Analysis 1 Problems (Michaelmas Term 2014)

Remarks:

- Some problems need some explanations. These explanations are usually given right before the questions and are highlighted in **boldface**.
- Questions which are particularly difficult are marked by a star "*". If they are extraordinarily difficult, we mark then by two stars "**".

1 Basic logic and sets

1. Decide whether the following statements are true or false and give an explanation.

   a) The product of two odd numbers is odd.

   b) We have for all real numbers p, q: If $p^2 - 4q \geq 0$ then $x^2 + px + q = 0$ has two different real solutions.

   c) For every $\epsilon > 0$ there exists a natural number n such that $\frac{1}{n} < \epsilon$.

   d) 1001 is a prime number or $\int_0^\pi sin(x^2)dx \geq 4$.

2. "Who is who?" There are three people, Anna, Max and Tom, with three different professions: builder, electrician and lecturer. Assuming that all of the following statements are true, find out who is who. Show that this puzzle has only one solution.

   a) Today is Monday or Wednesday.

   b) If Tom is not the lecturer then Max is the lecturer.

   c) Today is Wednesday or Anna is the lecturer.

   d) If Tom is the lecturer then Anna is the electician.

   e) If today is Wednesday then Tom or Anna is the lecturer.

   f) If Tom is not the builder then today is Wednesday.

   g) If Max is the electrician then today is Wednesday.

3. Let A, B be statements. Construct truth tables for the two statements "(not A) or B" and "(A and B) or (not A)". Are these statements equivalent or not?

4. Use De Morgan's Laws to express the following combined statements only with the connectives "not" and 'or" (in other words, eliminate all "and" connectives):

a) $\text{not}\,(\text{not}\,(A)\,\text{and}\,(B\,\text{and}\,(\text{not}\,C)))$.

b) $(A\,\text{and}\,\text{not}\,(B))\,\text{or}\,\text{not}\,(A\,\text{and}\,\text{not}\,(C))$.

c) $A\,\text{and}\,\text{not}\,(B)\,\text{and}\,\text{not}\,(C)\,\text{and}\,D$.

5. A *tautology* is a statement such that the truth table has true for all outputs. Show that each of the following statements are tautologies:

a) $A\,\text{or}\,\text{not}\,(A)$.

b) $A\,\text{or}\,(A\,\text{or}\,B)\,\text{or}\,\text{not}\,(B)$.

c) $\text{not}\,(((A\,\text{and}\,B)\,\text{or}\,(B\,\text{and}\,C))\,\text{and}\,(\text{not}\,B))$.

The equivalence symbol "$\Leftrightarrow$" can also be considered as a connective between two statements, having the following truth table:

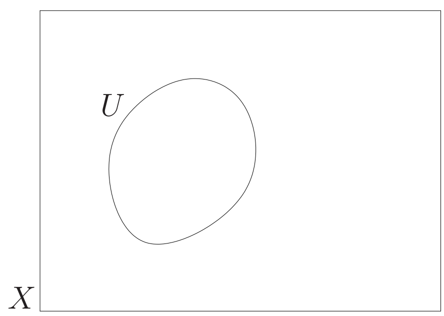
A	B	$A \Leftrightarrow B$
false	false	true
false	true	false
true	false	false
true	true	true

This means "$A \Leftrightarrow B$" is only true if both statements A and B have the same truth value. We also say "A if and only if B". Then the fact that A and B are equivalent translates into the fact that the combined statement "$A \Leftrightarrow B$" is a tautology (see Problem 5).

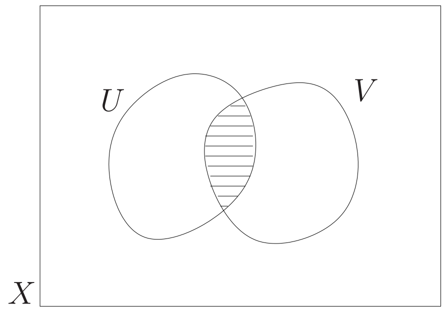
6. * Express the combined statement "$A \Leftrightarrow B$" using only the connectives and, or and not.

7. Find the union and intersection of $\{x \in \mathbb{R} \mid x^2 - 9x + 14 = 0\}$ and $\{y \in \mathbb{Z} \mid 3 \leq y < 10\}$.

