Horospherical limit points of locally symmetric spaces

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Abstract. Fix a point x in the symmetric space X associated to $\mathrm{SL}(n,\mathbb{R})$. A point ξ on the visual boundary of X is a *horospherical limit point* if the $\mathrm{SL}(n,\mathbb{Z})$-orbit of x intersects every horoball based at ξ. In the special case of the upper half-plane model of X for $n = 2$, it is well known that the horospherical limit points are precisely the irrational numbers on the real line. For larger n, it was proved by T. Hattori that every horospherical limit point satisfies a certain irrationality property. We prove the converse, by applying a special case of Ratner's Theorem on unipotent flows that was established by S. G. Dani. Furthermore, $\mathrm{SL}(n,\mathbb{R})$ can be replaced with any semisimple Lie group and $\mathrm{SL}(n,\mathbb{Z})$ can be replaced with any S-arithmetic subgroup, if we replace X with the corresponding Bruhat-Tits building.

This is joint work with G. Avramidi and K. Wortman.

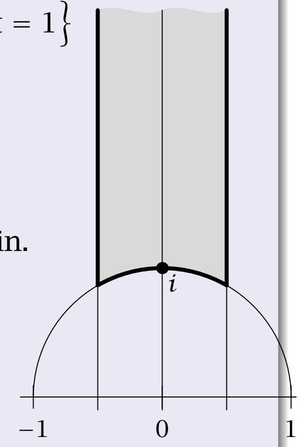
Recall

$G = \mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a,b,c,d \in \mathbb{R},\ \det = 1 \right\}$
acts on the upper half-plane $\mathfrak{H} = X$
by Möbius transformations $\frac{az+b}{cz+d}$.

$\Gamma = \mathrm{SL}(2,\mathbb{Z})$ has fundamental domain.

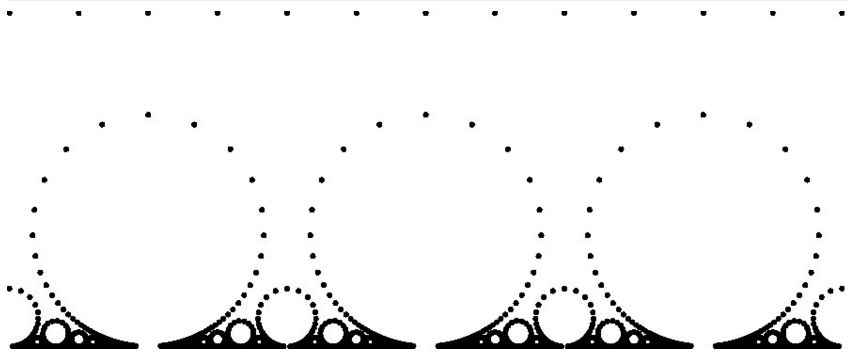
Fix $x \in X$.
Γx has 1 point in each copy of fund domain.
So it is discrete in X.

But every point on $\partial X = \mathbb{R} \cup \infty$
is an accumulation point.
However, there are big holes near ∂X.



$\xi \in \mathbb{Q} \Rightarrow \Gamma x$ is disjoint from some disk tangent to ∂X at ξ.
= **horoball** based at ξ

$\xi \in \mathbb{Q} \Rightarrow \Gamma x$ is disjoint from some horoball based at ξ.



Proof. $\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} x \} = x + \mathbb{Z}$ hits all horiz translates of fund dom.
So no other points in Γx can be in these translates.
Therefore Im Γx is **bounded**.
I.e., Γx is disjoint from horoballs based at ∞.
$\Gamma \infty = \mathbb{Q} \cup \{\infty\}$.

$\xi \in \mathbb{Q} \Rightarrow \Gamma x$ is disjoint from some horoball based at ξ.

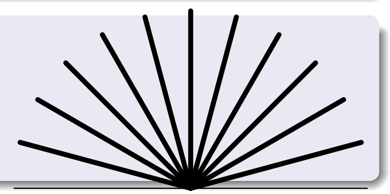
Converse is true (well known)

ξ irrational $\iff$ Γx intersects every horoball based at ξ.
ξ is ~~a horospherical limit point~~ **good**

I.e., ξ irrational, $g\xi = \infty$ $\Rightarrow$ Im $g\Gamma x$ is unbounded.

