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Extension of coherent lower previsions to unbounded random variables

Matthias C. M. Troffaes and **Gert de Cooman**
Universiteit Gent, Onderzoeksgroep SYSTeMS
Technologiepark – Zwijnaarde 9, 9052 Zwijnaarde, Belgium
{matthias.troffaes,gert.decooman}@rug.ac.be

Abstract

We study the extension of coherent lower previsions from the set of bounded random variables to a larger set. An *ad hoc* method in the literature consists in approximating an unbounded random variable by a sequence of bounded ones. Its 'extended' lower prevision is then defined as the limit of the sequence of lower previsions of its approximations. We identify the random variables for which this limit does not depend on the details of the approximation, and call them previsible. We thus extend a lower prevision to previsible random variables, and we study the properties of this extension. We also consider the special case of super-modular lower previsions.

Keywords: Coherence, imprecise probabilities, Choquet integral, Dunford integral, Lebesgue dominated convergence theorem, lower prevision, unbounded random variable.

1 Introduction

In the literature, a number of theories are available for modelling uncertainty [5]. From a foundational point of view, the most satisfactory of these seems to be Walley's behavioural theory of imprecise probabilities [12], which can be formulated in terms of so-called coherent lower previsions.

One important shortcoming of the existing theory is that it only deals with random variables that are bounded, whereas in engineering, for instance, applications involving unbounded random variables abound [10]. To give only a few examples, the following classes of problems would certainly benefit from an extension of imprecise probability theory able to deal with unbounded random variables: (i) the estimation of unbounded quantities, such as the time to failure of a component in a system [11]; and (ii) optimisation involving an unbounded (e.g., quadratic) cost [4].

Loosely speaking, an intuitive, *ad hoc* way of dealing with an unbounded random variable is to approximate it by a sequence of bounded ones, and to use limit arguments in order to extend notions defined in the context of the bounded random variables to their unbounded counterparts, in the hope that the eventual result will not depend on the exact form of the approximation. Similar types of construction exist in the theory of integration—we shall use them as a source of inspiration.

Our main objectives in this paper are: (i) to construct an extension of coherent lower previsions from bounded random variables to a larger set; (ii) to study of the properties of this extension in order to motivate that it can be seen as a coherent lower prevision in its own right; and (iii) to provide a justification for the so-called cut-off method, where an unbounded random variable is approximated by a sequence of bounded cuts.

The paper is organised as follows. We give a brief introduction to the theory of imprecise probabilities in Section 2. In Section 3 we outline the basic concepts of our theory, after a short survey of relevant results in the existing theory of integration. Important properties of our extension method are listed in Section 4, and Section 5 contains a justification of the cut-off method through a Lebesgue dominated convergence theorem for coherent lower previsions. Finally, in Section 6 we show that for super-modular coherent lower previsions, there is a Choquet integral representation for their extension to unbounded random variables.

Throughout, given the limitations of space, we have preferred to stress the underlying ideas rather than to present detailed proofs. Readers interested in the proofs and in the exact details of the mathematical reasoning, are referred to [3].

2 Imprecise probabilities

We start with a brief introduction to the most important aspects of the existing behavioural theory of imprecise probabilities that are relevant to the problem at hand. More details can be found in [12].

Let us consider an agent who is uncertain about something, say, the outcome of some experiment. If the set of possible outcomes is Ω, then a *random variable* is a mapping from Ω to $\mathbb{R}$, and it is interpreted as an uncertain reward: if ω turns out to be the true outcome of the experiment then the agent receives the amount $X(\omega)$, expressed in units of some linear utility. Bounded random variables are also called *gambles*. They play a very important part in the existing theory. The set of all gambles is denoted by $\mathscr{L}(\Omega)$.