A *Venn Diagram* is useful to represent operations of sets geometrically. If U is a subset of X, we can illustrate this as follows (the shape of U is not important):

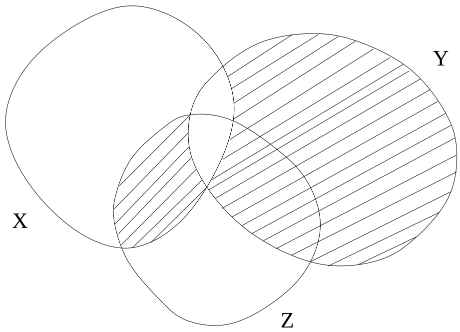


If V is another subset of X, we can express $U \cap V$ is the following way (the intersection set is marked):



This allows us to illustrate complex set theoretic expressions. But be aware that a Venn Diagram never replaces a proof for a set theoretic identity.

8. Describe the set illustrated in the following Venn diagram.



9. * For two real numbers $x, y \in \mathbb{R}$, the minimum of x and y is denoted by $\min\{x, y\}$. Let $a, b \in \mathbb{R}$ be two real numbers. Show the following identity:

$$\{x \in \mathbb{R} \mid x \leq a\} \cap \{x \in \mathbb{R} \mid \min\{x, a\} \leq b\} = \{x \in \mathbb{R} \mid x \leq \min\{a, b\}\}.$$

10. Show the following facts:

    a) If $X \cup Y = Y$ then $X \subset Y$.
    b) If $X \cap Y = X$ then $X \subset Y$.
    c) If $X \subset Y$ then $X \cup Y = Y$.

11. The *symmetric difference* of two sets X, Y is defined as

$$X \Delta Y = (X \backslash Y) \cup (Y \backslash X).$$

    a) Draw a Venn Diagram to illustrate $X \Delta Y$.
    b) Draw Venn Diagrams for $(X \Delta Y) \Delta Z$ and $X \Delta (Y \Delta Z)$.
    c) Show that $X \Delta Z \subset (X \Delta Y) \cup (Y \Delta Z)$.

If X is a set, then the *power set* $\mathcal{P}(X)$ of a set is the set of all subsets of X. For example, if $X = \{1, 2\}$, then

$$\mathcal{P}(X) = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}.$$

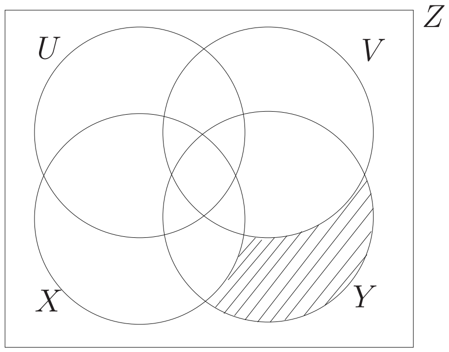
Another example: We have $\mathbb{N} \in \mathcal{P}(\mathbb{Q})$.

12. * Think about the following statements about power sets and determine which of them are true, which are false. Try to explain why you think so.

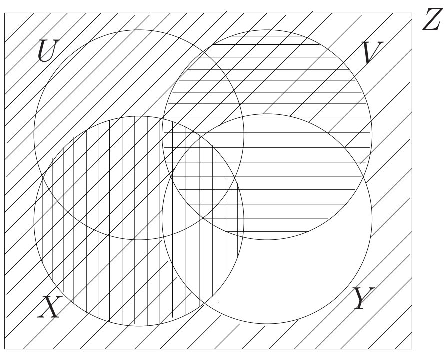
    1. If X is a finite set and has n elements, then $\mathcal{P}(X)$ is also finite and has 2^n elements.
    2. If $Z = X \cap Y$, then
    $$\mathcal{P}(Z) = \mathcal{P}(X) \cap \mathcal{P}(Y).$$
    3. If $Z = X \cup Y$, then
    $$\mathcal{P}(Z) = \mathcal{P}(X) \cup \mathcal{P}(Y).$$

13. Show that the following two sets are equal: $X = \{(\cos(t), \sin(t)) \mid t \in [0, 2\pi)\}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. You may use without proof that for every $x \in [-1, 1]$ there exists $t \in [0, \pi]$ such that $\cos(t) = x$ (this can be seen from the two facts that $\cos(t)$ is continuous and monotone decreasing from 1 to -1 on the interval $[0, \pi]$)) and also other basic properties of sin and cos. Give a geometric description of these sets.

14. ** *(Jack's Dilemma)* Let U, V, X, Y be four subsets of Z. Jack draws the following Venn Diagrams to show that
$$Y \cap (U^c \cap V^c \cap X^c) = (V \cup X \cup Y^c)^c, \tag{1}$$
where complements are taken with respect to the set Z.



$Y \cap (U^c \cap V^c \cap X^c)$



$V \cup X \cup Y^c$

Then he looks at the following example:
$$U = \{1\}, V = \{3, 4\}, X = \{3\}, Y = \{1, 2, 3\}$$
and $Z = \{1, 2, 3, 4\}$ and obtains

- $Y \cap (U^c \cap V^c \cap X^c) = \{2\}$,
- $(V \cup X \cup Y^c)^c = \{1, 2\}$.

