

The
Knuth-Bendix
algorithm and
conjugacy
problems in
monoids

Fabienne
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The
Knuth-Bendix
algorithm

The
conjugacy
problems in
monoids

A cyclical
rewriting
system

Solution to
the conjugacy
problems in
monoids

The algorithm
of cyclical

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Semigroup forum 2011

Fabienne Chouraqui

University of Haifa, Campus Oranim

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Definition of a rewriting system

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Let $M = \text{Mon}\langle X \mid l_1 = r_1, l_2 = r_2, \dots, l_m = r_m \rangle$, with $l_i, r_i \in X^*$.

String rewriting system

- $\mathfrak{R} \subseteq X^* \times X^*$ is a string rewriting system.

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$abaab$ is irreducible

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Some more definitions

- \rightarrow is *terminating* if there is no infinite sequence

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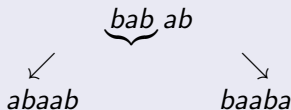
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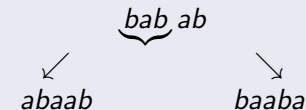
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\mathcal{R} is not confluent!

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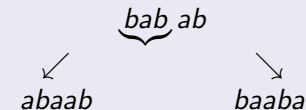
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A crucial idea

Assume \rightarrow is terminating. Then, \rightarrow is confluent if and only if \rightarrow is locally confluent.

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- An *overlap* occurs in uvw if $uv \rightarrow r_1$ and $vw \rightarrow r_2$.
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Assume \mathfrak{R} is terminating. Then, \mathfrak{R} is confluent if and only if all the critical pairs resolve.

Sketch of the algorithm

- First step: find all the critical pairs, order the pairs and add new rules to \mathfrak{R} to get \mathfrak{R}_1 .

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- Next step: find all the critical pairs in \mathcal{R}_1 , order the pairs and add new rules to \mathcal{R}_1 to get \mathcal{R}_2 .
- It may succeed with a finite or infinite equivalent complete rew.syst \mathcal{R}' or it may fail.

Application of the Knuth-Bendix algorithm of completion

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- By induction, there exists an equivalent infinite complete rew.syst: $\mathfrak{R}' = \{ba^nba \rightarrow aba^2b^{n-1}, n \geq 1\}$.

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\mathfrak{R}' is equivalent to \mathfrak{R} means: $u \leftrightarrow_{\mathfrak{R}} v$ if and only if $u \leftrightarrow_{\mathfrak{R}'} v$

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\rightarrow is complete if

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If \rightarrow is complete

- Every element reduces to a unique normal form.
- There exists a simple algorithm to solve the word problem: compare the normal forms.

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Examples of groups with finite complete rew.syst

Coxeter groups (finite-LeChenadec, Hermiller), graphs of groups (Hermiller-Meier)...

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- There exists a simple algorithm to solve the word problem: compare the normal forms.
- The monoid (group) is FP_∞

Examples of groups with finite complete rew.syst

Coxeter groups (finite-LeChenadec, Hermiller), graphs of groups (Hermiller-Meier)... Not known for hyperbolic or automatic groups.

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The algorithm
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Let M be a monoid generated by X , let $u, v \in X^*$.

- RConj: if there is a word $w \in X^*$ such that $uw =_M wv$.

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- Trans: if there are words w, w' in the free monoid such that $u =_M ww'$ and $v =_M w'w$. Trans is reflexive and symmetric, but not transitive.

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- $\text{Trans} = \text{Conj} = \text{LConj} = \text{RConj}$ for free monoids (Lentin-Schutzenberger).

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- $\text{Trans} = \text{Conj} = \text{LConj} = \text{RConj}$ for free monoids (Lentin-Schutzenberger).
- $\text{Trans} = \text{Conj} = \text{LConj}$ and solvable for a monoid with a special, finite and complete rewriting system (Otto).