The information the agent has about the outcome of the experiment will lead him to accept or reject transactions whose reward depends on this outcome, and we can formulate a model for his uncertainty by looking at a specific type of transaction: buying gambles. The agent's *lower prevision* (or supremum acceptable buying price) $\underline{P}(X)$ for a gamble X is the highest price s such that he is disposed to buy the gamble X for any price strictly lower than s. If the agent assesses a supremum acceptable buying price for every gamble X in a subset $\mathcal{K}$ of $\mathscr{L}(\Omega)$, the resulting mapping $\underline{P} : \mathcal{K} \to \mathbb{R}$ is called a *lower prevision*.

It can be argued that $\underline{P}$ must satisfy the following rationality constraint: for every $n \in \mathbb{N}$, every $\lambda_0, \ldots, \lambda_n \geq 0$, and every $X_0, \ldots, X_n \in \mathcal{K}$ we must have that[1]

$$\sup\left[\sum_{i=1}^{n} \lambda_i X_i - \lambda_0 X_0\right] \geq \sum_{i=1}^{n} \lambda_i \underline{P}(X_i) - \lambda_0 \underline{P}(X_0).$$

Here and elsewhere, we denote by $\sup[X]$ the supremum value $\sup_{\omega\in\Omega} X(\omega)$ of the gamble X (and similarly for $\inf[X]$). If the lower prevision $\underline{P}$ satisfies this constraint, we say that it is *coherent*. If $\mathcal{K}$ is a linear space, e.g., when $\mathcal{K} = \mathscr{L}(\Omega)$, then $\underline{P}$ is coherent if and only if

$$\underline{P}(X) \geq \inf[X], \quad \underline{P}(\lambda X) = \lambda \underline{P}(X), \text{ and} \quad \underline{P}(X+Y) \geq \underline{P}(X) + \underline{P}(Y),$$

for all gambles X, Y in $\mathcal{K}$ and $\lambda \geq 0$.

It can be shown that if $\underline{P}$ is coherent, there always exists a (unique) smallest coherent extension of $\underline{P}$ from its domain $\mathcal{K}$ to $\mathscr{L}(\Omega)$. This extension is called the *natural extension* of $\underline{P}$ and it is given by

$$\underline{E}(X) = \sup\left\{\inf\left[X - \sum_{i=1}^{n} \lambda_i (X_i - \underline{P}(X_i))\right]\right\}$$

where $X \in \mathscr{L}(\Omega)$ and the supremum runs over $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \geq 0$ and $X_1, \ldots, X_n$ in $\mathcal{K}$. This shows that without loss of generality, we may from now on assume that $\underline{P}$ is a coherent lower prevision defined on all of $\mathscr{L}(\Omega)$.

[1] For example, take $n = 0$ and $\lambda_0 = 1$, then we find that $\underline{P}(X) \geq \inf[X]$, which means that the agent should be willing to pay at least the lowest possible reward.

$\overline{P}$ will denote the conjugate *upper prevision* of $\underline{P}$. It is defined by $\overline{P}(X) = -\underline{P}(-X)$ for every $X \in \mathscr{L}(\Omega)$. $\overline{P}(X)$ represents the agent's infimum acceptable selling price for the gamble X. The difference $\overline{P}(X) - \underline{P}(X)$ is a measure for the amount of imprecision in the agent's behavioural dispositions towards the gamble X.

An *event* A is a subset of Ω. It will be identified with its indicator I_A, which is a gamble.[2] The *lower probability* $\underline{P}(A)$ is then defined as the lower prevision $\underline{P}(I_A)$ of its indicator I_A, and similarly for the upper probability $\overline{P}(A)$.

If it so happens that $\overline{P}(X) = \underline{P}(X)$ for every gamble X, then $\underline{P}$ is called a *linear prevision*, and it is denoted by P. Linear previsions are linear functionals on the linear space $\mathscr{L}(\Omega)$ that are positive and have unit norm ($P(I_\Omega) = 1$). They are the *fair prices* or *previsions* in the sense of de Finetti [6, 7]. The restriction of a linear prevision P to events is a finitely additive probability (also called a probability charge, see Section 3.2) and $P(X)$ is equal to the expected value of the bounded random variable X with respect to this probability charge (see Theorem 1 further on). In this way, any Bayesian model can be considered to be a linear prevision, which is a special kind of lower prevision. The set of all linear previsions on $\mathscr{L}(\Omega)$ is denoted by $\mathscr{P}(\Omega)$.

