more than a decade. It contains a wealth of useful, beautiful and intriguing facts that the author has discovered during this study. Many of these results are of fundamental importance for those studying complex hyperbolic geometry. As well as pioneering new areas of the subject, the book is anchored into the existing literature (for example it contains a commentary on Giraud's seminal paper). There is a long bibliography and references for further reading are provided throughout the text. There are many different conventions and systems of notation in the literature. This potential source of confusion is minimised by Goldman, who fixes conventions and notation throughout. It is to be hoped that (unless there are clear reasons for not doing so) writers of future papers and books in this subject will either adopt Goldman's notation or else provide a clear means of translating between their conventions and his.

The unit ball in $\mathbb{C}^n$ has a natural metric of constant negative holomorphic sectional curvature, called the Bergman metric. As such it forms a model for *complex hyperbolic n-space* $\mathbf{H}^n_{\mathbb{C}}$ analogous to the ball model of (real) hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$. The main difference is that the (real) sectional curvature is no longer constant, but is pinched between two negative numbers whose ratio is 4. Goldman normalises so that the holomorphic sectional curvature is -1 which means that the sectional curvatures lie in the interval $[-1, -1/4]$. (There seems to be no consensus about which interval to take. Different choices lead to awkward factors of 2 or 4 in various key places.) The geometry of $\mathbf{H}^n_{\mathbb{C}}$ is not a completely straightforward generalisation of $\mathbf{H}^n_{\mathbb{R}}$. Aspects such as negative curvature and the fact that maximal parabolic subgroups are nilpotent rather than Abelian tend to make it hard to generalise real hyperbolic results to the complex case. However, the complex structure gives more tools for solving problems (for example complex hyperbolic space is a Kähler manifold). These two effects tend to cancel one another out. It is not the case that results from $\mathbf{H}^n_{\mathbb{R}}$ either generalise to $\mathbf{H}^n_{\mathbb{C}}$ or else break down. Using analogy as a guide, one can often formulate qualitatively similar results, but the methods of proof a usually rather different. This makes the subject rich and leads to many surprises when trying to make these analogies more concrete.

As a first step to understanding the geometry of the unit ball with the Bergman metric, it is natural to consider the totally geodesic submanifolds. Goldman shows that these are either embedded copies of $\mathbf{H}^m_{\mathbb{C}}$ or $\mathbf{H}^m_{\mathbb{R}}$ for $1 \leq m \leq n$. Thus, the real dimension of a totally geodesic submanifold is either at most n or else is even. In particular, there are no totally geodesic real hypersurfaces in complex hyperbolic n-space (for $n \geq 2$). This increases the difficulty of constructing polyhedra (for example fundamental polyhedra for discrete groups of complex hyperbolic isometries). One of the major themes of this book is the study of a particular class of real hypersurfaces which are a good substitute for totally geodesic ones. These hypersurfaces are bisectors: the locus of points equidistant from a particular pair of points. The boundary of a bisector is called a spinal sphere. Although not totally geodesic, bisectors are foliated by totally geodesic real and complex submanifolds. Important consequences of their not being totally geodesic are that bisectors have non-trivial intersection properties and polyhedra whose boundaries are made up of pieces of bisectors are not convex. A goal of this book is to classify bisector intersections. In the process, many beautiful aspects of the geometry of complex hyperbolic geometry are developed. In the light of recent developments, bisectors and spinal spheres may not play such an important role as was first thought. Bisectors are rather inflexible and polyhedra whose faces are pieces of bisector tend to be rather complicated. These difficulties have recently been overcome by Schwartz who constructs "hybrid spheres" and Falbel who constructs "$\mathbb{C}$-spheres". These generalise spinal spheres and form the boundaries of richer classes of hypersurfaces from which one may construct polyhedra. Importantly, they are much more flexible and promise to be much more powerful tools.

The boundary of complex hyperbolic n-space is the one point compactification of the $(2n-1)$-dimensional Heisenberg group in the same way that the boundary of real hyperbolic n-space is the one point compactification of Euclidean $(n-1)$-space. Just as the internal geometry of real hyperbolic space may be studied using conformal geometry on the boundary, so the internal geometry of complex hyperbolic space may be studied using CR-geometry on the Heisenberg group. The foliations of bisectors by real and complex totally geodesic subspaces give rise to two foliations of spinal spheres by lines of longitude and latitude. In addition, when $n = 2$, the boundary is 3-dimensional and so it is easy to illustrate geometrical objects. This book is very profusely illustrated and many of the figures show objects on the boundary of complex hyperbolic space in this way. In order to do this the author, together with Mark Phillips and Robert Miner, developed a computer programme called HEISENBERG. This is very useful for gaining insight into the geometry of objects in $\mathbf{H}^2_{\mathbb{C}}$. These illustrations are fascinating and beautiful in their own right. Often Goldman gives a sequence of views of the same geometric objects, for example Figs. 4.5–4.8. It is quite a challenge to reconstruct these three dimensional objects from such a sequence of illustrations – and then to visualise the four dimensional objects of which these are the boundaries. Most of the figures are very well drawn, but there are exceptions (such as Fig. 4.4).

The book has a coherent structure and is well organised – even if the organisation is, of necessity, rather complicated at times. In order to make the book easier to use, it would have been helpful to have an index of formulae. A very useful feature, particularly for those using the book to learn the subject, is the first chapter where some of the material is worked out in the classical case of $n = 1$. After this chapter, Goldman spends time in outlining the diverse background material he will use from algebra, geometry and analysis. Only at this point does the main discussion of objects in complex hyperbolic space and on its boundary begin. There are sections on the two main models of complex hyperbolic space and their boundaries followed by an extensive discussion of bisectors and spinal spheres, automorphisms and numerical invariants. The amount of material it contains makes the book rather daunting, particularly to those learning the subject. Furthermore, much of the material is treated from a very refined point of view. This can make parts of the book rather terse and it may seem unmotivated. In fact, this is not the case: there are many applications contained in the bibliography. For example, one of the main motivating forces behind the book is the study of discrete groups of complex hyperbolic isometries and fundamental polyhedra for such groups. Though isometries are mentioned throughout the book, there is relatively little material about discrete groups, and fundamental polyhedra only make it into the last section.

The field of complex hyperbolic geometry is wide open and is currently enjoying more interest than for many years. This book will certainly be of paramount importance in future progress. In the various preprint versions the book has become a standard reference and, now that a definitive version has been published, it is a necessary item for the library of everyone working in this field. It makes a rather challenging introduction to the subject but is an invaluable source of useful facts. I strongly recommend it to all those who work on related fields from differential geometry to several complex variables and from symplectic topology to discrete groups.

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