Imprecise Probabilities — Discussion and Open Problems

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Abstract— **If only little, vague or conflicting information is available about a system, it is hard to construct and motivate a model based on the so-called precise probability theory. In this case, generalisations of this theory—called in the literature imprecise probability theories—prove to be useful. We discuss the most satisfactory of these models, the unifying behavioural theory of lower previsions, we identify a number of open problems in the theory, and we briefly point to work that is being done to solve them.**

Keywords— **uncertainty, systems, modelling.**

I. Introduction

In modelling a system, it often occurs that some of its aspects, or some of the influences acting on it, are not well known. The uncertainty this produces about the system's behaviour is usually modelled by a probability measure, and treated using techniques from probability theory—we could call this the Bayesian approach. It can deal efficiently with uncertainty in many situations (Bayesian learning, data compression, pattern recognition, reliability, population models in biology, etc.).

Such a model will often not be adequate—it might be unreasonable to identify a specific and unique probability measure. This can happen because

- the information is scarce ("Tossing this coin three times resulted in tail, tail and tail. What will be the result of the next toss?");
- the information is vague, for example when only linguistic information is available ("Mary is young."); or
- the information is conflicting, for example when there is a conflict between prior information and statistical data, or when trying to combine the information of a number of experts, or a group, in order to make group decisions.

In particular, the following classes of problems would benefit from a generalisation of the theory of probability:

- decision making with little information [1] (e.g., processing subjective knowledge and expert estimates [2]);
- optimisation using an imprecise cost criterion (e.g., selecting an optimal consumption path in a complex economical society trying to avoid a ecological catastrophe [3]);
- the estimation of quantities which depend on parameters that are not well known (e.g., the time to failure of a component in a system [4]; modelling an opponent in a game-theoretical setting [5], [6]);
- modelling systems based on linguistic information [4], [7];
- classification [8], [9] and pattern recognition; and
- constructing, and reasoning on, hierarchical models [10] (e.g., in order to make group decisions).

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II. Imprecise Probabilities

Generalisations of probability theory have been developed and applied in order to represent and manipulate the really available knowledge about the system. These extensions are called imprecise probabilities—most commonly found in the literature are: comparative probability orderings [11], [12], [13]; Choquet capacities [14]; belief functions [15], [16]; possibility measures [17]; fuzzy measures [18]; lower (and upper) previsions, sets of desirable gambles or convex sets of probability distributions [19]; and risk measures [20].

It has also been shown that in some situations, particularly when only little information is available, using imprecise probabilities leads to better results than using a Bayesian model. As an illustrative example [6], consider a two-player zero-sum game, in which the second player plays a constant strategy. Simulations show that, a (first) player which uses an imprecise probability model to represent the constant player, makes more profit than a player which uses a Bayesian model. The difference is most striking at the start of the game, and as the players play longer, the difference becomes negligible. This shows that, when little information is available, an imprecise probability model may perform better, and it also justifies the use of a Bayesian model when a lot of information is available. This of course motivates further research in the—far from complete—theory of imprecise probabilities.

III. Lower Previsions

Walley's theory of lower previsions [19] unifies most of the above-mentioned models, and from a foundational point of view it seems to be the most satisfactory one by far. Briefly, it consists of

- an assessment of the behaviour of the system in certain situations—this represents the available knowledge about the system;
- rationality criteria—these are used to identify conflicts in the information and to determine whether the model is consistent or not; and
- a reasoning/inference method that calculates how the system should behave in other situations—this tells us how to draw conclusions from, and make decisions based on, the available knowledge.

From a mathematical point of view, specifying a consistent lower prevision is equivalent to specifying a set of (finitely) additive probability distributions—thus the theory of lower previsions can in certain respects be interpreted in terms of Bayesian sensitivity analysis.

However, the theory of lower previsions can be developed without such an interpretation, and without a prior need for precise Bayesian models. Precise probabilities are a special case of lower previsions, and the laws of precise probability theory follow from the above-mentioned rationality criteria, as special cases. Classical probability theory can be used as a source of inspiration for the development of new analytical methods in the theory of imprecise probabilities. Conversely, as a number of studies (see for instance [21]) have already shown, looking at classical probability theory from a broader perspective provides additional insights for that theory as well.

IV. Open Problems

Despite the fact that the theory of lower previsions is very appealing, it has a few shortcomings in the following respects:

- A technical problem is that lower previsions only deal with bounded random variables, whereas applications involving unbounded random variables abound. In particular, the following classes of problems would benefit from a generalisation of imprecise probability theory:
  - optimisation using an imprecise cost criterion, with an unbounded (e.g., quadratic) cost; and
  - the estimation of unbounded quantities which depend on parameters that are not well known.

  We are studying this problem in the SYSTeMS group and we have constructed an extension to unbounded random variables on which the Bayesian sensitivity interpretation also continues to hold. We are still working on the study of consistency and inference for lower previsions on unbounded random variables.
- A dynamical approach to imprecise systems is still in its developing phase. It should however be mentioned that imprecise probability theory has a learning tool, called the *imprecise Dirichlet model* [22] which allows a type of inductive reasoning that has distinct advantages over its Bayesian counterpart. It has already been applied with success in classification problems [9], game-theoretic learning [5], [6] and we are currently studying its application to learning in hidden Markov models. There are also indications that the theory of differential inclusions might provide mathematical tools for studying so-called imprecise dynamical systems [23].
- The relation of the theory of lower previsions to the classical von Neumann utility theory [24] and its generalisations [1], [25], commonly applied in economics, needs to be clarified. There are strong indications that this could lead to interesting insights in epistemology and the study of logic [21].

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