$\mathcal{M}(\underline{P})$ will denote the set of all linear previsions that dominate $\underline{P}$ point-wise on $\mathscr{L}(\Omega)$:

$$\mathcal{M}(\underline{P}) = \{Q \in \mathscr{P}(\Omega) \colon Q \geq \underline{P}\}.$$

One can show that $\mathcal{M}(\underline{P})$ is a non-empty, convex and compact[3] subset of $\mathscr{P}(\Omega)$, and that $\underline{P}$ is the lower envelope of $\mathcal{M}(\underline{P})$, that is,

$$\underline{P}(X) = \min_{Q \in \mathcal{M}(\underline{P})} Q(X),$$

for all $X \in \mathscr{L}(\Omega)$. This equality, and the fact that the lower envelope of any non-empty set of linear previsions is a coherent lower prevision, gives rise to what is called the *Bayesian sensitivity analysis interpretation*, or *Quasi-Bayesian interpretation* of lower previsions: specifying a coherent lower prevision is formally equivalent to specifying a non-empty, convex and compact set of linear previsions (or probability charges).

3 Previsibility

In the previous section, we have seen that it is possible to extend a given coherent lower prevision to the set of all bounded random variables. We now investigate whether it can be extended still further to unbounded random variables.

3.1 Integration: a brief review

In the theory of integration [1, 9] one usually starts with a set Ω, a σ-field $\mathfrak{F}$ on Ω, and a measure μ on $\mathfrak{F}$.[4] A random variable X of the form $\sum_{i=1}^{n} a_i I_{A_i}$ with $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $A_i \in \mathfrak{F}$ and $\mu(A_i) < \infty$ whenever $a_i \neq 0$, is called *simple*. The integral of X is then defined as $\int X \,\mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(A_i)$. A random variable Y is called *measurable* if $Y^{-1}(M) \in \mathfrak{F}$ for every M in the Borel σ-field $\mathfrak{B}(\mathbb{R})$ on $\mathbb{R}$.[5] Obviously, every simple random variable is measurable. A sequence (X_n) of simple random variables is said to be *mean fundamental* if $\int |X_n - X_m| \,\mathrm{d}\mu \to 0$ as $n, m \to \infty$. A sequence (X_n) of random variables is said to *converge in measure* to a measurable random variable Y if for every $\epsilon > 0$ we have that

$$\mu(\{\omega \in \Omega \colon |X_n(\omega) - Y(\omega)| > \epsilon\}) \to 0$$

as $n \to \infty$. The measurability of Y guarantees that $\{\omega \in \Omega \colon |X_n(\omega) - Y(\omega)| > \epsilon\} \in \mathfrak{F}$.

A measurable random variable Y is called *integrable* if there is a mean fundamental se-

[2] I_A is the random variable that takes the value 1 on A and 0 elsewhere.

[3] We assume in this paper that $\mathscr{P}(\Omega)$ is provided with its topology of point-wise convergence: the relativisation to $\mathscr{P}(\Omega)$ of the weak*-topology on the topological dual $\mathscr{L}(\Omega)^*$, where $\mathscr{L}(\Omega)$ is provided with the supremum norm topology.

[4] $\mathfrak{F}$ is a set of subsets of Ω that contains $\emptyset$ and that is closed under the formation of complements and countable unions. μ is an extended real-valued, non-negative, and countably additive set function, defined on $\mathfrak{F}$, such that $\mu(\emptyset) = 0$.