While Jack's Venn Diagrams confirm (1), his example shows that (1) is not true.

Check that Jack's Venn Diagram agrees with (1) and that there is no mistake in his example. Can you explain this discrepancy?

2 Numbers and Inequalities

15. Find all those values of x for which $\dfrac{3x+4}{2} \leq \dfrac{6-x}{4}$.

16. Find all those values of x for which $x^2 - x < 2$.

17. Find all those values of x for which $\dfrac{-3}{x-4} \leq x$.

18. Find all those values of x for which $\dfrac{3}{x-4} < -x$.

19. Find all real values of x such that $|x^2 + x - 4| = 2$.

20. For all real x show that $|8x - 9| < 7x - 6$ if and only if $|x - 2| < 1$.

21. For all real x show that $|2x + 1| < 3x$ if and only if $x > 1$.

22. Find all values of x for which $|2x + 5| > 4$.

23. Find all values of x for which $|2x + 1| \leq |3x - 6|$.

24. Find all real numbers x for which $|x - 1| + |x - 2| > 1$.

25. Find all real numbers x for which $|x - 1| + |x + 1| < 2$.

26. Using the triangle inequality, prove that

$$|a| + |b| \leq |a + b| + |a - b|.$$

3 Basics about sequences and limits

27. Calculate $\lim_{n\to\infty} x_n$ in each of the following cases (or show that no limit exists).
(a) $x_n = \cos(n^2)/\sqrt{n^2+n}$ (b) $x_n = (3n+1)^2(4n^4+1)^{-1/2}$
(c) $x_n = [(1+2n)/(2n)]^n$ (d) $x_n = [(2n+1)/(n+1)]^{2n}$ *[Hint: show* $\frac{2n+1}{n+1} \geq \frac{3}{2}$*]*
(e) $x_n = (n^2 + \log n)/\sqrt{2n^3 - 1}$ (f) $x_n = (n^5 + \log n)^{2/n}$
(g) $x_n = n^2[1/(n+1) - 1/(n-1)]$.

28. Calculate $\lim_{n\to\infty} x_n$ in each of the following cases.
(a) $x_n = (2n+1)^2/\sqrt{n^4+1}$ (b) $x_n = n(\sqrt{1+n^2} - n)$ (c) $x_n = \log(n) - \log(n+1)$
(d) $x_n = (n^2 + \mathrm{e}^{-n})/(\log(n) + 5n^3)$ (e) $x_n = (n!)^2/[(n-2)!(n+2)!]$ (f) $x_n = n!\, n^{-n}$ (g) $x_n = 2^n/n!$ (h) $x_n = n\sin(\pi/n)$ *[Use* $\frac{\sin\theta}{\theta} \to 1$ *as* $\theta \to 0$*]*
(i) $x_n = (1+n^2)^{1/n}$
(j) $x_n = (n+3)!/(n!\, n^3)$ (k) $x_n = n^2[n^{-1} - (n+1)^{-1}]$.

29. Let x_n be as in (d) of the previous question. Given $\epsilon > 0$, show that the distance from x_n to its limit is less than ϵ if $n > 2/(5\epsilon)$.

30. Calculate $\lim_{n\to\infty} x_n$ in each of the following cases (or show that no limit exists).
(a) $x_n = (n + \log n^2)/\sqrt{n^2+2}$ (b) $x_n = \sqrt{n}\,(n + \mathrm{e}^{-n})^{-1}\sin(\mathrm{e}^n)$

31. Find the limit of each of the following sequences as $n \to \infty$, or show that no limit exists. (a) $x_n = (n^2 + e^n)^{1/n}$ (b) $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$ (c) $x_n = \left(\frac{n-1}{n+1}\right)^n$

32. Compute $\lim_{n\to\infty} x_n$ for the following: (a) $x_n = (n^2 + n)^{1/n}$ (b) $x_n = n\left(\sqrt{n+1} - \sqrt{n}\right)^2$

33. If $\{x_n\}$ is a sequence such that $x_n \to x^*$ as $n \to \infty$, and $x_n < 0$ for all n, prove that $x^* \leq 0$. Is it necessarily true that $x^* < 0$?

