Matthias C. M. Troffaes and Gert de Cooman. The combination of conflicting information. In *Proceedings of the 3rd FTW PhD Symposium*, Dec 2002.

The Combination of Conflicting Information

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Abstract— **We argue that the theory of imprecise probabilities can be used to represent expert information. We indicate how conflicting assessments may be combined through a second-order uncertainty model. As an illustration, we show that our method can be used to deal with Poincaré's paradox.**

Keywords— **imprecise probabilities, lower prevision, vacuous, Bayesian probability, natural extension, marginal extension, conflict, second-order model.**

I. Introduction

When modeling a system, one must often rely on expert information. Such information may be represented in various forms, and the modeler has to manipulate it in such a way that he can represent it in a uniform manner, combine it, and make inferences about the system.

It is clearly an impossible task to model all types of information that experts might use, but here we suggest a representation that is able to model the 'standard' types of expert assessment. Any reasonable representation of expert assessments should satisfy the following requirements:
• *inference*—it should enable us to infer useful conclusions about the values of physical variables such as temperature, pressure, distance, time, concentration, etc. In order to stress that these variables are not always well-known we call them *random* variables (random temperature, etc.).
• *conditional inference*—it should allow us to infer useful conclusions about the values of physical, real-valued variables, conditional on observations of other variables.
• *(independent) product*—it should allow us to combine expert assessments about the values of two or more different (independent) variables.
• *conjunction*—it should allow us to combine two or more assessments about the value of a single variable.
A number of commonly used representations of expert assessments, and in particular propositional logic and classical (Bayesian) probability theory, satisfy the first three of these requirements. But they all have a problem with the last requirement: conjunction rules easily lead to contradiction, as is evidenced by Poincaré's paradox in propositional logic [1], and by conflicting priors in Bayesian probability theory. In the present work, we propose a method for dealing with such conflicts in a systematic way.

II. Lower Previsions

Our method is based on a powerful class of methods for representing and handling uncertain information, namely the fairly recent theory of imprecise probabilities [2]. Very roughly speaking, it attempts to model a subject's beliefs about the values of a random variable of interest, say $a \in \mathcal{A}$, through the assessment of a supremum buying price $\underline{P}(X)$ for a bounded random utility X, also called a *gamble*, whose value depends on a. The domain of $\underline{P}$, i.e., the *finite* set of those gambles X for which assessments $\underline{P}(X)$ are given, is denoted by $\mathcal{K}$. The set of all gambles that depend on the random variable a is denoted by $\mathcal{L}(\mathcal{A})$. The map $\underline{P}$ is called a *lower prevision.* Examples of lower previsions are:
• If "a belongs to the set $A \subseteq \mathcal{A}$" then $\underline{P}_A(X) = \inf_{a\in A} X(a)$: the lowest possible reward given that $a \in A$. We call $\underline{P}_A$ the vacuous lower prevision relative to A.
• If "a has probability density ϕ" we should pay $P(X) = \int_{\mathcal{A}} X(a)\phi(a)\mathrm{d}a$, the expectation w.r.t. ϕ [3], [4]. This is called the linear prevision induced by the density ϕ.
• Similarily, "a has conditional probability density $\phi(\cdot|b)$" induces a conditional linear prevision $P(X|b) = \int_{\mathcal{A}} X(a)\phi(a|b)\mathrm{d}a$.
• If "a has a probability density that belongs to the set Φ" we pay at most $\underline{P}(X) = \inf_{\phi\in\Phi} \int_{\mathcal{A}} X(a)\phi(a)\mathrm{d}a$.
These examples indicate that lower previsions are uncertainty *representations* that are expressive enough to generalise both propositional logic and Bayesian probability theory (for more details see [2]). We now introduce a method of *inference*, associated with lower previsions, that also generalises the inference methods of classical propositional logic and Bayesian probability theory.