[5] $\mathfrak{B}(\mathbb{R})$ is the σ-field generated by all open sets of $\mathbb{R}$.

quence of simple random variables that converges in measure to Y. The idea behind this definition is that all mean fundamental sequences of simple random variables that converge in measure to the same measurable random variable, eventually have the same integral. In this way, one can define an integral as a limit of integrals of simple random variables. This extended integral has all the properties that we expect from an integral, in particular, it is linear, positive and σ-additive.

3.2 Charges and linear previsions

When we relax the σ-additivity in the definition of a measure, and only require finite additivity, we enter the domain of the theory of *charges*, or finitely additive measures. It turns out that the course of reasoning outlined above can still be used, with minor modifications, to associate an integral with a charge. Let us give a brief account of how this is done, but restrict ourselves to the essentials that are needed to understand this paper. In particular, we shall focus on so-called probability charges on $\wp(\Omega)$ as we do not need the more general theory. General and detailed accounts can be found in [1] and [8].

A *probability charge* is a real-valued mapping on a field $\mathfrak{F}$ on Ω such that $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, $\mu(A) \geq 0$ whenever $A \in \mathfrak{F}$, $\mu(A) + \mu(B) = \mu(A \cup B)$ whenever A, $B \in \mathfrak{F}$ and $A \cap B = \emptyset$. From now on, $\mathfrak{F}$ is assumed to be the power set $\wp(\Omega)$.

A random variable X of the form $\sum_{i=1}^{n} a_i I_{A_i}$ with $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and $A_i \subseteq \Omega$, is called *simple*. The (Dunford) integral of X is then defined as $\mathrm{D} \int X \, \mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(A_i)$.

A sequence of random variables (X_n) is said to *converge hazily* (cf. convergence in measure) to a random variable Y if for every $\epsilon > 0$ we have that

$$\mu(\{\omega \in \Omega \colon |X_n(\omega) - Y(\omega)| > \epsilon\}) \to 0$$

as $n \to \infty$. If there is a sequence of simple random variables that converges hazily to Y, then Y is said to be T_1-*measurable*. Finally, Y is called D-*integrable* if there is a sequence (X_n) of simple random variables that converges hazily to Y, such that moreover

$$\mathrm{D} \int |X_n - X_m| \, \mathrm{d}\mu \to 0$$

as $n, m \to \infty$. Any such sequence is called a *determining sequence* for Y.

The idea behind this definition is that all determining sequences for Y eventually have the same integral. In this way, one defines the *Dunford integral* of Y as the limit of integrals of simple random variables:

$$\mathrm{D} \int Y \, \mathrm{d}\mu = \lim_{n \to \infty} \mathrm{D} \int X_n \, \mathrm{d}\mu.$$

The Dunford integral has all the properties that we expect from an integral, in particular, it is linear (and therefore finitely additive) and positive.

The following theorem is a special case of [1, Theorem 4.7.4].

Theorem 1. *There is a canonical one-to-one correspondence between probability charges on $\wp(\Omega)$ and linear previsions on $\mathscr{L}(\Omega)$. The correspondence is given by $\mu(A) = P(I_A)$ for every $A \subseteq \Omega$, and $P(X) = \mathrm{D} \int X \, \mathrm{d}\mu$ for every $X \in \mathscr{L}(\Omega)$.*

3.3 Extension of a lower prevision by a limit procedure

If we carefully examine the ideas behind the introduction of an integral in the preceding sections, we see that the starting point is a functional defined on some domain (the simple random variables) that is then extended by a limit procedure, where the necessary care is taken to ensure that it yields a unique result. Let us now show that the same ideas can be used to extend a coherent lower prevision $\underline{P}$ from $\mathscr{L}(\Omega)$ to certain unbounded random variables.

We define the $\underline{P}$-*norm* of a gamble X by $\|X\|_{\underline{P}} = \overline{P}(|X|)$. Using the coherence of $\underline{P}$, we can show that $\|\cdot\|_{\underline{P}}$ is a semi-norm on $\mathscr{L}(\Omega)$. A sequence (X_n) of gambles is called $\underline{P}$-*fundamental* (cf. mean fundamental) if it is Cauchy with respect to $\|\cdot\|_{\underline{P}}$, i.e., if $\|X_n - X_m\|_{\underline{P}} \to 0$ as $n, m \to \infty$.