A different approach: a cyclical rewriting system

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Definition of \circlearrowleft^i , \rightarrow

- $u \circlearrowleft^1 v$ if v is a cyclic conjugate of u obtained by moving the first letter of u to be the last letter of v .

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Definition of \circlearrowleft^i , \rightsquigarrow

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- $u \rightsquigarrow v$ if $u \circlearrowleft^i \tilde{u} \rightarrow v$.

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- u is cyclically irreducible if there is no v s.t $u \rightsquigarrow v$ (unless v a cyclic conjugate of u).

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Same definitions of \rightsquigarrow^* , terminating, confluent ... for \rightsquigarrow

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Same definitions of \rightsquigarrow^* , terminating, confluent ... for \rightsquigarrow

$\rho(u)$ cyclically irreducible form of u

Definition of a cyclical rew.system \curvearrowright

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Example: $\mathfrak{R} = \{ab \rightarrow bc, cd \rightarrow da\}$

- \mathfrak{R} is a finite complete rew.system.

Definition of a cyclical rew.system \mathcal{R}

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Example: $\mathcal{R} = \{ab \rightarrow bc, cd \rightarrow da\}$

- \mathcal{R} is a finite complete rew.system.
- \mathcal{R} is not terminating, since $bcd \mathcal{R}^* bcd$.
 $bcd \rightarrow bda \circlearrowright^2 abd \rightarrow bcd$.

Definition of a cyclical rew.system \mathcal{R}

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A local approach: Definition of $\text{Allseq}(w)$

$\text{Allseq}(w)$ is the set of all the possible sequences of cyclical reductions that begin by each word from $\{w_1, \dots, w_k\}$, where $w_1 = w, w_2, \dots, w_k$ are all the cyclic conjugates of w .

Definition of a cyclical rew.system \rightarrow

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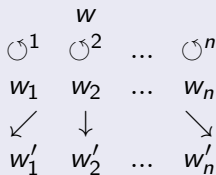
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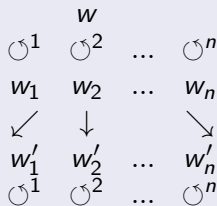
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Definition of a cyclical rew.system $\varphi \rightarrow$

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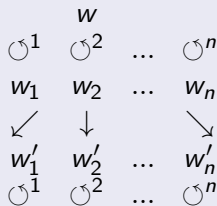
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A local approach: Definition of $\text{Allseq}(w)$



$\text{Allseq}(w)$ terminates if

there is no infinite sequence of cyclical reductions in $\text{Allseq}(w)$.

Definition of a cyclical rew.system $\varphi \rightarrow$

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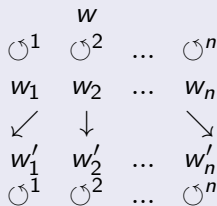
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A local approach: Definition of $\text{Allseq}(w)$



$\text{Allseq}(w)$ converges if

a unique cyclically irreducible form is achieved in $\text{Allseq}(w)$ (up to cyclic permutation)

When does $\text{Allseq}(w)$ converge?

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Definition of a triple \tilde{c} -defined

Let $R_1, R_2 \in \mathfrak{R}$ s.t. R_1 can be applied on a cyclic conjugate of w and R_2 can be applied on another one.

When does $\text{Allseq}(w)$ converge?

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Definition of a triple \tilde{c} -defined

Let $R_1, R_2 \in \mathfrak{R}$ s.t. R_1 can be applied on a cyclic conjugate of w and R_2 can be applied on another one. *The triple (w, R_1, R_2) is \tilde{c} -defined* if there is a cyclic conjugate \tilde{w} of w such that both rules R_1 and R_2 can be applied on \tilde{w} .

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When does Allseq(w) converge?

Assume Allseq(w) terminates. If all the triples occurring in Allseq(w) are \tilde{c} -defined, then Allseq(w) converges.