A. Inference

Through a procedure called *natural extension*, we are able to derive from the assessments embodied in $\underline{P}$ a supremum buying price $\underline{E}(X)$ for each gamble X in $\mathcal{L}(\mathcal{A})$; $\underline{E}$ is the smallest (and therefore most conservative) lower prevision that satisfies, for any gambles X and Y
• $\underline{E}(X) \geq \inf[X]$ (avoiding sure loss)
• $\underline{E}(\lambda X) = \lambda\underline{E}(X)$ whenever $\lambda > 0$ (scale independence)
• $\underline{E}(X + Y) \geq \underline{E}(X) + \underline{E}(Y)$ (super-additivity)
• $\underline{E}(X) \geq \underline{P}(X)$ (compatibility)
If $\underline{E}$ exists, $\underline{P}$ is said to *avoid sure loss.* The natural extension $\underline{E}(X)$ can be easily calculated: it is equal to the maximum achieved by the free variable α subject to

$$X(a) - \alpha \geq \sum_{Y\in\mathcal{K}} \lambda_Y\big(Y(a) - \underline{P}(Y)\big)$$

for each $a \in \mathcal{A}$, with variables $\lambda_Y \geq 0$ for each $Y \in \mathcal{K}$—if $\mathcal{A}$ is finite, this is a linear program. If the maximum is $\alpha = +\infty$, then the natural extension does not exist; this identifies a conflict in the assessments. If $\underline{E}$ and $\underline{P}$ coincide on $\mathcal{K}$, then $\underline{P}$ is called *coherent.* All the above-mentioned example lower previsions are coherent.

Inference about other variables, say a variable b, that depend on a, is achieved through the pull-back of random utilities X depending on b: $\underline{E}^b(X) = \underline{E}(X \circ b)$. This returns

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the supremum buying price $\underline{E}^b$ for gambles X that depend on b in terms of the natural extension $\underline{E}$ of a lower prevision $\underline{P}$ defined on gambles depending on a.

B. Conditional Inference

If we have a coherent lower prevision $\underline{P}(\cdot|b)$ on $\mathcal{L}(\mathcal{A})$, conditional on the variable $b \in \mathcal{B}$, and a coherent lower prevision $\underline{Q}$ on $\mathcal{L}(\mathcal{B})$, the *marginal extension theorem* [2] tells us that

$$\underline{E}(X) = \underline{Q}(\underline{P}(X|\mathcal{B})),$$

with $\underline{P}(X|\mathcal{B})\colon b \mapsto \underline{P}(X|b)$, is the supremum buying price for a gamble $X \in \mathcal{L}(\mathcal{A})$. Observe that this generalises Kolmogorov's definition of conditional probability [5].

C. Product

Given the coherent lower previsions $\underline{P}_i$ on $\mathcal{K}_i \subseteq \mathcal{L}(\mathcal{A}_i)$ for $i \in \{1,\dots,n\}$, the *product* of $(\underline{P}_i)_{i=1}^n$ gives supremum buying prices for gambles which depend on all $a_i \in \mathcal{A}_i$. It is defined as the smallest coherent lower prevision on $\mathcal{L}(\times_{i=1}^n \mathcal{A}_i)$ with marginals $\underline{P}_i$. It is equal to the maximum achieved by the free variable α subject to

$$X(a_1,\dots,a_n) - \alpha \geq \sum_{i\in I}\sum_{Y_i\in\mathcal{K}_i} \lambda_{Y_i}\big(Y_i(a_i) - \underline{P}_i(Y_i)\big)$$

for each $a_i \in \mathcal{A}_i$, with variables $\lambda_{Y_i} \geq 0$ for each $Y_i \in \mathcal{K}_i$. It is possible, but beyond the scope of this extended abstract, to take independence of the variables into account.

