Analysis 1 Problems (Epiphany Term 2015)

Remarks:

- Some problems need some explanations. These explanations are usually given right before the questions and are highlighted in **boldface**.
- Questions which are particularly difficult are marked by a star "*". If they are extraordinarily difficult, we mark then by two stars "**".

8 Differentiable functions

111. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = c$ with $f(c) \neq 0$. Show that $1/f$ is also differentiable at c and that

$$(1/f)'(c) = -\frac{f'(c)}{f^2(c)}.$$

112. Let $f : (\alpha, \beta) \to (a, b)$ and $g : (a, b) \to (\alpha, \beta)$ be inverse functions of each other, i.e., $f \circ g = \mathrm{id}_{(a,b)}$.

(a) Let $c \in (a, b)$. Assume that f is differentiable at $g(c)$ and that g is differentable at c. Show that

$$g'(c) = \frac{1}{f'(g(c))}. \quad (1)$$

(b) Let $f : (0, \pi) \to (-1, 1)$ be $f(x) = \cos x$. Using (a) and the derivative of $\cos x$ and assuming differentiability of f and of its inverse function $\arccos(x)$, calculate the derivative of the function $\arccos(x)$ at $x = c$.

113. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable at $x = 0$. Check whether f' is continuous at $x = 0$.

114. Show that between any two real solutions of $e^x \sin x = 1$ is at least one real solution of $e^x \cos x = -1$. *[Hint: Consider the function $e^{-x} - \sin x$.]*

115. Let $f_n(x) = (x^2 - 1)^n$. The Legendre polynomial of order $n \in \mathbb{N}$ is defined by

$$p_n(x) = \frac{1}{2^n n!} f_n^{(n)}(x).$$

Using Rolle's Theorem, show that p_n has exactly n pairwise different zeroes in $(-1, 1)$.

116. Verify the classical Mean Value Theorem for $f(x) = 2x^2 - 7x + 10$ on $[a, b] = [2, 5]$.

117. (a) Let $0 < a < b$. Prove that

$$1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1.$$

(b) Use (a) to show that

$$\frac{1}{6} < \log(1.2) < \frac{1}{5}.$$

118. * Let $f(x)$ be a real-valued differentiable function on (a, b).
(a) Show that if $f'(x) \equiv 0$ then f is a constant function.
(b) Show that if $f'(x) > 0$ for $x \in (a, b)$ then f is strictly monotone increasing.
(c) Show that if f is continuous on $[a, b]$ and satisfies $t \leq f'(x) \leq T$ on (a, b) then $t(y - x) \leq f(y) - f(x) \leq T(y - x)$ for all x, y such that $a \leq x \leq y \leq b$.

119. Let $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Note that $\sinh'(x) = \cosh(x)$ and $\cosh^2(x) = 1 + \sinh^2(x)$ and, by the previous exercise, $\sinh : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing and, therefore, invertible. Its inverse function is denoted by $\mathrm{Ar}\sinh : \mathbb{R} \to \mathbb{R}$.

(a) Calculate $\mathrm{Ar}\sinh$ via the explicit expression of of sinh and derive $\mathrm{Ar}\sinh'(y)$.

(b) Using the formula (1), calculate $\mathrm{Ar}\sinh'(y)$ from the derivative of sinh.

120. In this question you may use without proof that $\arctan'(x) = 1/(1 + x^2)$.

(a) Let $0 < a < b$. Prove that

$$\frac{b - a}{1 + b^2} < \arctan(b) - \arctan(a) < \frac{b - a}{1 + a^2}.$$

(b) Show that

$$\frac{\pi}{4} + \frac{3}{25} < \arctan(4/3) < \frac{\pi}{4} + \frac{1}{6}.$$

121. * Prove L'Hôpital's Rule using the Generalised Mean Value Theorem or, more precisely, the identity

$$0 = (g(b) - g(a))f'(c) - (f(b) - f(a))g'(c)$$

for some $c \in (a, b)$. In your proof, make sure that you do not carry out a division by zero.

122. Evaluate $\lim_{x \to 1} \frac{1 + \cos(\pi x)}{x^2 - 2x + 1}$.

123. Evaluate $\lim_{x\to 0} \frac{x-\sin x}{x^3}$.

124. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and

$$f(x) = \begin{cases} \frac{g(x)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and $g(0) = g'(0) = 0$ and $\lim_{x\to 0} g''(x) = 17$. Find $f'(0)$.

