A mechanical system with quasiergodic orbits

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In this article, we consider a simple mechanical system with two degrees of freedom, which has quasiergodic orbits. This system turned up in an exchange of letters with Mr. G. Herglotz.

We want to construct a surface whose geodesics are 'almost all' quasiergodic, i.e. get arbitrarily close to each point, and in every given direction. We consider the upper half plane $y > 0$ of the complex numbers $z = x + iy$ and use the Poincaré metric with line element

$$ds = \frac{d\sigma}{y} \qquad (1)$$

where $d\sigma = |dz|$ is the normal Euclidean line element. This metric will then have two well known simple properties:

1. It is invariant under all linear maps $z_1 = \frac{az+b}{cz+d}$ with real coefficients and positive determinant $ad - bc > 0$. It is also invariant under the reflection $z \mapsto -\bar{z}$, where $\bar{z}$ denotes the complex conjugate to z.

2. The geodesic lines are all half-circles in the upper half plane, which touch the real axis perpendicularly.

The homogeneity of our metric follows from 1. Calculation leads to curvature $K = -1$.

From now on we identify all points in the upper half plane that are mapped to each other by a substitution with integer coefficients and determinant 1; we also regard two points z_1 and z_2 as 'equal' when

$$z_1 = \frac{az_2 + b}{cz_2 + d} \qquad (2)$$

with integers a, b, c, d which satisfy the condition $ad - bc = 1$.

The fundamental domain of this group of transformations is the well-known modular domain, consisting of points between the straight lines $x = -1/2$ and $x = +1/2$, that lie outside the unit circle. In this 'triangle' there is only one representative of each set of points that we consider to be 'equal' (except that on the boundary of the triangle opposite points are identified). The boundary points of the triangle are identified in a way which is easy to understand. We obtain an abstract surface, which we can think of as originating from 'glueing together' the modular form. Because of the homogeneity of the Poincaré metric, the metric on our surface is regular, up to the boundary of the modular domain. We will later discuss the physical feasibility.

To trace a geodesic line in our surface, which is originally given by an orthogonal circle K, we have to find all circles 'identical' to K, i.e. all circles that arise from K through linear transformations by an integer and unimodular substitutions. It suffices here to consider those 'representatives' of this circle, i.e. those parts of circles, that lie in the modular domain. In the (above) picture, we have traced a geodesic line in the modular domain.

There is exactly one geodesic through a given point in a given direction. So quasiergodicity implies that among the circles 'identical' to K, there are circles that come arbitrarily close to a given circle C.

If we now note that an orthogonal circle is determined by its intersection points ξ, η with the real axis, that ξ, η will be mapped to the endpoints of the image circle, and that two circles are 'near each other' when their endpoints are close to each other, we obtain the following fact:

The endpoints ξ, η of K are mapped by

$$\xi' = \frac{a\xi + b}{c\xi + d} \quad , \quad \eta' = \frac{a\eta + b}{c\eta + d} \qquad (3)$$

to the endpoints ξ', η' of the image circle. It will be useful to introduce a ξ, η plane. Then every point in the ξ, η plane corresponds to a circle in the z-plane whose endpoints are precisely the coordinates of this point, whereas conversely two points of the ξ, η plane are assigned to each circle, since there is still choice in naming the endpoints ξ or η.

With this transformation the point ξ, η is mapped to the simultaneously transformed coordinates ξ', η' (or η', ξ'). The points obtained by all such transformations must now get arbitrarily close to every point, so that in the z-plane the transformed circles come arbitrarily close to every circle. However, this implies that the points arising from ξ, η must densely cover the plane.

A short investigation shows that the dense covering of the subset

$$\{1 < \xi, \ -1 < \eta < 0\} \qquad (4)$$

is both necessary and sufficient.

Necessity is clear. If the condition is fulfilled, then we apply the transformation

$$\xi' = \xi + n \qquad\qquad \eta' = \eta + n \qquad (5)$$

to all points obtained, for any integer n, and find that the area $n < \xi$, $n-1 < \eta < n$ is also densely covered. If we then fix the ambiguity of the notation of the endpoints by demanding that $\eta < \xi$, and observe that we only need to demand that the area which consists of circles that actually intersect the modular domain is densely covered, then the easy-to-find boundary of this domain shows that it is already covered by the domains which we already looked at.

In particular, we can assume that our initial circle belongs to the area $\xi > 1$, $-1 < \eta < 0$.

We now develop ξ and η into continued fractions:

$$\xi = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}} \qquad ; \qquad -\eta = \cfrac{1}{a_{-1} + \frac{1}{a_{-2} + \dots}} \qquad (6)$$

and define for any integer n;

$$\xi_n = a_n + \cfrac{1}{a_{n+1} + \frac{1}{a_{n+2} + \dots}} \qquad ; \qquad -\eta_n = \cfrac{1}{a_{n-1} + \frac{1}{a_{n-2} + \dots}} \qquad (7)$$

From the theory of continued fractions we can easily obtain the following formulas

$$\xi = \frac{P_n \xi_n + P_{n-1}}{Q_n \xi_n + Q_{n-1}}, \qquad \eta = \frac{P_n \eta_n + P_{n-1}}{Q_n \eta_n + Q_{n-1}} \qquad (8)$$

where P_i, Q_i are integers that satisfy the known recursion formulas and the equation

$$P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n \qquad (9)$$

It is essential here that for ξ and η the same numbers P_n and Q_n appear.

