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Contours and Hyperbolic Geometry II

Problems, set 1.

Exercise 1.
Let $S = (0, 0, -1) \in S^2$ be the south pole and identify $\mathbb{C}$ this time with $\mathbb{R}^2 \times \{0\}$ via $u + iv \mapsto (u, -v, 0)$. Show that stereographic projection from the south pole

$$\Phi_S : S^2 \backslash \{S\} \to \mathbb{C} \subset \mathbb{R}^3$$

is given by

$$\Phi_S(X, Y, Z) = \left(\frac{X}{1+Z}, \frac{Y}{1+Z}, 0 \right) = \frac{X - Yi}{1+Z}.$$

Calculate Φ_S^{-1} explicitly, as well.

Exercise 2. Using your results from Exercise 1, prove that

$$\Phi_S \circ \Phi_N^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}, \quad \Phi_S \circ \Phi_N^{-1}(z) = \frac{1}{z}$$

and

$$\Phi_N \circ \Phi_S^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}, \quad \Phi_N \circ \Phi_S^{-1}(z) = \frac{1}{z}.$$

Exercise 3. Only using your imagination, describe without any calculation

(a) the preimages of straight lines through the origin plus ∞ under Φ_N^{-1},

(b) the images of circles $C \subset S^2$, obtained by intersections with horizontal planes (i.e., planes parallel to the (X, Y)-plane) under Φ_N.

Exercise 4. Let $C \subset S^2$ be the great circle through the antipodal points $(0, 1, 0)$ and $(0, -1, 0)$ given by

$$C = \{\cos(\alpha)(0, 1, 0) + \sin(\alpha)(\sqrt{1-h^2}, 0, h) \mid \alpha \in [0, 2\pi)\}$$

for some $h \in (-1,1)$. Show that $\Phi_N(C)$ is the circle in the (X,Y)-plane with centre

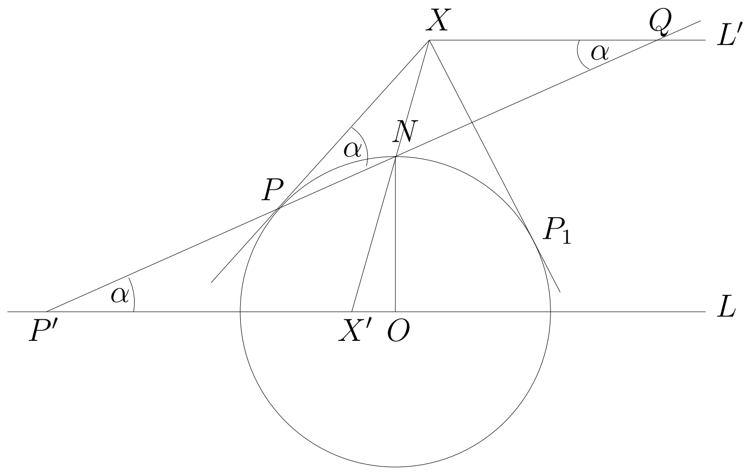
$$\left(\frac{1}{2}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right), 0, 0\right)$$

and radius

$$\frac{1}{2}\left(\sqrt{\frac{1+h}{1-h}} + \sqrt{\frac{1-h}{1+h}}\right).$$

Exercise 5. In this exercise we show that a large class of circles in S^2 are mapped to circles in $\overline{\mathbb{C}}$. The exercise needs some geometric imagination and arguments from elementary geometry. Let $C = S^2 \cap F$, where $F \subset \mathbb{R}^3$ is a plane not passing through $(0,0,0)$ and such that $(0,0,0)$ and N lie on different sides of the plane F. Let X be the tip of the cone touching the sphere S^2 in the circle C and X' the intersection of the line through N and X with the (X,Y)-plane. We show that the image of C under Φ_N is a circle in the (X,Y)-plane with centre X'.

Let $O = (0,0,0)$ be the origin. Let P be a point of the circle C and $P' = \Phi_N(P)$. The following picture is a two-dimensional section of the plane through the three points X, N, P.



In this two-dimensional section the sphere S^2 appears as a circle, the cone as the two rays XP and XP_1 and the (X,Y)-plane as the line L. Draw a line L' parallel to L through X and denote the intersection point with the line through N, P by Q. Then the angles $NP'X'$ and XQN are equal. We denote this angle by α. Prove the following statements:

(a) The angle XPN is also α. Use the fact that NPO is an isosceles triangle.

(b) The triangle PQX is isosceles and we have $|XP| = |XQ|$.

(c) Derive from (b) that

$$|X'P'| = \frac{|XP| \cdot |X'N|}{|XN|}.$$

(d) Conclude from (c) that $|X'\Phi_N(P)|$ is constant for all $P \in C$.

Exercise 6. Let $C \subset \mathbb{C}$ be the circle with centre $1 + i$ and radius 3. Derive the equation of C of the form

$$|z|^2 + \beta z + \bar{\beta}\bar{z} + \gamma = 0$$

with $\beta \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $\gamma \leq |\beta|^2$.

Exercise 7. Let $C \subset \mathbb{C}$ be the circle given by

$$|z|^2 + (-3 - i)z + (-3 + i)\bar{z} + 6 = 0.$$

Find the centre and the radius of C.