34. Compute $\lim_{n\to\infty} \left[(n+1)^2 - (n-1)^2\right] / (n + \sqrt{n})$, or show that the limit does not exist.

35. Find $\lim_{n\to\infty}$ for each of the following sequences, or show that no limit exists. (a) $x_n = (n^2+2)^{1/n}$ (b) $x_n = [(n+2)/(n+1)]^{2n}$ (c) $x_n = n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right)$

36. Compute $\lim_{n\to\infty} \left[(n+1)^2 - n^2\right] / (n + \log n)$, or show that the limit does not exist.

37. Find the limit of each of the following sequences as $n \to \infty$, or show that the limit does not exist.
(a) $x_n = \left(\frac{n}{n+1}\right)^n$ (b) $x_n = \left(\frac{1}{n+1} - \frac{1}{n-1}\right) / \sin\left(\frac{1}{n^2}\right)$ (c) $x_n = \left(\frac{3n+2}{2n+1}\right)^n$

38. Prove that limits are unique: i.e. if $x_n \to x^*$ and $x_n \to x'$ as $n \to \infty$, then $x^* = x'$.

39. Let (x_n) be a sequence. Show the following fact. If $(|x_n|)$ is convergent with $\lim_{n\to\infty} |x_n| = 0$ then (x_n) is also convergent with $\lim_{n\to\infty} x_n = 0$.

The following problem deals with fundamental and important facts about the *geometric series*. for $c \in \mathbb{R}$, $c \neq 1$, the geometric series is defined by

$$x_n = 1 + c + c^2 + \cdots + c^{n-1}.$$

We can also define (x_n) recursively by $x_1 = 1$ and $x_{n+1} = x_n + c^n$.

40. Let $c \in \mathbb{R}$ and $c \neq 1$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ via

$$x_n = 1 + c + c^2 + \cdots + c^{n-1}.$$

a) Calculate $(1-c)x_n$ and conclude from it that x_n satisfies the explicit formula

$$x_n = \frac{1-c^n}{1-c}.$$

b) Assume that $|c| < 1$. Show that (x_n) is convergent with limit

$$\lim_{n\to\infty} x_n = \frac{1}{1-c}.$$

41. Show that if $\{x_n\}$ is a sequence with $x_n \leq b$ for all n, and $\lim_{n\to\infty} x_n = x^*$, then $x^* \leq b$.

42. * If $\{x_n\}$ is a bounded sequence (i.e., there is a number K such that $|x_n| \leq K$ for all n), and if $y_n \to 0$ as $n \to \infty$, prove that $x_n y_n \to 0$ as $n \to \infty$.

43. Prove that one of the following statements is true and that the other is false.
(a) If $x_n \to 1$ as $n \to \infty$, then $(x_n)^n \to 1$ as $n \to \infty$.
(b) * If $0 < r < 1$ and $x_n \to r$ as $n \to \infty$, then $(x_n)^n \to 0$ as $n \to \infty$.

44. Compute $\lim_{n\to\infty}(t+1/n)^n$ for each positive real value of t for which this limit exists.

45. Determine for which real values of x the sequence $\{n^{-1}x^n\}$ tends to a limit as $n \to \infty$.

46. Compute $\lim_{n\to\infty} x_n$, or show that the limit does not exist, for each of the following.
(a) $x_n = (p^n + q^n)^{1/n}$, with $p \geq q \geq 0$
(b) $x_n = n \int_0^n \mathrm{e}^{-nx}\, dx$ *[Hint: do the integral]*
(c) $x_n = (1 + 1/n)^{n^2}$ *[Hint: log]*.

47. Prove the following facts:

   a) Let (a_n) be a real sequence. (a_n^2) is convergent if and only if $(|a_n|)$ is convergent.

   b) If (a_n) is convergent then $a_{n+1} - a_n \to 0$ as $n \to \infty$.

4 More logic: Quantifiers, negation and proof techniques

48. Give an indirect proof for the following fact: If $X \cap Y \subset Z$ and $x \in Y$, then $x \notin X \backslash Z$.

49. For each of the following statements, restate it without using the abbreviating quantifiers. Explain in your own words what it means. Finally, write down its negation.

   a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The first statement reads as follows:
$$\forall\, C > 0 \quad \exists\, n \in \mathbb{N} : \quad x_n > C.$$

   b) Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The second mathematical statement reads as follows:
$$\forall\, x \in \mathbb{R} \quad \forall\, y \geq x : \quad f(x) \leq f(y).$$

   c) Let X, Y be two sets and $g : X \to Y$ be a function. The third mathematical statement reads as follows:
$$\forall\, y \in Y \quad \exists\, x \in X : \quad y = g(x).$$

50. Formulate the contrapositive statements of the following true statements (you do not need to prove them):

a) If the sides lengths a, b, c of a triangle satisfy $a^2+b^2 = c^2$ then the triangle is right-angled.

b) If four points in the plane lie on a common circle then the opposite angles of the corresponding quadrilateral add up to 180^o.