125. Use Newton's method to to calculate a positive root of $5 \sin x = 4x$ to three decimal digits.

9 Infinite series

126. Determine whether or not the series $\Sigma_{n=1}^{\infty}(2 + n)/\sqrt{4n^4 - 1}$ converges.

127. Determine whether or not the series $\Sigma_{n=1}^{\infty}\sqrt{n}/(n^3 + 1)$ converges.

128. Determine whether or not the series $\sum_{n=1}^{\infty} \sin(2^n)/2^n$ converges.

129. Determine whether or not the series $\sum_{n=1}^{\infty}(n - 3)(2 + 9n^6)^{-1/2}$ converges.

130. Use the comparison test to decide whether or not $\sum_{n=1}^{\infty} x_n$ converges in each of the following cases. (You may assume that $\sum_{n=1}^{\infty} n^{-\alpha}$ converges iff $\alpha > 1$.)
(a) $x_n = n/\sqrt{1 + n^6}$ (b) $x_n = 1/(n + \sqrt{n})$ (c) $x_n = (3 - n\sqrt{n})/n^6$
(d) $x_n = n!\, n^2/(n + 3)!$ (e) $x_n = n^2 \exp(-\sqrt{n})$ (f) $x_n = (n \cos n)/(n^3 + \log n)$
(g) $x_n = n^{-1} \sin(n^{-1})$ *[Use $\sin\theta < \theta$ for $\theta > 0$]* (h) $x_n = n^{-2}(\log n)^4$
(i) $x_n = \sqrt{1 + n^2} - n$.

131. For which values of α do the following series converge?
(a) $\sum_{n=1}^{\infty}(n^2 + 1)^{-\alpha} \log(1 + \frac{1}{n})$ (b) $\sum_{n=1}^{\infty} n^{\alpha} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$.

132. Show that the series $(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + \ldots$ converges if and only if the sequence $\{x_n\}$ tends to a limit as $n \to \infty$.

133. If $\sum_{n=1}^{\infty} x_n$ converges to s, and $y_n = (x_n + x_{n+1})/2$ for all n, does $\sum_{n=1}^{\infty} y_n$ converge, and if so to what?

134. Given that $\sum_{n=1}^{\infty} x_n$ converges, and $\sum_{n=1}^{\infty} y_n$ converges absolutely, prove that $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely. If we knew only that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converged, would it follow that $\sum_{n=1}^{\infty} x_n y_n$ converged as well?

135. Determine whether or not each of the following series converges.
(a) $\sum_{n=3}^{\infty} \tan(\pi/n)\cos(n\pi)$ (b) $\sum_{n=2}^{\infty} n^{-1}(\log n)^{-3}$ (c) $\sum_{n=1}^{\infty}(2n)!\, 5^{-n}(n!)^{-2}$

136. Determine whether or not each of the following series converges.

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{(-1)^n\sqrt{n}\tanh n} \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n(2n)!}{9^n(n!)^2} \qquad (c)\ \sum_{n=1}^{\infty} \frac{n-1}{(n^2+2)(n^2+1)^{1/4}}.$$

137. Discuss whether or not $\sum_{n=1}^{\infty} x_n$ converges in each of the following cases.
(a) $x_n = (n!)^2/(2n)!$ (b) $x_n = 1/[(n+1)\log(n+1)]$
(c) $x_n = (\cos \pi n)/(n \log(n+1))$.

138. For what values of α does the series $\sum_{n=1}^{\infty} x_n$ converge, in each of the following cases? *[Be careful to investigate all real values of α. In each case except (c), use the ratio test first, and then deal with the remaining values of α separately.]*
(a) $x_n = \alpha^n n^\alpha$ (b) $x_n = \alpha^{n-1}/(n3^n)$ (c) $x_n = n^{-1}(\log(n+1))^{-\alpha}$ (d) $x_n = n!\alpha^n$
(e) $x_n = n\alpha^n/(2^n(3n-1))$. *[For (c): first compare with $y_n = (n+1)^{-1}(\log(n+1))^{-\alpha}$].*

139. Find values of z for which the series $\sum a_n(z - z_0)^n$ converges in the following cases:
(a) $z_0 = 0$, $a_n = 1/n!$; (b) $z_0 = 1$, $a_n = 1/(n-1)!$, $n > 1$; (c) $z_0 = 0$, $a_n = c^n$; (d) $z_0 = 0$, $a_n = n$; (e) $z_0 = 0$, $a_n = n!$.