D. Conjunction

Given the coherent lower previsions $\underline{P}_i$ on $\mathcal{K}_i \subseteq \mathcal{L}(\mathcal{A})$ for $i \in \{1,\dots,n\}$, the *conjunction* of $(\underline{P}_i)_{i=1}^n$ is defined as the smallest coherent lower prevision on $\mathcal{L}(\mathcal{A})$ that is compatible with all the $\underline{P}_i$. It is equal to the maximum achieved by the free variable α subject to

$$X(a) - \alpha \geq \sum_{i\in I}\sum_{Y_i\in\mathcal{K}_i} \lambda_{Y_i}\big(Y_i(a) - \underline{P}_i(Y_i)\big)$$

for each $a \in \mathcal{A}$, with variables $\lambda_{Y_i} \geq 0$ for each $Y_i \in \mathcal{K}_i$. If the maximum is $\alpha = +\infty$, then the conjunction does not exist: we say that the assessments $(\underline{P}_i)_{i=1}^n$ are conflicting.

III. Resolving Conflicting Assessments

We have just seen that conjunction need not always exist, because of conflicting assessments. The ultimate reason for this is that we require the conjunction to be compatible with all the assessments $\underline{P}_i$. If the conjunction does not exist, our trust in at least one of the assessments is misplaced, and hence, we should rather drop one or more of them. The problem is that it is not always clear which of the assessments should be dropped.

In order to solve this problem we suggest associating a degree of trust t_i with each of the assessments $\underline{P}_i$ (degrees of trust can be given an operational, behavioural definition). Applying the techniques form the theory of imprecise probabilities, it can be shown that the corresponding natural extension $\underline{E}^1(X)$ is now given by the minimum achieved by $\sum_{j\in J}\alpha_j \underline{R}_j(X)$ subject to $\sum_{j\in J}\alpha_j = 1$ and

$$\sum_{j\in J}\alpha_j \delta_{\underline{P}_i}(\underline{R}_j) \geq t_i,$$

for each $i \in \{1,\dots,n\}$, with variables $\alpha_j \geq 0$ for each $j \in J$, and where $(\underline{R}_j)_{j\in J}$ denotes an enumeration of all conjunctions of subsets of $(\underline{P}_i)_{i=1}^n$ and $\delta_{\underline{P}_i}(\underline{R}_j)$ is equal to 1 whenever $\underline{P}_i \leq \underline{R}_j$, and is 0 otherwise. If there is no feasible solution for $(\alpha_j)_{j\in J}$, then the degrees of trust are too large to resolve the inconsistencies. If all degrees of trust equal 1, we recover the usual formula for conjunction.

IV. Example: Poincaré's Paradox

Poincaré's paradox [1] arises when we consider three objects a, b and c, such that a cannot be distinguished from b, b cannot be distinguished from c, but clearly a is not equal to c. It thus consists of the assessments $a = b$, $b = c$ and $a \neq c$. We investigate to what extent these assessments are consistent within the present approach.

To this end, we assign an equal degree of trust t to each assessment. The conjunctions $\underline{R}_j$ and the coefficients $\delta_{\underline{P}_i}(\underline{R}_j)$ are listed in Table I. The corresponding linear programming problem has a feasible solution only for $t \leq \frac{2}{3}$. This means that Poincaré's paradox can be resolved only if we trust each assessment up to a degree of $66,7\%$. Since this conclusion only depends on the values of the $\delta_{\underline{P}_i}(\underline{R}_j)$, any three conflicting assessments that are pair-wise consistent, are actually consistent up to an equally distributed degree of trust of $66,7\%$.

TABLE I

$\delta_{\underline{P}_i}(\underline{R}_j)$ for Poincaré's Paradox

	$a=b$	$b=c$	$a\neq c$
no assessment	0	0	0
$a=b$	1	0	0
$b=c$	0	1	0
$a\neq c$	0	0	1
$a=b\wedge a\neq c$	1	0	1
$b=c\wedge a\neq c$	0	1	1
$a=b\wedge b=c$	1	1	0

Acknowledgments

The authors would like to thank the FWO for funding the research reported in this paper.

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