The following three questions deal with *unions and intersections of infinitely many sets*. Let X_n with $n \in \mathbb{N}$ be a family of sets. The union of these sets is then denoted by $\bigcup_{n\in\mathbb{N}} X_n$ and is defined as follows:

$$\bigcup_{n\in\mathbb{N}} X_n = \{x \mid \exists\, n \in \mathbb{N} : x \in X_n\}.$$

In other words, $\bigcup X_n$ consists of all elements which are contained in at least one of the sets X_n. Similarly, the intersection of these sets is denoted by $\bigcap_{n\in\mathbb{N}} X_n$ and is defined as follows:

$$\bigcap_{n\in\mathbb{N}} X_n = \{x \mid \forall\, n \in \mathbb{N} : x \in X_n\}.$$

In other words, $\bigcap X_n$ consists of all elements which are contained in all of the sets X_n.

Note also that we can also take unions and intersections over other index sets, so $\bigcup_{p \textbf{ prime}} X_p$ and $\bigcap_{q\in\mathbb{Q}} U_q$ make perfect sense.

51. Let X_n with $n \in \mathbb{N}$ be sets. Assume that all these sets X_n are subsets of a set X. Show the following Laws of De Morgan for infinitely many sets:

$$X \backslash \left(\bigcup_{n\in\mathbb{N}} X_n \right) = \bigcap_{n\in\mathbb{N}} (X \backslash X_n)$$

and

$$X \backslash \left(\bigcap_{n\in\mathbb{N}} X_n \right) = \bigcup_{n\in\mathbb{N}} (X \backslash X_n).$$

52. The following infinite unions and intersections represent relatively simple subsets of $\mathbb{R}$. Find these sets in each case and justify your answer:
(i) $\bigcup_{n\in\mathbb{N}} [1/n, 1)$ (ii) $\bigcap_{n\in\mathbb{N}} (-1/n, 2/n)$ (iii) $\bigcup_{n\in\mathbb{N}} [1, n)$.

53. Express the following sets as infinite unions/intersections of concrete sets:
(a) $\{x \in \mathbb{R} \mid 0 < \sin(x) \leq 1\}$ (b) all natural numbers which are not squares or cubes of primes.

54. * Show that the following statements about the real numbers x and y are equivalent:
(a) $x \geq y$; (b) For every $\epsilon > 0$ we have $x > y - \epsilon$; (c) For every $\epsilon > 0$ we have $x + \epsilon > y$.

The following questions deal with an important proof technique called *Proof by Induction*. Assume you have a conditional statement $A(n)$ depending on a natural number $n \in \mathbb{N}$. Induction is a

method to prove that "$A(n)$ is true for all $n \in \mathbb{N}$". Proof by Induction goes as follows:
(a) Start of Induction: You show that $A(1)$ is true.
(b) Induction Step: You show that if $A(n)$ is true then $A(n+1)$ is also true.
The start value does not need to be 1. Here is a concrete example: Let $A(n)$ be "$2^n \geq n^2$". We want to show that $A(n)$ holds for all $n \geq 4$.
(a) Start of Induction: We have $2^4 = 16 = 4^2$.
(b) Induction Step: Assume that $n \geq 4$ and we have $2^n \geq n^2$. Then

$$2^{n+1} = 2 \cdot 2^n \geq 2n^2 \geq n^2 + 4n \geq n^2 + 2n + 1 = (n+1)^2.$$

(a) and (b) together imply that $A(n)$ holds for all $n \geq 4$.

55. Show the following formulas by induction:

    i) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
    ii) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

56. Prove the following inequality by Induction.
    Bernoulli's Inequality: Let $x > -1$ and $n \in \mathbb{N}$. Then we have

    $$(1+x)^n \geq 1 + nx.$$

57. Use Bernoulli's Inequality introduced in Problem 56 to prove the following facts about sequences:

    a) If $c > 1$ then the sequence $(c^n)_{n \in \mathbb{N}}$ becomes arbitrarily large as $n \to \infty$, i.e., for every $K > 0$ there exists $N \in \mathbb{N}$ such that $c^n \geq K$ for all $n \geq N$. *[Hint: Set $c = 1 + x$.]*
    b) If $0 < c < 1$ then the sequence $(c^n)_{n \in \mathbb{N}}$ is convergent with $\lim_{n \to \infty} c^n = 0$. *[Hint: Set $c = 1/(1+x)$.]*

