The Knuth-Bendix algorithm and conjugacy problems in monoids

Semigroup forum 2011

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Definition of a rewriting system

Let $M = \text{Mon}\langle X \mid l_1 = r_1, l_2 = r_2, ..., l_m = r_m \rangle$, with $l_i, r_i \in X^*$.

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Sketch of the algorithm

- First step: find all the critical pairs, order the pairs and add new rules to $\Re$ to get $\Re_1$.

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Assume $\mathfrak{R}$ is terminating. Then, $\mathfrak{R}$ is confluent if and only if all the critical pairs resolve.

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- It may succeed with a finite of infinite equivalent complete rew.syst $\Re'$ or it may fail.

Application of the Knuth-Bendix algorithm of completion

Example: $\Re = \{bab \rightarrow aba\}$

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$\Re'$ is equivalent to $\Re$ means: $u \leftrightarrow_{\Re} v$ if and only if $u \leftrightarrow_{\Re'} v$

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- $\text{Trans} = \text{Conj} = \text{LConj} = \text{RConj}$ for free monoids (Lentin-Schutzenberger).
- $\text{Trans} = \text{Conj} = \text{LConj}$ and solvable for a monoid with a special, finite and complete rewriting system (Otto).

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$\rho(u)$ cyclically irreducible form of u

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  $bcd \rightarrow bda \circlearrowleft^{2} abd \rightarrow bcd$.

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Allseq(w) is the set of all the possible sequences of cyclical reductions that begin by each word from $\{w_1, .., w_k\}$, where $w_1 = w, w_2, .., w_k$ are all the cyclic conjugates of w.

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Allseq(w) *terminates* if

there is no infinite sequence of cyclical reductions in Allseq(w).

Definition of a cyclical rew.system $\looparrowright$

A local approach: Definition of Allseq(w)

Allseq(w) is the set of all the possible sequences of cyclical reductions that begin by each word from $\{w_1, .., w_k\}$, where $w_1 = w, w_2, .., w_k$ are all the cyclic conjugates of w.

A local approach: Definition of Allseq(w)

$$
\begin{array}{cccc}
 & w & & \\
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w_1 & w_2 & \dots & w_n \\
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\end{array}
$$

Allseq(w) *converges* if

a unique cyclically irreducible form is achieved in Allseq(w) (up to cyclic permutation)

When does Allseq(w) converge?

Definition of a triple $\widetilde{c}$-defined

Let $R_1, R_2 \in \Re$ s.t. R_1 can be applied on a cyclic conjugate of w and R_2 can be applied on another one.

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Let $R_1, R_2 \in \Re$ s.t. R_1 can be applied on a cyclic conjugate of w and R_2 can be applied on another one. *The triple* (w, R_1, R_2) *is $\widetilde{c}$-defined* if there is a cyclic conjugate $\widetilde{w}$ of w such that both rules R_1 and R_2 can be applied on $\widetilde{w}$.

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When does Allseq(w) converge?

Assume Allseq(w) terminates. If all the triples occurring in Allseq(w) are $\widetilde{c}$-defined, then Allseq(w) converges.

Given $\Re$ complete and cyclically terminating, when $\Re$ is cyclically confluent?

Definition of cyclical critical pairs

- A *cyclical overlap* occurs in $xuyv$ if $xuy \rightarrow r_1$ and $yvx \rightarrow r_2$.

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- A *cyclical overlap* occurs in $xuyv$ if $xuy \to r_1$ and $yvx \to r_2$.
- A *cyclical inclusion* occurs in u if $u \to r_1$, $u' \to r_2$, and u' is a subword of a cyclic conjugate of u.

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Proposition

Let $\Re$ be a complete and cyclically terminating rewriting system. If there are no cyclical overlaps or cyclical inclusions between the rules in $\Re$, then $\Re$ is cyclically confluent.

Example of cyclical inclusion

Example: $\Re' = \{ba^n ba \rightarrow aba^2 b^{n-1}, n \geq 1\}$

- There is a cyclical inclusion in ba^2ba, since $ba^2ba \rightarrow aba^2b$, $bab \rightarrow aba$.

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Note there is no cyclical inclusion in ba^2bab

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Example: $\Re = \{xy \rightarrow zx, yz \rightarrow zx, xz^n x \rightarrow zxzy^{n-1}, n \geq 1\}$

There is a cyclical overlap in xz^2xz^3, since $xz^2x \rightarrow zxzy$, $xz^3x \rightarrow zxzy^2$.

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The triple $(xz^2xz^3x, xz^2x \to zxzy, xz^3x \to zxzy^2)$ is $\widetilde{c}$-defined.

Solution to the conjugacy problems in monoids

Theorem

Let M be the finitely presented monoid $\mathrm{Mon}\langle X \mid R\rangle$ and let $\Re$ be a complete rewriting system for M. Let u and v be words in X^*. Assume that $\looparrowright$ is terminating and confluent. Then

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$$u \,\text{Trans}\, v \Rightarrow \rho(u) = \rho(v) \Rightarrow u \,\text{Conj}\, v.$$

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Diekert-Duncan-Myasnikov 2012

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- They apply their technique to the conjugacy problem in f.g of graphs of groups and others
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  a f.g virtually free group has conjugacy problem solvable in linear time

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- do the same with other cyclical overlaps and inclusions.
- if there is a contradiction, the algorithm fails.
- otherwise, $\Re^+$ is cyclically complete and cyclically equivalent to $\Re$.

$\Re^+$ is cyclically equivalent to $\Re$ means: $u \, \mathrm{Conj}_{\Re^+} \, v$ iff $u \, \mathrm{Conj}_{\Re} \, v$

Application of the algorithm of cyclical completion

Example of $\langle a, b \mid aba = bab \rangle$, using another set of generators- Hermiller and Meier

- $\{a, b, \underline{ba}, \underline{ab}, \Delta = \underline{aba}\}$
  The complete and finite rewriting system is
  $\mathfrak{R} = \{ab \to \underline{ab}, ba \to \underline{ba}, a\underline{ba} \to \Delta, \underline{ab}a \to \Delta, b\underline{ab} \to \Delta, \underline{ab}\,\underline{ab} \to a\Delta, \underline{ba}b \to \Delta, \underline{ba}\,\underline{ba} \to b\Delta, \Delta a \to b\Delta, \Delta b \to a\Delta, \Delta\underline{ab} \to \underline{ba}\Delta, \Delta\underline{ba} \to \underline{ab}\Delta\}$.

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- $ab \to \underline{ab}$ and $ab \circlearrowleft^1 ba \to \underline{ba}$. That is, $ab \looparrowright \underline{ab}$ and $ab \looparrowright \underline{ba}$, where both $\underline{ab}$ and $\underline{ba}$ are cyclically irreducible.

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- $ab \to \underline{ab}$ and $ab \circlearrowleft^1 ba \to \underline{ba}$. That is, $ab \looparrowright \underline{ab}$ and $ab \looparrowright \underline{ba}$, where both $\underline{ab}$ and $\underline{ba}$ are cyclically irreducible.
- decide arbitrarily whether $\underline{ab} \looparrowright^+ \underline{ba}$ or $\underline{ba} \looparrowright^+ \underline{ab}$.

The end

Thank you!