140. Determine whether or not the following series converge:
(a) $\Sigma_{n=1}^{\infty} n^2\, 2^{-n}$ (b) $\Sigma_{n=1}^{\infty}[1 + \exp(-n)]/[(n+1)^2 - (n-1)^2]$
(c) $\Sigma_{n=1}^{\infty} n^{-2} \log n$ (d) $\Sigma_{n=1}^{\infty} n!\, 2^n\, n^{-n}$

141. * Test the following series for convergence:

$$\sum_{n=1}^{\infty} \left[n^4 \sin^2 \left(\frac{2n}{3n^3 - 2n^2 + 5} \right) \right]^n .$$

142. * Test the following series for convergence:

$$\sum_{n=1}^{\infty} \frac{(3n-1)! - 4^{n+1}}{(3n)!}.$$

143. ** Test the following series for convergence:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n + (-1)^n}.$$

144. * This problem is dedicated to the proof of **Raabe's Test** for series $\sum a_n$, stating the following: Assume that we have

$$\lim_{k\to\infty} k \left(1 - \frac{|a_{k+1}|}{|a_k|} \right) = L > 1. \tag{2}$$

Then the series $\sum a_n$ converges absolutely.

(a) Let (2) be satisfied. Show that then there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that for all $k \geq N$:

$$|a_k| \leq \frac{1}{\epsilon} \left((k-1)|a_k| - k|a_{k+1}| \right).$$

(b) Conclude from (a) that we have for all $M \geq N$:

$$\sum_{k=N}^{M} |a_k| \leq \frac{N-1}{\epsilon} |a_N|.$$

(c) Conclude from (b) that $c_M = \sum_{k=N}^{M} |a_k|$ is convergent and, therefore, $\sum a_k$ is absolute convergent.

145. Let $\sum a_k$ be given by

$$a_k = \left(\frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{3 \cdot 6 \cdot 9 \cdots (3k)} \right)^2.$$

(a) Check that the Ratio Test fails for this series.

(b) Apply Raabe's Test introduced in the previous problem and prove convergence of the series.

146. * Assume that $\sum_{k=1}^{\infty} a_k$ is conditionally convergent. Let $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \min\{a_n, 0\}$ and $s_n^+ = \sum_{k=1}^{n} a_k^+$ and $s_n^- = \sum_{k=1}^{n} a_k^-$. Show that both $\sum a_n^+$ and $\sum a_n^-$ contain infinitely many nonzero terms and that neither of them is convergent. Therefore, the sequence of partial sums s_n^+ is not bounded above and s_n^- is not bounded below.

147. * Using the results from the previous problem, we now give an explanation why conditionally convergent series $\sum a_k$ can be rearranged to converge to any given limit $s^* \in \mathbb{R}$. Since s_n^+ is monotone increasing and unbounde, there exists a smallest index n_1 with $s_{n_1-1}^+ \leq s^* \leq s_{n_1}^+$. Assuming that all the zero terms in a_n^+ and a_n^- have been discarded, the first n_1 terms in the rearrangement are then $a_1^+ + a_2^+ + \cdots + a_{n_1}^+$. We denote this value by

$$U_1 = a_1^+ + a_2^+ + \cdots + a_{n_1}^+ \geq s^*.$$

Now we add terms from a_k^-, stopping at the smallest index when the sum satisfies

$$U_1 + a_1^- + a_2^- + \cdots + a_{m_1}^- < s^*,$$

and we introduce

$$L_1 = a_1^- + a_2^- + \cdots + a_{m_1}^-.$$

Find now the right arguments to define $U_2 = a_{n_1+1}^+ + \cdots + a_{n_2}^+$ and $U_3, U_4, \ldots$ and $L_2, L_3, \ldots$. Explain that this procedure never stops and that the sequences

$$s_k^U = U_1 + L_1 + U_2 + L_2 + \cdots + U_k$$

and

$$s_k^L = U_1 + L_1 + U_2 + L_2 + \cdots + U_k + L_k$$

converge from above and below to s^*. This provides you with the rearrangement of $\sum a_k$ converging to s^*.

