Dr. Norbert Peyerimhoff, Durham University 24/01/2006

Contours and Hyperbolic Geometry II

Problems, set 2.

Exercise 8. Let $f(z) = \frac{4z-i}{7}$ and $g(z) = \frac{2-i}{3-z}$. Write $f(z) = Az$ and $B(z) = Bz$ and check that $A, B \in \mathrm{GL}(2, \mathbb{C})$. Use this to calculate $f \circ g(z)$ and $g^{-1}(z)$.

Exercise 9. Assume that

$$f(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$ and $c \neq 0$. Find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ in terms of a, b, c, d such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Use this to show that $f(z)$ can be written as the composition of Möbius transforms of the type $f_{u,v}(z) = uz + v$ and $J(z) = \frac{1}{z}$. (Compare with the proof of Theorem 3.3.)

Exercise 10.

(a) Find a Möbius transformation $f(z) = Az$ which maps $i, \infty, 3i$ to $0, 1, \infty$, respectively.

(b) Find a Möbius transformation $g(z) = Bz$ which maps $1, 1+i, 2$ to $0, 1, \infty$, respectively.

(c) Use (a) and (b) to find a Möbius transformation $h(z)$ which maps $1, 1+i, 2$ to $i, \infty, 3i$, respectively.

Exercise 11. Prove the uniqueness statement of Corollary 3.7. You may use the following fact from the proof of Theorem 3.6: If a Möbius transformation $f(z)$ fixes $0, 1, \infty$, then $f = \mathrm{id}_{\overline{\mathbb{C}}}$.

Exercise 12.

(a) Find a Möbius transformation $f(z) = Az$ which maps $2, \infty, i$ to $0, 1, \infty$, respectively.

(b) Find a Möbius transformation $g(z) = Bz$ which maps $0, 1, \infty$ to $1, \infty, 0$, respectively.

(c) Use (a) and (b) to find a Möbius transformation $h(z)$ which maps $2, \infty, i$ to $\infty, 0, 1$, respectively.

Exercise 13. Let $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (note that this is a "circle" in $\overline{\mathbb{C}}$). Assume that

$$f(\overline{\mathbb{R}}) = \overline{\mathbb{R}}.$$

We want to determine what this means for the coefficients $a, b, c, d \in \mathbb{C}$.

(a) Assume that $c = 0$. This forces $a, d \neq 0$. By rescaling we can assume that $d = 1$. Show that in this case a, b have to be real, as well.

(b) Assume that $c \neq 0$. By rescaling, we can assume that $c = 1$. By calculating, e.g., $f(\infty)$, $f^{-1}(\infty)$ and $f(1-d)$, conclude that a, b, d have to be real, as well.

The conclusion from this exercise is that the Möbius transformations, which preserve $\overline{\mathbb{R}}$ as a set (not pointwise) are precisely those of the form

$$f(z) = \frac{az+b}{cz+d}$$

with **real** coefficients a, b, c, d and $ad - bc \neq 0$. By rescaling, we can even assume $ad - bc = -1$ or $ad - bc = 1$.