58. * Let $a_1, \ldots, a_n > 0$. Then the *arithmetic mean* is defined by

    $$A(a_1, \ldots, a_n) = \frac{a_1 + \cdots + a_n}{n},$$

    and the *geometric mean* is defined by

    $$G(a_1, \ldots, a_n) = \sqrt[n]{a_1 \ldots a_n}.$$

    We now prove the *Inequality of the Arithmetic and Geometric Means*, namely

    $$G(a_1, \ldots, a_n) \leq A(a_1, \ldots, a_n).$$

    Let $S(n)$ be the statement

    $$\forall\, a_1, \ldots, a_n > 0 : \quad G(a_1, \ldots, a_n) \leq A(a_1, \ldots, a_n).$$

    Proceed with the proof via the following steps:

    a) Prove that $S(2)$ is true.
    b) Let $n \geq 2$. Prove that if $S(2)$ and $S(n)$ are true then also $S(2n)$.
    c) Prove for all $n \geq 2$ that if $S(n+1)$ is true, then also $S(n)$. *[Hint: Use $G(a_1, \ldots, a_n, G(a_1, \ldots, a_n)) = G(a_1, \ldots, a_n)$.]*
    d) Finally, explain why this show that $S(n)$ is true for all $n \geq 2$.

5 The Completeness Axiom for $\mathbb{R}$

59. Determine the sup and inf, where they exist, of the following sets:
    (a) $S = \{x : |2x - 1| < 11\}$ (b) $S = \{x + |x - 1| \mid x \in \mathbb{R}\}$.

60. Determine infimum and supremum of the following sets:
    (i) $\{x \mid x < 0 \,\text{and}\, x^2 + x - 1 < 0\}$
    (ii) $\{1/n + (-1)^n \mid n \in \mathbb{N}\}$.

61. Compute the sup and the inf of the function $f(x) = e^x/(1 + e^x)$, where $x \in \mathbb{R}$.

62. Compute the sup and the inf of the function $f(x) = x/(1 + |x|)$, where $x \in \mathbb{R}$.

63. Determine the least upper bound and the greatest lower bound of each of the following sets:
    (a) $X = \{(n^2 - n)/(n^2 + 1) \mid n \in \mathbb{N}\}$ (b) $Y = \{(2m + n)/(m + 3n) \mid n, m \in \mathbb{N}\}$.

64. Determine the supremum of $g(x) = (x \cos^2 x - 2)/(2x \cos^2 x + 2)$ for $x \geq 0$.

65. Determine the least upper bound and the greatest lower bound of each of the following functions (where x ranges over the real numbers):
    (a) $f(x) = (1 + x^2 \cos x)/(2 + x^2)$ (b) $g(x) = x^2 \exp(-x^2)$.

66. If g is bounded above, and $f(x) < g(x)$ for all x, prove that $\sup(f) \leq \sup(g)$. Is it necessarily true that $\sup(f) < \sup(g)$?

67. Compute the supremum and the infimum of the function
    $f(x) = x^2/(1 + x^2)$ on $\mathbb{R}$.

68. Compute the supremum and the infimum of the function
    $f(x) = \sqrt{x}/(2 + x)$ for $x > 0$.

69. Compute the supremum and the infimum of the function
    $f(x) = x/(x^2 + 1)$ for $x > 0$.

70. ** Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to [a, b]$ be a monotone increasing function, i.e., we have $f(x) \leq f(y)$ for all $x, y \in [a, b]$ with $x \leq y$. Moreover, we assume that $f(a) > a$ and $f(b) < b$. Show that there exists $x^* \in (a, b)$ with $f(x^*) = x^*$ (i.e., there exists a fixed point x^* of the map $f : [a, b] \to [a, b]$). *[Hint: Consider* $\sup(\{x \in [a, b] : f(x) \geq x\})$*.]*

6 More on limits of sequences

71. ** The sequence $\{x_n\}$ is defined recursively by $x_1 = 10$ and $x_{n+1} = \sqrt{6 + x_n}$. Find $\lim_{n\to\infty} x_n$. *[Hint: first find the fixed points of the iteration.]*

72. Let $x_n = \left(1 + \dfrac{1}{n}\right)^n \geq 1$. Here is a proof that (x_n) is convergent.

a) Show for $n \in \mathbb{N}$ that

$$\frac{x_{n+1}}{x_n} = \left(1 - \frac{1}{(n+1)^2}\right)^n \left(\frac{n+2}{n+1}\right).$$

b) * Use Bernoulli's Inequality (see Problem 56) to show that

$$\frac{x_{n+1}}{x_n} \geq \frac{1+(n+1)^3}{(n+1)^3} > 1.$$

This means that the sequence (x_n) is monotone increasing.

c) Use similar arguments to show that the sequence $y_n = \left(1 - \frac{1}{n+1}\right)^{n+1} \geq 0$ is also monotone increasing.

d) Show that $x_{n+1}y_n \leq 1$ for all $n \in \mathbb{N}$ and that (x_n) is therefore bounded above by 4.

e) Conclude that (x_n) is convergent.