148. Show that the Cauchy product is not necessarily convergent if both series $\sum a_k$ and $\sum b_k$ are only conditionally convergent by discussing the choice $a_k = b_k = \frac{(-1)^n}{\sqrt{n+1}}$.

149. Calculate $\lim_{n\to\infty} z_n$ in each of the following cases (or show that no limit exists).
(a) $z_n = \exp(in^2)/(1+in^2)$ (b) $z_n = n^2\exp(in^2 - n)$
(c) $z_n = \exp(i\pi n/\sqrt{16n^2+1})\sqrt{2n^2+1}/(n+i)$ (d) $z_n = 2n\exp(in\pi)/(n+i)$

150. Determine whether or not $\Sigma_{n=1}^{\infty} z_n$ converges, in each of the following cases.
(a) $z_n = 1/(n+i)$ (b) $z_n = 1/(n^2+i)$
(c) $z_n = (5+2i)^n/n!$ (d) $z_n = (n+2i)^4\exp(in^4 - n)$

10 Integrals

151. Let $\mathcal{P}_n$ denote the partition of $[0,1]$ into n subintervals of equal length (so that $\mathcal{P}_n = \{0, 1/n, 2/n, \ldots, 1\}$).
(a) Write down the upper and lower Riemann sums for the function $f(x) = \mathrm{e}^{-x}$ with respect to $\mathcal{P}_n$.
(b) Use these to show that e^{-x} is Riemann integrable on $[0,1]$.
(c) Evaluate $\int_0^1 \mathrm{e}^{-x}\,dx$ directly, express $L(f,\mathcal{P}_n)$ as a partial sum of a geometric series, and and use the fact that $L(f,\mathcal{P}_n) \to \int_0^1 \mathrm{e}^{-x}\,dx$ as $n\to\infty$ to deduce that $\lim_{n\to\infty} n(\mathrm{e}^{1/n}-1) = 1$.

152. Let $\mathcal{P}_n$ denote the partition of $[1,2]$ into n subintervals of equal length. Write down the upper and lower Riemann sums of $f(x) = \log x$ with respect to $\mathcal{P}_n$, and use these to show that f is Riemann integrable on $[1,2]$. Show that $\lim_{n\to\infty} L(f,\mathcal{P}_n) = \int_1^2 f(x)\,dx$, evaluate the integral directly, and deduce that

$$\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n-1}{n}\right)\right]^{1/n} \to \frac{4}{\mathrm{e}} \text{ as } n\to\infty.$$

153. Let $\mathcal{P}_n$ denote the partition of the interval $[1,2]$ into n subintervals of equal length. For $n=2$ and $n=4$ compute the Riemann sums $U(f,\mathcal{P}_n)$ and $L(f,\mathcal{P}_n)$ which approximate $I = \int_1^2 dx/x$. Tabulate the difference between I and each of these four approximations, working to 4 decimal places.

154. Show that

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin(\frac{\pi}{2n}) + \sin(\frac{2\pi}{2n}) + \sin(\frac{3\pi}{2n}) + \cdots + \sin(\frac{n\pi}{2n})\right) = \frac{2}{\pi}.$$

155. Show that

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right) = \log(2).$$

156. Let a, b be two real numbers and $a < b$. Recall that a function $f : [a,b] \to \mathbb{R}$ is called uniformly continuous if, for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [a,b]$ with $|y - x| < \delta$ we have

$$|f(y) - f(x)| < \epsilon.$$

Show that every uniformly continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable. (Since every continuous function f on a compact interval is uniformly continuous, we see that all continuous functions defined on compact intervals are Riemann integrable.)

157. * Let $f : [0, 1] \to \mathbb{R}$ be defined as follows: $f(x) = 0$ if $x \in [0, 1]$ is irrational and $f(x) = 1/q$ if $x \in [0, 1]$ is rational of the form $x = p/q$ with $p, q \in \mathbb{N} \cup \{0\}$ without common factors. Show that f is Riemann integrable and that $\int_0^1 f(x)dx = 0$. *[Hint: We always have $L(f, \mathcal{P}) = 0$. So you need to find partitions which make the upper Riemann sum arbitrarily small.]*

158. Prove that $\displaystyle\lim_{k\to\infty} \int_0^{2\pi} \frac{\sin(kx)}{x^2 + k^2} dx = 0.$

159. Prove that $\displaystyle\left| \int_1^{\sqrt{3}} \frac{e^{-x} \sin(x)}{x^2 + 1} dx \right| \leq \frac{\pi}{12e}.$

160. * In this problem we present a clever way to calculate directly the integral $\int_a^b x^p dx$ for $0 < a < b$ and $p \in \mathbb{N}$. We use partitions $\mathcal{P}_n = \{a = x_0, x_1, \ldots, x_n = b\}$ for which the ratios x_i/x_{i-1} are constant (and not the differences $x_i - x_{i-1}$).