We say that a sequence of gambles (X_n) converges $\underline{P}$*-hazily* to the random variable X (cf. convergence in measure, hazy convergence) if for every $\epsilon > 0$ we have that[6]

$$\lim_{n\to\infty} \overline{P}(\{\omega \in \Omega\colon\ |X(\omega) - X_n(\omega)| > \epsilon\}) = 0.$$

Observe that we do not need to impose any measurability conditions, since $\overline{P}(A)$ is defined for every $A \subseteq \Omega$. If there is a sequence of simple random variables that converges $\underline{P}$-hazily to the random variable X, then X is said to be $\underline{P}$*-measurable* (cf. T_1-measurable). The following lemma is the basic result that will guarantee the unicity of the extension introduced in Definition 1.

Lemma 1. *If (X_n) and (Y_n) are $\underline{P}$-fundamental sequences of gambles converging $\underline{P}$-hazily to the same random variable Z, then* $\lim_{n\to\infty} \underline{P}(X_n) = \lim_{n\to\infty} \underline{P}(Y_n)$.

The proof of Lemma 1 is too long to be included here. It should be mentioned, however, that it is based on similar ideas as its counterpart in the theory of charges (see for instance the proof of Proposition 4.4.10 in [1]).

Definition 1. *A random variable Z is said to be $\underline{P}$-*previsible *if there is a $\underline{P}$-fundamental sequence (X_n) of gambles that converges $\underline{P}$-hazily to Z. We then define $\underline{P}^{\mathbf{x}}(Z) = \lim_{n\to\infty} \underline{P}(X_n)$, and (X_n) is called a* determining sequence *for Z.*

By Lemma 1, the limit $\underline{P}^{\mathbf{x}}(Z)$ is independent of the details of the determining sequence (X_n). The set of all $\underline{P}$-previsible random variables contains all gambles (bounded random variables) and it is a linear lattice with respect to the point-wise order. It will be denoted by $\mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$.

Since $\mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$ is a linear lattice, we know that $|Z|$ is $\underline{P}$-previsible if Z is, and therefore we can extend the semi-norm $\|\cdot\|_{\underline{P}}$ introduced above to $\mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$ through $\|Z\|_{\underline{P}} = \overline{P}^{\mathbf{x}}(|Z|)$ for all Z in $\mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$. It is not difficult to show that $\|\cdot\|_{\underline{P}}$ is also a semi-norm on $\mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$. Moreover, if (Z_n) is a sequence of $\underline{P}$-previsible random variables and $\lim_{n\to\infty} \|Z - Z_n\|_{\underline{P}} = 0$ then $\lim_{n\to\infty} \underline{P}^{\mathbf{x}}(Z_n) = \underline{P}^{\mathbf{x}}(Z)$. This shows that topologically indistinguishable random variables are also behaviourally indistinguishable, that is, they have the same extended lower (and upper) prevision.

[6] Recall that we denote $\overline{P}(I_A)$ by $\overline{P}(A)$.

4 Properties

Let us now study the properties of the extension $\underline{P}^{\mathbf{x}}$. We shall see that they are similar to the properties of coherent lower previsions.

4.1 Coherence

In particular, we have for X and Y in $\mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$, and for $\lambda \geq 0$ that

$$\underline{P}^{\mathbf{x}}(X) \geq \inf[X], \quad \underline{P}^{\mathbf{x}}(\lambda X) = \lambda \underline{P}^{\mathbf{x}}(X), \text{ and}$$
$$\underline{P}^{\mathbf{x}}(X+Y) \geq \underline{P}^{\mathbf{x}}(X) + \underline{P}^{\mathbf{x}}(Y).$$

Actually, all of the properties of coherent lower previsions listed in [12, Section 2.6.1] remain valid for the extension $\underline{P}^{\mathbf{x}}$.