Given \mathcal{R} complete and cyclically terminating, when \mathcal{R} is cyclically confluent?

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Definition of cyclical critical pairs

- A *cyclical overlap* occurs in $xuyv$ if $xuy \rightarrow r_1$ and $yvx \rightarrow r_2$.

Given \mathcal{R} complete and cyclically terminating, when \mathcal{R} is cyclically confluent?

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- A *cyclical overlap* occurs in $xuyv$ if $xuy \rightarrow r_1$ and $yvx \rightarrow r_2$.
- A *cyclical inclusion* occurs in u if $u \rightarrow r_1$, $u' \rightarrow r_2$, and u' is a subword of a cyclic conjugate of u .

Given \mathcal{R} complete and cyclically terminating, when \mathcal{R} is cyclically confluent?

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Proposition

Let \mathcal{R} be a complete and cyclically terminating rewriting system. If there are no cyclical overlaps or cyclical inclusions between the rules in \mathcal{R} , then \mathcal{R} is cyclically confluent.

Example of cyclical inclusion

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Example: $\mathcal{R}' = \{ba^n ba \rightarrow aba^2 b^{n-1}, n \geq 1\}$

- There is a cyclical inclusion in $ba^2 ba$, since $ba^2 ba \rightarrow aba^2 b$, $bab \rightarrow aba$.

Example of cyclical inclusion

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- Also, $\text{Allseq}(ba^2 ba)$ does not terminate:
 $ba^2 ba \rightarrow aba^2 b \circlearrowleft^1 ba^2 ba \dots$

Example of cyclical inclusion

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- Nevertheless, $\rho(ba^2 ba) = aba^3$:
 $ba^2 ba \circlearrowleft^1 a^2 bab \rightarrow a^3 ba \dots$

Example of cyclical inclusion

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- Nevertheless, $\rho(ba^2 ba) = aba^3$:
 $ba^2 ba \circlearrowleft^1 a^2 bab \rightarrow a^3 ba \dots$

Note there is no cyclical inclusion in $ba^2 bab$

Example of cyclical overlap

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Example: $\mathfrak{R} = \{xy \rightarrow zx, yz \rightarrow zx, xz^n x \rightarrow zxzy^{n-1}, n \geq 1\}$

There is a cyclical overlap in xz^2xz^3 , since $xz^2x \rightarrow zxzy$,
 $xz^3x \rightarrow zxzy^2$.

Example of cyclical overlap

The
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conjugacy
problems in
monoids

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Example: $\mathfrak{R} = \{xy \rightarrow zx, yz \rightarrow zx, xz^n x \rightarrow zxzy^{n-1}, n \geq 1\}$

There is a cyclical overlap in xz^2xz^3 , since $xz^2x \rightarrow zxzy$,
 $xz^3x \rightarrow zxzy^2$.

Example: $\mathfrak{R} = \{xy \rightarrow zx, yz \rightarrow zx, xz^n x \rightarrow zxzy^{n-1}, n \geq 1\}$

The triple $(xz^2xz^3x, xz^2x \rightarrow zxzy, xz^3x \rightarrow zxzy^2)$ is \tilde{c} -defined.

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Theorem

Let M be the finitely presented monoid $\text{Mon}\langle X \mid R \rangle$ and let \mathfrak{R} be a complete rewriting system for M . Let u and v be words in X^* . Assume that \mathfrak{R} is terminating and confluent. Then

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(i) If u and v are transposed, then $\rho(u) = \rho(v)$ (up to cyclic permutation).

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- (i) If u and v are transposed, then $\rho(u) = \rho(v)$ (up to cyclic permutation).
- (ii) If $\rho(u) = \rho(v)$ (up to cyclic permutation), then u and v are left and right conjugates.

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(i) If u and v are transposed, then $\rho(u) = \rho(v)$ (up to cyclic permutation).