    (a) Show that we have $x_i = ac^{i/n}$ with $c = b/a$.

    (b) Using $f(x) = x^p$, show that

$$\begin{aligned} U(f, \mathcal{P}_n) &= a^{p+1}(1 - c^{-1/n}) \sum_{i=1}^{n} \left(c^{(p+1)/n}\right)^i \\ &= (a^{p+1} - b^{p+1})c^{(p+1)/n} \frac{1 - c^{-1/n}}{1 - c^{(p+1)/n}} \\ &= (b^{p+1} - a^{p+1}) \cdot c^{p/n} \cdot \frac{1}{1 + c^{1/n} + c^{2/n} + \cdots + c^{p/n}}. \end{aligned}$$

    Find a similar formula for $L(f, \mathcal{P}_n)$.

    (c) Conclude that we have

$$\int_a^b x^p dx = \frac{b^{p+1} - a^{p+1}}{p + 1}.$$

161. Show that there exist values $\xi_1, \xi_2 \in [0, 1]$ such that

$$\int_0^1 \frac{\sin(\pi x)}{1 + x^2} dx = \frac{2}{\pi(\xi_1^2 + 1)} = \frac{\pi}{4} \sin(\xi_2 \pi).$$

162. (a) Use the Mean Value Theorem for Integrals (Theorem 10.9) to derive the following fact: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

    (b) Prove the following fact directly using (a): Let $f : [a, b] \to \mathbb{R}$ be continuous and $F(c) = \int_a^c f(x)dx$. Then F is continuous on $[a, b]$.

163. Use the results in the previous problem to prove the Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be continuous and $F(c) = \int_a^c f(x)dx$. Then F is differentiable and we have $F'(c) = f(c)$. It is sufficient if you restrict your arguments to the case $c \in (a, b)$.

164. Calculate $\lim_{c\to 0} \dfrac{\int_0^c \sin(x^3)dx}{c^4}$.

165. Calculate $\lim_{x\to\pi/2} \dfrac{ex^2/\pi - e\pi/4 + \int_x^{\pi/2} e^{\sin t} dt}{1 + \cos(2x)}$.

166. Let $f(x) = \displaystyle\int_1^{x^3+x} (t^2 + t + 1)dt$. Show that

$$f'(x) = 3x^8 + 7x^6 + 3x^5 + 5x^4 + 4x^3 + 4x^2 + x + 1.$$

167. Let $f, g : [a, b] \to \mathbb{R}$ be continuous. This problem ist concerned with the proof of Schwarz's inequality:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx. \tag{3}$$

(a) Let $A = \int_a^b (f(x))^2 dx$, $B = \int_a^b (g(x))^2 dx$ and $C = \int_a^b f(x)g(x)dx$. W.l.o.g. assume that $B \neq 0$. Derive from

$$\int_a^b (f(x) + \lambda g(x))^2 dx \geq 0 \qquad \text{for all } \lambda \in \mathbb{R}$$

that $C^2 - AB \leq 0$. *[Hint: Here you may use the fact that the discriminant $b^2 - 4ac$ of a non-negative quadratic polynomial $p(x) = ax^2 + bx + c$ with $p(x) \geq 0$ for all $x \in \mathbb{R}$ must be non-positive.]*

(b) Conclude Schwarz's inequality from (a).

168. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and g be not identically zero. Use the proof of the previous problem that equality in (3) implies that there exists $\lambda \in \mathbb{R}$ with $f = \lambda g$.