73. Let (a_n) be the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1}$.

    (a) Show that if $1 \leq a_n \leq (1+\sqrt{5})/2$ then $1 \leq a_{n+1} \leq (1+\sqrt{5})/2$, which implies that (a_n) is bounded below by 1 and above by $(1+\sqrt{5})/2$.

    (b) Show that (a_n) is monotone increasing.

    (c) Prove that $\lim_{n\to\infty} a_n = (1+\sqrt{5})/2$.

    The number $(1+\sqrt{5})/2$ is called the *golden ratio*.

74. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be a family of a non-empty closed intervals (where $n \in \mathbb{N}$) with the following properties:

    (a) We have $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.

    (b) We have $b_n - a_n \to 0$ as $n \to \infty$.

    Show that there exists a unique number $c \in \mathbb{R}$ with $c \in I_n$ for all $n \in \mathbb{N}$. This fact is called the *principle of nested intervals*.

75. * Give an example that the statement in Problem 74 is no longer true if we choose open intervals $I_n = (a_n, b_n)$ instead of closed intervals.

76. (a) Prove the following fact (Proposition 6.4): Let (x_n) be convergent with limit $x^* = \lim_{n\to\infty} x_n$ and (x_{n_j}) be a subsequence. Then (x_{n_j}) is also convergent and we have

    $$\lim_{j\to\infty} x_{n_j} = x^*.$$

    (b) Give the contrapositive formulation of the statement proved in (a).

77. Prove the following fact (Theorem 6.8): If (x_n) is a Cauchy sequence then (x_n) is bounded.

78. Prove the following fact (Theorem 6.9): If (x_n) is convergent then (x_n) is also a Cauchy sequence.

79. (a) Let $a \leq b$ be real numbers. Show that the sequence $a_1 = a, a_2 = b$ and $a_{n+2} = (a_{n+1} + a_n)/2$ for $n \geq 1$ is a Cauchy sequence.
(b) ** Find the limit of the sequence introduced in (a).

80. a) ** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying $0 \leq u_{n+1} \leq u_n/2 + 1/n$ for all $n \in \mathbb{N}$. Show that (u_n) is convergent and $\lim_{n\to\infty} u_n = 0$. *[Hint: Give an indirect proof.]*

    b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_1 \geq 1$ and $a_{n+1} = \sqrt{a_n} + 1/n$. Show that (a_n) is convergent and $\lim_{n\to\infty} a_n = 1$. *[Hint: Define $u_n = a_n - 1$ and use fact a).]*

81. Let (u_n) be defined by $u_1 = 1$ and

$$u_{n+1} = u_n + \frac{(-1)^n}{n} \quad \forall n \in \mathbb{N}.$$

Show that the subsequence $(u_{2k-1})_{k\in\mathbb{N}}$ is monotone decreasing and that the subsequence $(u_{2k})_{k\in\mathbb{N}}$ is monotone increasing. Conclude from this that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

82. Give an indirect proof for the following fact: The real sequence (w_n), given by $w_{n+1} = w_n^2 + 1$ for all $n \geq 1$, does not have a limit for any real initial value $w_1 \in \mathbb{R}$.

83. * Prove the following fact: Let (u_n) be a bounded real sequence and $c \in \mathbb{R}$. If we have for every convergent subsequence $(u_{n_j})_{j\in\mathbb{N}}$ that $\lim_{j\to\infty} u_{n_j} = c$, then (u_n) is convergent and we have

$$\lim_{n\to\infty} u_n = c.$$

[Hint: Use an indirect proof.]

84. Use the Newton method to calculate iteratively $\sqrt{5}$ up to an error $\leq 10^{-4}$.

7 Functions, Limits and continuity

85. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 + y^2 + z^2$. Describe the preimages $f^{-1}(\{-1\})$, $f^{-1}(\{0\})$, $f^{-1}(\{1\})$ and $f^{-1}([1, 2])$ geometrically.

86. Let $f : [0, 4] \to \mathbb{R}$, $f(x) = \sin(\pi x)$. Sketch the graph of f and determine the preimage $f^{-1}([0, 1)) \subset \mathbb{R}$.

87. Let $f : X \to Y$ be a function and $X_0 \subset X$ and $Y_0 \subset Y$. Show the following facts:

    (a) $f(f^{-1}(Y_0)) \subset Y_0$,

    (b) $f^{-1}(f(X_0)) \supset X_0$.