4.2 Increasing domain under increasing precision

Consider a coherent lower prevision $\underline{Q}$ that *dominates* another coherent lower prevision $\underline{P}$ on the domain $\mathcal{K}$ of $\underline{P}$: $\underline{Q}(X) \geq \underline{P}(X)$ for all gambles X in $\mathcal{K}$. This means that an agent with lower prevision $\underline{Q}$ is willing to pay higher prices for the gambles in $\mathcal{K}$, and therefore to take more risks, than an agent whose lower prevision is $\underline{P}$. We then also say that $\underline{Q}$ is more precise or more informative than $\underline{P}$. It turns out that the extension $\underline{Q}^{\mathbf{x}}$ will then also be more informative than $\underline{P}^{\mathbf{x}}$.

Theorem 2. *If $\underline{Q}$ dominates $\underline{P}$, then $\mathscr{L}^{\mathbf{x}}_{\underline{Q}}(\Omega) \supseteq \mathscr{L}^{\mathbf{x}}_{\underline{P}}(\Omega)$, and $\underline{Q}^{\mathbf{x}}$ dominates $\underline{P}^{\mathbf{x}}$.*

Thus, as the precision of a coherent lower prevision increases, more random variables become previsible. For the vacuous prevision $\underline{P}_v\colon X \mapsto \inf[X]$, that is, in case of complete ignorance, we have that $\mathscr{L}^{\mathbf{x}}_{\underline{P}_v}(\Omega) = \mathscr{L}(\Omega)$. Thus the set of all vacuously previsible random variables is exactly the set of bounded random variables.

4.3 Weak*-compactness and a lower envelope theorem

Let $\mathscr{P}_{\underline{P}^\mathsf{x}}(\Omega)$ denote the set of all real-valued linear functionals on the linear space $\mathscr{L}^\mathsf{x}_{\underline{P}}(\Omega)$ that dominate $\underline{P}^\mathsf{x}$ point-wise. These linear functionals have all the properties of a linear prevision—they are linear and positive, and have unit norm. It turns out that $\mathscr{P}_{\underline{P}^\mathsf{x}}(\Omega)$ is non-empty, convex and compact with respect to the weak* topology on the topological dual $\mathscr{L}^\mathsf{x}_{\underline{P}}(\Omega)^*$, where the topology on $\mathscr{L}^\mathsf{x}_{\underline{P}}(\Omega)$ is induced by the semi-norm $\|\cdot\|_{\underline{P}}$. Moreover, it is not so difficult to establish the following, quite remarkable result.

Theorem 3. *There is a canonical one-to-one correspondence between $\mathcal{M}(\underline{P})$ and $\mathscr{P}_{\underline{P}^\mathsf{x}}(\Omega)$, given by*

$$\mathscr{P}_{\underline{P}^\mathsf{x}}(\Omega) \to \mathcal{M}(\underline{P})\colon\ R \mapsto R|_{\mathscr{L}(\Omega)}$$

and

$$\mathcal{M}(\underline{P}) \to \mathscr{P}_{\underline{P}^\mathsf{x}}(\Omega)\colon\ Q \mapsto Q^\mathsf{x}|_{\mathscr{L}^\mathsf{x}_{\underline{P}}(\Omega)}$$

Note that we denote by $f|_A$ the restriction of a mapping f to the subset A of its domain. Through this correspondence, it is easy to prove the following lower envelope theorem.

Theorem 4. *For any $\underline{P}$-previsible random variable X we have that*

$$\underline{P}^\mathsf{x}(X) = \min_{Q \in \mathcal{M}(\underline{P})} Q^\mathsf{x}(X).$$

This shows that the Bayesian sensitivity analysis interpretation still holds for our extension. More details about the extensions of linear previsions will be given in Section 6, and in particular in Theorems 7 and 8.