Generalize by translating to Lie groups.

$a^s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \Rightarrow a^s(z) = e^{2s} z$.
Orbit of i ($\perp \mathbb{R}$) is a geodesic.
Endpoints are ∞, 0: **bad** pts on ∂X.



Prop. *Endpoints of geodesic from diagonal matrices are bad. All bad points on* ∂X*: orbit of these under* $\mathrm{SL}(2,\mathbb{Q})$.

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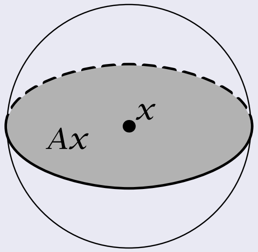
$G = SL(3,\mathbb{R})$ acts by isometries on **symmetric space** $X \sim \mathbb{R}^5$
non-positive curvature CAT(0)

$A = \{\text{diag mats}\}$, Ax is a **flat** $\cong \mathbb{R}^2$, $\exists x$.

[Hattori 2005] Known bad points on ∂X:
∂Ax (endpoints of geodesics in Ax)
and orbit of these under $SL(3,\mathbb{Q})$.



Theorem (Avramidi-Morris [2014], Rehn's Conjecture [2007])
Every bad point on ∂X is known.

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Proof. Apply **Ratner's Theorem on Unipotent Flows**.
Bad points are defined by **Geometry**,
but **Ratner's Theorem** is about actions of **Lie groups**,
so we need to **translate** into the language of Lie groups
(and interpret the result using **Algebraic Groups**).

Stronger version

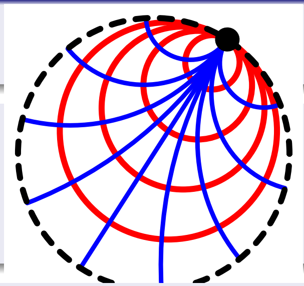
Assume $\xi \in \partial X$ is not known to be a bad point.
$SL(2,\mathbb{Z})$: Theorem says $g\xi = \infty \Rightarrow \operatorname{Im} g\Gamma x$ is unbounded.
Actually prove: $\operatorname{Im} g\Gamma x$ is **dense** in $\mathbb{R}^+$.
I.e., $g\Gamma x \approx$ every horiz line S (= horosphere based at ∞).
I.e., $x \approx \Gamma g S$ if $\xi = g\infty$.
I.e., if $g\infty$ is not known to be bad, then $\Gamma g S$ is dense in X.
gS is dense in $\Gamma\backslash X$

Translate to Lie groups

What is a horosphere?

Ans: boundary of a horoball
(or level set of Busemann function).

Fix $\xi \in \partial X$.
Geodesics to ξ foliate X.
Horospheres based at ξ foliate X.
These foliations are orthogonal.



Tangent space of horosphere $\perp$ *geodesic to* ξ.

Example ($SL(2,\mathbb{Z})$)

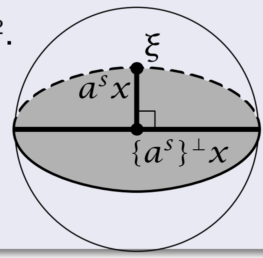
$n^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} = N$. $\quad n^t z = t + z$
N-orbit $= Nz =$ horizontal line $=$ horosphere based at ∞.
$gNz =$ horosphere based at $g\infty$.

$SL(2,\mathbb{Z})$: $gNz =$ horosphere based at $g\infty$.

General case ($SL(3,\mathbb{Z})$)

$$N = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} = A \cong \mathbb{R}^2.$$

- Ax is a flat in X (for appropriate x).
- $\xi \in \partial Ax$: $\exists \{a^s\} \subset A$, $a^s x \to \xi$.
- $\{a^s\}^\perp \subseteq A$.



Horosphere based at $g\xi$ is $gN\{a^s\}^\perp x$.

To show: If $g\infty$ is not a known bad point,
then gS is dense in $\Gamma\backslash X$. I.e., $gN\{a^s\}^\perp x$ is dense in $\Gamma\backslash X$.