169. Determine whether or not the following integrals converge.
(a) $\int_0^\infty (\cos x)/(x+e^x)\, dx$ (b) $\int_1^\infty (x+\sqrt{x})^{-1}\, dx$ (c) $\int_1^\infty \sqrt{(6 + x)/(1 + x^6)}\, dx$
(d) $\int_0^\infty x^2 e^{-x}\, dx$ [Do in two different ways.] (e) $\int_1^\infty (1+x^3)^{-1/2}\, dx$ (f) $\int_0^1 x^{-3/2} e^{-x}\, dx$
(g) $\int_0^1 e^{-x}/\sqrt{x}\, dx$ (h) $\int_0^1 x/\sqrt{1 - x^2}\, dx$ (i) $\int_0^1 x^{-1/3} \cos x\, dx$ (j) $\int_0^1 \sqrt{x - x^2}/x\, dx$

170. If $\int_1^\infty f(x)\, dx$ converges and $\lim_{x\to\infty} f(x) = L$, prove that $L = 0$.

171. Let $f(x)$ be continuous for $x \geq 0$ and differentable for $x > 0$. Suppose that you want to use integration by parts on $[0, R]$ to express $\int_0^\infty x f'(x)\, dx$ in terms of f and $\int_0^\infty f(x)\, dx$. (Here f' denotes the derivative of f.) Find conditions on f which make such a formula valid.

172. Determine whether or not $\int_0^2 x(16 - x^4)^{-1/2}\, dx$ converges, by (a) actually doing the integral *[Hint: $u = x^2$]*; and (b) comparison, without doing the integral.

173. Determine whether or not $\int_0^1 (\log x)^2\, dx$ converges, by (a) actually doing the integral *[Look it up if necessary]*; and (b) comparison, without doing the integral *[Hint: use the fact that $x^{1/4}\log x \to 0$ as $x \to 0$]*.

174. Determine whether or not $\int_0^{\pi/2} (\tan x)^3\, dx$ converges. *[Hint: do $\int_0^c$ for $c < \pi/2$: look at up, or use* $\tan^2 = \sec^2 - 1$. *Alternatively, use comparison.]*

175. For each of the following integrals, determine all the values of α for which the integral converges. (a) $\int_0^1 x^{-\alpha} \cos x\, dx$ (b) $\int_0^1 (x+1/x)^\alpha\, dx$ (c) $\int_0^1 (\sin x)^{-\alpha}\, dx$ (d) $\int_0^1 x^{-\alpha} \sin x\, dx$ (e) $\int_0^\infty x^{\alpha-1}/(1+x)\, dx$.

176. Discuss the convergence of the integral $\int_0^\infty x^{-4/3} \sin x\, dx$.
[Use $\sin x \le x$ for $0 \le x \le 1$.]

177. For which values of the real parameter c does $\int_0^\infty x^c/\sqrt{x^2+x}\, dx$ converge?

178. For which real values of p does $\int_0^\infty x^{-p} \mathrm{e}^{-x} \sin(x)\, dx$ converge?
[Use $2x/\pi \le \sin x \le x$ for $0 \le x \le \pi/2$.]

179. For which real values of p does $\int_0^\infty (x+x^2)^{-p}\, dx$ converge?

180. * Assume that $f : [0,\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = a$. Show that

$$\lim_{c\to\infty} \frac{1}{c} \int_0^c f(x)dx = a.$$

11 Sequences of functions and uniform convergence

181. Find the pointwise limit of the functions $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \le n, \\ x-n & \text{if } n < x \le n+1, \\ 0 & \text{if } n+1 < x, \end{cases}$$

and decide whether the convergence is uniform.

182. Find the pointwise limit of the functions $f_n : (1,\infty) \to \mathbb{R}$,

$$f_n(x) = \frac{e^x}{x^n}$$

and decide whether the convergence is uniform.

183. Find the pointwise limit of the functions $f_n : [-1,1] \to \mathbb{R}$,

$$f_n(x) = e^{-nx^2}$$

and decide whether the convergence is uniform.

184. Find the pointwise limit of the functions $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = 1 - \frac{e^{-x^2}}{n}$$

and decide whether the convergence is uniform.