5 An *ad hoc* approximation

One very simple and intuitive way of approximating an unbounded random variable Z is by considering a sequence (Z_n^*) of so-called *cuts*:

$$Z_n^*(\omega) = \begin{cases} -a_n & \text{if } Z(\omega) < -a_n \\ Z(\omega) & \text{if } -a_n \leq Z(\omega) \leq b_n \\ b_n & \text{if } Z(\omega) > b_n. \end{cases}$$

where (a_n) and (b_n) are non-negative sequences converging to $+\infty$. It turns out that Z is $\underline{P}$-measurable if and only if at least one (and therefore all) of its cut sequences (Z_n^*) converges $\underline{P}$-hazily to Z.[7] We can show that every $\underline{P}$-previsible random variable is $\underline{P}$-measurable. In particular, every gamble is $\underline{P}$-measurable. It turns out that we may restrict our attention to approximations of the form (Z_n^*) in order to investigate the $\underline{P}$-previsibility of Z.

The proof of this statement depends strongly on a generalisation of the Lebesgue dominated convergence theorem, which is of some interest in itself.[8]

Theorem 5 (Lebesgue Dominated Convergence)**.** *Let Y be a $\underline{P}$-previsible random variable. Let (Z_n) be a sequence of $\underline{P}$-measurable random variables such that $|Z_n| \leq |Y|$ for every $n \in \mathbb{N}$. Let Z be a random variable. Then the following statements are equivalent.*

(i) (Z_n) converges $\underline{P}$-hazily to Z.

(ii) Z is $\underline{P}$-previsible and moreover

$$\lim_{n\to\infty} \|Z - Z_n\|_{\underline{P}} = 0$$

(whence $\underline{P}^\mathsf{x}(Z) = \lim_{n\to\infty} \underline{P}^\mathsf{x}(Z_n)$).

The proof of Theorem 5 is rather long, but it is not very different from the proof of its counterpart in the theory of charges (see Theorem 4.6.14 in [1]). It is quite easy to show that it implies the following.

Theorem 6. *A random variable Z is $\underline{P}$-previsible if and only if one (and therefore all) of its cut sequences (Z_n^*) is $\underline{P}$-fundamental and converges $\underline{P}$-hazily to Z, i.e., if and only if one (and therefore all) of its cut sequences is a determining sequence for Z.*

[7] By "one cut sequence" we mean a cut sequence for a particular choice of (a_n) and (b_n).

[8] A reader familiar with the Lebesgue dominated convergence theorem will notice that this is a slightly simplified version. We only mention that a stronger result holds, which requires the introduction of "almost everywhere equality with respect to a coherent lower prevision", which is a quite interesting concept with a direct behavioural interpretation.

6 Examples

6.1 The linear case: charges and Dunford integrals

As a special case of our theory, we recover the theory of D-integrability with respect to probability charges.[9] Indeed, let P be a linear prevision on $\mathscr{L}(\Omega)$. As described in Section 3.2, this linear prevision is uniquely determined through the probability charge μ defined by $\mu(A) = P(I_A)$. The linear prevision $P(X)$ of a gamble X is then the Dunford integral of X with respect to μ.

But it is well known that the set of all random variables that are D-integrable with respect to μ is much larger. It is worth mentioning that it is nothing but the set $\mathscr{L}_P^{\mathsf{x}}(\Omega)$ of P-previsible random variables, and that it coincides with the so-called *Lebesgue space* $L_1(\Omega, \wp(\Omega), \mu)$ of those random variables X that are T_1-measurable and whose absolute value $|X|$ is D-integrable. This follows at once from the following correspondence between the extension of a linear prevision P on $\mathscr{L}(\Omega)$ and the Dunford integral with respect to the probability charge μ. It is an immediate consequence of Theorems 1 and 6.