Because of (9), for even n the points (ξ_n, η_n) are equivalent to (ξ, η), which belongs to our domain by (7). Because of the ambiguity in the notation of the endpoints, the points $(-\frac{1}{\eta_n}, -\frac{1}{\xi_n})$ will also belong to it, likewise for even n.

Moreover, it follows from results of Minkowski that these are all the points equivalent to (ξ, η) that belong to our domain.

Now in order for these points to be everywhere dense, it is necessary that the continued fraction expansions agree arbitrarily far with any other given pair of continued fraction expansions, for a suitable choice of n. This implies the following.

Given two sequences $b_1, b_2, ..., b_m; c_0, c_1, ...c_m$, there must be an even index n, so that the sequence $a_{n-m}, a_{n-m+1}, ..., a_{n-1}, a_n, a_{n+1}, ..., a_{n+m}$ either coincides with $b_m, b_{m-1}, ..., b_1, c_0, c_1, ..., c_m$ or the reversed sequence.

Now we want to call a sequence $a_\nu, a_{\nu+1}, ..., a_\mu$ a detail of our a-sequence. Then clearly any other conceivable sequence $d_1, d_2, ..., d_r$ of positive integers must occur in our a-sequence.

We see that this condition must be fulfilled when we take $c_i = b_i$. This condition is also sufficient. [Because] if it is fulfilled, then we look for a detail of the a-sequence that coincides with the sequence $b_m, b_{m-1}, ..., b_1, c_0, c_1, ..., c_m$, $b_m, b_{m-1}, ..., b_1, c_0, c_1, ..., c_m$. One of the c_0 that occurs then definitely has an even index.

We briefly summarize what we have proved:

Our geodesic has an a-sequence which belongs to it

$$..., a_{-1}, a_{-1}, a_0, a_1, a_2, ...$$

We can find the essential orthogonal circles using formula (7). All other circles that intersect the modular domain arise from these circles (in the z-plane) by shifting by integers.

In order for our geodesic to be quasiergodic, it is necessary and sufficient that in its associated a-sequence every conceivable finite sequence of positive integers appears as a detail.

It is easy to see that there actually exists such an a-chain. The lattice points in the 'positive' octant of n-dimensional space are countable. If we now think of these sequences as written for every dimension, then using the diagonal method we can make a single sequence out of all the sequences. Now if instead of the points we write their coordinates in some fixed order, then we obtain a sequence $a_0, a_1, ...$ of positive numbers with the required property. We can still choose $a_{-1}, a_{-2}, ...$ arbitrarily.

This should imply that there are uncountably many quasiergodic chains. Even more is true! By results from Mr Celestyn Burstin, nearly all numbers ξ have a 'quasiergodic' continued fraction expansion. So out of all the geodesics that pass through a point of the plane almost all [of them] are quasiergodic.

In fact, we also obtain an insight into the other types of orbits. We content ourselves by listing some examples:

1. Periodic orbits belong to periodic chains and vice versa.

2. If the chain eventually becomes periodic on the right and likewise eventually periodic on the left, then it describes a doubly assymptotic orbit. The orbit 'branches away' from a periodic orbit, and eventually approaches a second (same or different) periodic orbit.

3. Our chain might not be quasiergodic, but ‘quasiperiodic’, i.e. any detail that appears once will appear infinitely many times, without given periodicity. Such a chain is obtained for example when we start with a quasiergodic chain and increase every digit by one. Its associated orbit then approximates itself without being quasiergodic. In other words, every state that occurs once will repeat itself infinitely many times up to arbitrary accuracy.

4. One case that doesn’t occur under type 3, might arise in the chain that we obtain from the chain constructed in 3 by writing a single 1 in the middle. Here, a state that occurs once does not recur, but the orbit approaches orbit type 3 more and more.

Now a few words on the physical feasibility. We obtain an offhand realizable case when besides the points identified by unimodular transformations we also consider points that are mapped to each other by the reflection $z_1 = -\bar{z}_2$ to be ‘equal’. The fundamental domain is then half of the modular domain, say the part lying inbetween $x = 0$ and $x = 1/2$. The geodesic lines are then simply circles, reflected at the boundaries. Since we now regard even more points as ‘equal’, our thoughts up till now certainly stay valid, so here almost all geodesic lines are also quasiergodic.

Now we consider the rotation surface of the tractrix of a thread of length one. It is known that this also has curvature $K = -1$, so that we can wrap part of our half plane around this surface. It is now easy to compute that we can surely take our half-modular domain as this part, so that it ‘fits’ into the rotation plane of the tractrix. Hence we have the physical realisation. First we draw a triangle congruent to the half-modular triangle on the surface of revolution of the tractrix. Our mechanical system can then be interpreted as a force-free movement of a point mass in this triangle (the point is constrained to stay on the surface), which will be reflected elastically from the edges of the triangle.