185. Find the pointwise limit of the functions $f_n : [0,1] \to \mathbb{R}$,

$$f_n(x) = x^n - x^{2n}$$

and decide whether the convergence is uniform. *[Hint: For each n, find the maximum of $f_n - f$ on $[0,1]$.]*

186. Find the pointwise limit of the functions $f_n : [0,\infty) \to \mathbb{R}$,

$$f_n(x) = \frac{nx}{1+n+x}$$

and decide whether the convergence is uniform. *[Hint: For each n, consider $|f_n(x) - f(x)|$ for large x.]*

187. Find the pointwise limit of the functions $f_n : [0,\infty) \to \mathbb{R}$,

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

and decide whether the convergence is uniform. *[Hint: Express $|f_n(x) - f(x)|$ as a fraction.]*

188. Give a proof of Theorem 11.5, i.e., the following fact: Let $I = [a,b]$ and $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions. If $f_n \to f$ uniformly, then we have for all $c \in [a,b]$

$$\int_a^c f_n(x)dx \to \int_a^c f(x)dx.$$

189. Let $a, b \in \mathbb{R}$ with $a < b$. For every $f \in C([a,b])$ we define

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

Show that $\|\cdot\|_\infty : C([a,b]) \to [0,\infty)$ satisfies the following properties (these are precisely the axioms of a norm):

(a) $\|f\|_\infty = 0$ if and only if $f \in C([a,b])$ is identically zero.

(b) $\|\lambda f\|_\infty = |\lambda| \cdot \|f\|_\infty$ for all $\lambda \in \mathbb{R}$ and all $f \in C([a,b])$.

(c) Triangle Inequality: $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in C([a,b])$.

190. ** Recall that $C([a,b])$ carries the structure of a real vector space. Moreover, $\|\cdot\|_\infty$ defines a norm on this vector space, which allows us to understand the expression $\|f-g\|_\infty$ as a kind of distance between the vectors f and g (like $\|v-w\| = \left(\sum_{i=1}^n (v_i - w_i)^2\right)^{1/2}$ can be understood as the distance between

the vectors v, w in the real vector space $\mathbb{R}^n$). The norm allows us to define convergence $f_n \to f$ in $C([a,b])$ and Cauchy sequences f_n in $C([a,b])$. For a sequence $f_n \in C([a,b]$ and a function $f \in C([a,b])$, we say f_n converges to f (in short "$f_n \to f$ in $C([a,b])$" if

$$\|f_n - f\|_\infty \to 0 \quad \text{as } n \to \infty.$$

A sequence $f_n \in C([a,b])$ is called a Cauchy sequence, if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have

$$\|f_n - f_m\|_\infty < \epsilon.$$

Prove the following fact about the normed real vector space $C([a,b])$:

Everg Cauchy sequence in $C([a,b])$ is convergent, i.e., if $f_n \in C([a,b])$ is a Cauchy sequence then there exist a function $f \in C([a,b])$ such that $f_n \to f$ in $C([a,b])$.

Normed vector spaces with this property are called *complete normed vector spaces* or *Banach spaces* and play an important role in Analysis. We already discussed Completeness of the space $\mathbb{R}$ of real numbers, and the Completeness Axiom can be reformulated as the fact that every Cauchy sequence in $\mathbb{R}$ is convergent.

12 Power series and Taylor series

191. If $\Sigma a_n z^n$ has finite radius of convergence R, what is the radius of convergence of $\Sigma a_n z^{2n}$? (Give a proof of your answer.)

192. Calculate the radius of convergence R of the power series $\Sigma a_n z^n$ in each of the following cases.
(a) $a_n = (2n)!/(n!)^2$ (b) $a_n = (3n+4)/2^n$ (c) $a_n = (2n)!/n^n$
(d) $a_n = (3n)!/[2^n(n!)^3]$ (e) $a_n = (-1)^n i^n n^2/3^n$ (f) $a_n = 2^{10n}/n!$
(g) $a_n = 2^n/(3^n+1)$

193. Calculate the radius of convergence R of the power series $\Sigma[(-1)^n/2^n]z^{2n}$.

194. Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}} z^n.$$

195. Use the n^{th} root test to find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$$

You may use without proof the following estimate for $n!$ (which is called Stirling's formula):

$$\sqrt{2\pi n} n^n e^{-n} < n! < \sqrt{2\pi n} n^n e^{-n} e^{1/(12n)}.$$

196. Use the n^{th} root test to find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} a_n z^n$$

where

$$a_n = \begin{cases} 2^k, & \text{if } n = k!, \\ 1, & \text{otherwise.} \end{cases}$$

197. Let $\sum a_n z^n$ and $\sum b_n z^n$ with $|a_n| \leq b_n$. Show that the radius of convergence of $\sum b_n z^n$ must be smaller or equal to the radius of convergence of $\sum a_n z^n$.