Theorem 7. *Let P be a linear prevision on $\mathscr{L}(\Omega)$, and let μ be its restriction to the set of events $\wp(\Omega)$. Then the following statements hold.*

(i) A random variable X is P-measurable if and only if it is T_1-measurable with respect to μ.

(ii) A random variable X is P-previsible if and only if it is D-integrable with respect to μ, in which case

$$P^{\mathsf{x}}(X) = \mathrm{D}\int X \,\mathrm{d}\mu.$$

[9] This result can immediately be extended to all bounded charges through application of Jordan's Decomposition Theorem [1, Theorem 2.2.2(1)], by which every bounded charge μ can be written as a linear combination of probability charges, that is, $\mu = a\mu^+ - b\mu^-$, where μ^+ and μ^- are probability charges and a, $b \geq 0$.

6.2 The 2-monotone case: Choquet integrals

Consider a coherent lower prevision $\underline{P}$ defined on $\mathscr{L}(\Omega)$ that is *super-modular*: for all gambles X and Y,

$$\underline{P}(\min\{X,Y\}) + \underline{P}(\max\{X,Y\}) \geq \underline{P}(X) + \underline{P}(Y),$$

where the maximum and minimum are pointwise. Its restriction to events is a coherent lower probability that is *2-monotone*:

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B),$$

for all A and B in $\wp(\Omega)$. Moreover, it can be shown that the natural extension of this lower probability to $\mathscr{L}(\Omega)$ is super-modular, and that it coincides with $\underline{P}$.[10] In other words, a coherent 2-monotone lower probability has a unique coherent super-modular extension to the set of all gambles. Using a result by Walley [12, Section 3.2.4] it can then be shown that $\underline{P}$ is actually the Choquet integral associated with the coherent 2-monotone lower probability:

$$\underline{P}(X) = \mathrm{C}\int_\Omega X \,\mathrm{d}\underline{P} = \int_{\inf[X]}^{\sup[X]} x \,\mathrm{d}\overline{F}_X(x),$$

for every $X \in \mathscr{L}(\Omega)$, where $\overline{F}_X(x) = 1 - \underline{P}(\{\omega \in \Omega \colon X(\omega) \geq x\})$ is the *upper distribution function* of X with respect to the lower probability $\underline{P}$, and the second integral is a Riemann-Stieltjes integral. This formula can be extended to $\mathscr{L}_{\underline{P}}^{\mathsf{x}}(\Omega)$, as the following theorem states.

Theorem 8. *Let $\underline{P}$ be a super-modular coherent lower prevision on $\mathscr{L}(\Omega)$. If a random variable X is $\underline{P}$-previsible then $\mathrm{C}\int_\Omega X \,\mathrm{d}\underline{P}$ and $\int_{-\infty}^{+\infty} x \,\mathrm{d}\overline{F}_X(x)$ exist, and*

$$\underline{P}^{\mathsf{x}}(X) = \mathrm{C}\int_\Omega X \,\mathrm{d}\underline{P} = \int_{-\infty}^{+\infty} x \,\mathrm{d}\overline{F}_X(x).$$

If in particular P is a linear prevision on $\mathscr{L}(\Omega)$, then for all X in $\mathscr{L}_P^{\mathsf{x}}(\Omega)$,

$$P^{\mathsf{x}}(X) = \int_{-\infty}^{+\infty} x \,\mathrm{d}F_X(x),$$

where $F_X(x) = P(\{\omega \in \Omega \colon X(\omega) \leq x\})$.

[10] An simple and elegant proof for this statement was suggested to us by Hugo Janssen.

7 Conclusions

The main message of this paper is that it is possible to define an extension of a coherent lower prevision to a linear space of previsible, not necessarily bounded random variables, and that this extension still has properties similar to those of coherent lower previsions. Previsibility coincides with the existing notion of D-integrability when the coherent lower previsions are linear, and an extended lower prevision can be written as the lower envelope of the extensions of the dominating linear previsions of the original. Finally, we have studied how the cut sequences of a random variable can help us determine whether or not it is previsible.

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