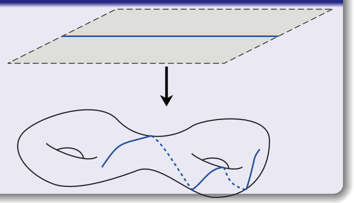
Contrapositive:
$gN\{a^s\}^\perp x$ **not** dense in $\Gamma\backslash X \Rightarrow g\infty$ is known bad point.

Ratner's Theorem on Unipotent Flows

Example

- $\{a^s\}$ = subgroup of A,
- $\{a^s\}x$ = geodesic in X,
- covering map $\pi \colon X \to \Gamma\backslash X$

Closure $\overline{\pi(\{a^s\}x)}$ can be a fractal.



To show: $\overline{\pi(gN\{a^s\}^{\perp}x)} \neq \Gamma\backslash X \;\Rightarrow\; g\xi$ is known to be bad.

Theorem (Ratner [1991])

N_1 = subgroup of N $\implies$ $\overline{\pi(gN_1x)}$ is a submfld of $\Gamma\backslash X$ (immersed)

In fact, $\exists$ closed subgroup $H \subseteq G$, such that
$\overline{\pi(gN_1x)} = \pi(gHx)$.

Also, $H \supseteq N_1$ (and $(gHg^{-1})_{\mathbb{Q}}$ dense in gHg^{-1})

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To show: $\overline{\pi(gN\{a^s\}^{\perp}x)} \neq \Gamma\backslash X \;\Rightarrow\; g\xi$ is known.

Ratner: $\overline{\pi(gNx)} = \pi(gHx)$, where $H \supseteq N$ and … [Dani 1986]

Well known: $P \supseteq AN = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ ("Borel subgroup")
$\Rightarrow P$ = "parabolic subgroup"
= stabilizer of point on ∂X or $= G$
$= AN,\ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix},\ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},\ G.$

Cor. $H = [P,P] = N,\ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},\ G.$

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Ratner: $\overline{\pi(gNx)} = \pi(gHx)$, $H = N,\ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},\ \not{G}.$

Eg., suppose $H = N$. (I.e., $\pi(gNx)$ is closed.)
Ratner: $gNg^{-1} \cap \mathrm{SL}(3,\mathbb{Q})$ is dense in gNg^{-1}
(gNg^{-1} is **defined over $\mathbb{Q}$**)
So $g \in \mathrm{SL}(3,\mathbb{Q}) \cdot \mathbf{N}_G(N) = \mathrm{SL}(3,\mathbb{Q}) \cdot AN$.

AN (Borel, so parabolic) fixes ξ, so
$g\xi \in \mathrm{SL}(3,\mathbb{Q})\,\xi \subseteq \mathrm{SL}(3,\mathbb{Q}) \cdot \partial Ax$.
Known bad point.

Motivation

Theorem (Avramidi-Morris [2014], Rehn's Conjecture [2007])

Every bad point on ∂X is known.

Definition (Bieri-Geoghegan [2016])

$\Sigma^0_\Gamma(X;\mathbb{Z})$ = {bad points on ∂X}
$= \{\, \xi \in \partial X \mid \mathcal{B}_\xi \cap \Gamma x$ is nonempty $\}$
= calculated for $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. [Avramidi-Morris]

$\Sigma^1_\Gamma(X;\mathbb{Z}) = \{\, \xi \in \Sigma^0_\Gamma(X;\mathbb{Z}) \mid \mathcal{B}_\xi \cap \Gamma x$ is $\overset{\text{coarsely}}{\text{connected}}\,\}$.
= ??? for $\Gamma = \mathrm{SL}(3,\mathbb{Z})$. *Open problem.*

Generalization

Thm. *Bad points for* $\mathrm{SL}(n,\mathbb{Z})$ *are:*
∂Ax *and* $\mathrm{SL}(n,\mathbb{Q})$*-orbit of these.*

Natural generalization [Avramidi-Morris 2014]:
G = semisimple Lie group,
Γ = lattice $G_{\mathbb{Z}}$,
A = maximal $\mathbb{Q}$-split torus,
X = symmetric space G/K.

Furthermore [Morris-Wortman 2014]:

- Γ = S-arithmetic subgroup,
  X = Bruhat-Tits building $(X \times X_{p_1} \times \cdots \times X_{p_r})$
- Including **characteristic p**.
  (Analogue of Dani's Thm proved by Mohammadi [2011].)

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