88. Compute $\lim_{x\to\infty} f(x)$ for the following functions, or show that no limit exists.
(a) $f(x) = x/\sqrt{4 + x^2}$ (b) $f(x) = (x + \log x^2)/(3x + 2)$
(c) $f(x) = x\sqrt{x^2 + 3} - x^2$ (d) $f(x) = x/(1 + x^2 \sin^2 x)$

89. Given $f(x) = \sqrt{5x+1}$. Find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ when $x > -\frac{1}{5}$.

90. Evaluate $\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2}$.

91. (a) Show that $\lim_{x\to\infty}(x - \sqrt{x^2-1}) = 0$.
(b) Show that the hyperbola $x^2/a^2 - y^2/b^2 = 1$ gets arbitrarily close to to the asymptote $y = (b/a)x$ as $x \to \infty$.

92. Find $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$.

93. Calculate $\lim_{x\to 1} \frac{2x^4-6x^3+x^2+3}{x-1}$.

94. Calculate $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

95. Calculate the limit if it exists:
(a) $\displaystyle\lim_{x\to 2}\left(\frac{1}{2-x} - \frac{12}{8-x^3}\right)$.
(b) $\displaystyle\lim_{x\to 0}\frac{x}{|x|}$.

96. Calculate the following limits:
(a) $\lim_{x\to 4} \frac{\sqrt{x}-2}{4-x}$.
(b) $\lim_{h\to 0} \frac{(2+h)^4-16}{h}$.

97. Let $f(x) = \frac{3x+|x|}{7x-5|x|}$. Evaluate (a) $\lim_{x\to\infty} f(x)$ (b) $\lim_{x\to-\infty} f(x)$ (c) $\lim_{x\to 0+} f(x)$ (d) $\lim_{x\to 0-} f(x)$.

98. Calculate the following limits:
(a) $\displaystyle\lim_{x\to\infty}\left(\frac{3x}{x-1} - \frac{2x}{x+1}\right)$.
(b) $\displaystyle\lim_{x\to 1}\frac{1}{x-1}\left(\frac{1}{x+3} - \frac{2x}{3x+5}\right)$.

99. Calculate $\lim_{h\to 0} \frac{\sqrt[3]{8+h}-2}{h}$. *[Hint: Let $x^3 = 8+h$.]*

100. Assume you know that $\lim_{x\to 0} \frac{e^x-1}{x} = 1$. Derive from this the following results:
(a) $\displaystyle\lim_{x\to 0}\frac{e^{-ax}-e^{-bx}}{x} = b-a$.
(b) $\displaystyle\lim_{x\to 0}\frac{a^x-b^x}{x} = \log(a/b)$ if $a, b > 0$.
(c) $\displaystyle\lim_{x\to 0}\frac{\tanh(ax)}{x} = a$, where $\tanh(x) = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

101. * Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. Use the contrapositive proof technique to show: If we have for every convergent sequence (x_n) with $\lim_{n\to\infty} x_n = c$ that
$$\lim_{n\to\infty} f(x_n) = f(c),$$
then f is continuous at c.

102. Give ϵ-δ proofs that the functions $f(x) = 3x+1$ with $x \in \mathbb{R}$ and $g(x) = 1/x$ with $x > 0$ are continuous.

103. Show that every function $f : \mathbb{N} \to \mathbb{R}$ is continuous.

104. Prove the following fact: If f is continuous on $[a, b] \subset \mathbb{R}$ and $c \in (a, b)$ with $A = f(c) \neq 0$, then there exists $\delta > 0$ with $a \leq c - \delta$, $c + \delta \leq b$ and $|f(x)| > |A|/2$ for all $x \in (c - \delta, c + \delta)$.

105. ** Give the details of the proof that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at a then their product is also continuous at a (without using COLT).

106. * Show that if the functions f and g are continuous on (a, b), then so are $m(x) = \min\{f(x), g(x)\}$ and $M(x) = \max\{f(x), g(x)\}$.

107. ** Let $g : [a, b] \to \mathbb{R}$ be continuous, and define $h(x) = \sup\{g(y) : a \leq y \leq x\}$ for $x \in [a, b]$. Prove that h is continuous on $[a, b]$.

108. Let $f(x) = 2x^3 - 3x^2 + 7x - 9$. Show that there exists a number $c \in (1, 2)$ with $f(c) = 1$.

109. Prove that $f(x) = \cos(x)e^x + 1$ has no zeros in $(-\infty, 0]$ and infinitely many zeros in $(0, \infty)$.

110. Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots a_1x + a_0$ be a real polynomial. Show the following facts:

    (a) If n is odd and $a_0 > 0$ then f has a zero in $(-\infty, 0)$.

    (b) If $a_0 < 0$ then f has a zero in $(0, \infty)$.

    (c) If n is even and $a_0 < 0$ then f has a zero in $(-\infty, 0)$.