198. Define $u_n(x) = x^3(1+x^2)^{-n}$ for $n = 1, 2, \ldots$. Show that $\Sigma_{n=1}^{\infty} u_n(x)$ converges to a rather simple function $f(x)$. *[Hint: geometric series.]* Compute $df/dx = d(\Sigma u_n)/dx$ and $\Sigma_{n=1}^{\infty} du_n/dx$ at $x = 0$: are they equal?

199. Define $u_n(x) = nx\exp[-nx^2] - (n-1)x\exp[-(n-1)x^2]$ for $n = 1, 2, \ldots$. Show that $\Sigma_{n=1}^{\infty} u_n(x)$ converges to a rather simple function $f(x)$. *[Hint: write out the partial sum $S_k(x)$.]* Compute $\int_0^1 f(x)\,dx = \int_0^1 \Sigma u_n(x)\,dx$ and $\Sigma_{n=1}^{\infty} \int_0^1 u_n\,dx$: are they equal?

200. Prove that

$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1}. \tag{4}$$

Deduce that

$$0 < e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right) < \frac{1}{100}$$

and conclude that $2.7083 < e < 2.7184$.

201. ** Prove the e is irrational. [Hint: Assume that $e = p/q$, where p, q are natural numbers and seek a contradiction using inequality (4).]

202. Prove that the following series converge uniformly in the given regions (a) $\sum_{n=1}^{\infty} \frac{\pi^n}{n^4} x^{2n}$, $|x| \leq 0.56$ (b) $\sum_{n=1}^{\infty} \frac{\sin(n|x|)}{n^2}$ all x
(c) $\sum_{n=1}^{\infty} \frac{n\,x^n}{n^3+|x|}$ $|x| \leq 1$

203. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

converges uniformly on $\mathbb{R}$.

204. Prove that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{nx}{1+n^4x^2}$$

converges uniformly on $[a, \infty)$ for $a > 0$. *[Hint: Find the maximum of $nx/(1+n^4x^2)$ on $[0, \infty)$.]*

205. Let f be as in the previous problem, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{nx}{1+n^4x^2}.$$

Show that $f(1/N^2) \geq (N^2/2) \sum_{n \geq N} 1/n^3$ and, by using an integral to estimate the sum, show that $f(1/N^2) \geq 1/4$. Conclude from this that the series does not converge uniformly on $\mathbb{R}$.

206. Use the Taylor series of e^x, $\log(1+x)$, $1/(1-x)$, $\sin x$ and $\cos x$ to derive the first three non-zero terms in the Taylor expansions about $x = 0$ of the following functions.
(a) $\cos^2 x$ (b) $\sin(x^2)$ (c) $e^x \sin x$ (d) $1/(1+x^2)$
(e) $x/(1+x^3)$ (f) $(1+x^2)^{-2}$ (g) $[\exp(x^4)-1]/x^3$
(h) $(1-x)^{-3}$ (i) $\exp(x^2)\sin(x^2)$ (j) $\exp[1/(1-2x)]$ (k) $\exp(\exp x)$
(l) $\log(1+2x^2)$ (m) $[\log(1+x)]^2$

207. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}.$$

Show that, for any $x \neq 0$,

$$f^{(k)}(x) = p_k(1/x)e^{-1/x^2},$$

where p_k is a polynomial of degree $3k$. Deduce that, for every $k \in \mathbb{N} \cup \{0\}$,

$$\frac{f^{(k)}(x)}{x} \to 0 \quad \text{as } x \to 0+.$$

Hence show that f can be differentiated infinitely many times at $x = 0$ and $f^{(k)}(0) = 0$ for all $k \in \mathbb{N} \cup \{0\}$. For which values of $x \in \mathbb{R}$ does the Taylor series of f converge to $f(x)$?

208. Evaluate the following infinite sums via manipulations of well-known power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.

(c) $\displaystyle\sum_{n=0}^{\infty} (2n+1) \left(\frac{1}{2}\right)^{2n+1}$.

209. By multiplying the Taylor series for $\sin(x)$ and $\cos(x)$, verify that

$$2\sin(x)\cos(x) = \sin(2x).$$

[Hint: You may use that $\sum_{k=0}^{n} \binom{2n+1}{2k+1} = 2^{2n}$ *which you can prove via the Binomial formula.]*

210. Let $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$. Determine $f^{(k)}(0)$ for all $k \in \mathbb{N}$.
[Hint: Find a power series representing $f(x)$*.]*