(ii) If $\rho(u) = \rho(v)$ (up to cyclic permutation), then u and v are left and right conjugates.

$$u \text{ Trans } v \Rightarrow \rho(u) = \rho(v) \Rightarrow u \text{ Conj } v.$$

Another approach of cyclic rewriting

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Diekert-Duncan-Myasnikov 2012

- They develop another approach of cyclic rewriting
- They apply their technique to the conjugacy problem in f.g of graphs of groups and others

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- They recover the result of Epstein-Holt:
a f.g virtually free group has conjugacy problem solvable in linear time

So, what to do if \Downarrow is terminating but not confluent?

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So, what to do if $\varphi \rightarrow$ is terminating but not confluent?

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Assume $\varphi \rightarrow$ is terminating and there is a cyclical overlap or inclusion.

- assume $w \varphi \rightarrow z_1$ and $w \varphi \rightarrow z_2$, where z_1, z_2 are cyclically irreducible and not cyclic conjugates.

So, what to do if $\varrho \rightarrow$ is terminating but not confluent?

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Assume $\varrho \rightarrow$ is terminating and there is a cyclical overlap or inclusion.

- assume $w \varrho \rightarrow z_1$ and $w \varrho \rightarrow z_2$, where z_1, z_2 are cyclically irreducible and not cyclic conjugates.
- add a new rule $z_1 \varrho \rightarrow^+ z_2$ or $z_2 \varrho \rightarrow^+ z_1$

So, what to do if $\varrho \rightarrow$ is terminating but not confluent?

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- do the same with other cyclical overlaps and inclusions.

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- do the same with other cyclical overlaps and inclusions.
- if there is a contradiction, the algorithm fails.

So, what to do if $\varrho \rightarrow$ is terminating but not confluent?

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- add a new rule $z_1 \varrho \rightarrow^+ z_2$ or $z_2 \varrho \rightarrow^+ z_1$
- do the same with other cyclical overlaps and inclusions.
- if there is a contradiction, the algorithm fails.
- otherwise, \mathbb{R}^+ is cyclically complete and cyclically equivalent to \mathbb{R} .

\mathbb{R}^+ is cyclically equivalent to \mathbb{R} means: $u \text{Conj}_{\mathbb{R}^+} v$ iff $u \text{Conj}_{\mathbb{R}} v$

Application of the algorithm of cyclical completion

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Example of $\langle a, b \mid aba = bab \rangle$, using another set of generators-
Hermiller and Meier

- $\{a, b, \underline{ba}, \underline{ab}, \Delta = \underline{aba}\}$

The complete and finite rewriting system is

$$\mathfrak{R} = \{ab \rightarrow \underline{ab}, ba \rightarrow \underline{ba}, \underline{aba} \rightarrow \Delta, \underline{aba} \rightarrow \Delta, \underline{bab} \rightarrow \Delta, \underline{abab} \rightarrow a\Delta, \underline{bab} \rightarrow \Delta, \underline{ba\ ba} \rightarrow b\Delta, \Delta a \rightarrow b\Delta, \Delta b \rightarrow a\Delta, \Delta \underline{ab} \rightarrow \underline{ba}\Delta, \Delta \underline{ba} \rightarrow \underline{ab}\Delta\}.$$

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- $ab \rightarrow \underline{ab}$ and $ab \circ^1 ba \rightarrow \underline{ba}$. That is, $ab \looparrowright \underline{ab}$ and $ab \looparrowright \underline{ba}$, where both \underline{ab} and \underline{ba} are cyclically irreducible.

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- $ab \rightarrow \underline{ab}$ and $ab \circ^1 ba \rightarrow \underline{ba}$. That is, $ab \varphi \underline{ab}$ and $ab \varphi \underline{ba}$, where both \underline{ab} and \underline{ba} are cyclically irreducible.
- decide arbitrarily whether $\underline{ab} \varphi^+ \underline{ba}$ or $\underline{ba} \varphi^+ \underline{ab}$.

The end

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Thank you!