Dr. Norbert Peyerimhoff, Durham University 01/02/2006

Contours and Hyperbolic Geometry II

Problems, set 3.

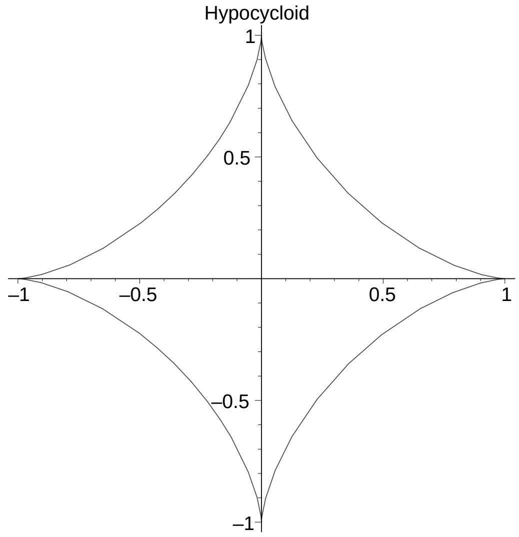
Exercise 14. Determine the type, parabolic, elliptic, or loxodromic, of each of the following Möbius transformations:

$$g_1(z) = \frac{2z+5}{3z-1}, \quad g_2(z) = 7z+6, \quad J(z) = \frac{1}{z}, \quad g_3(z) = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} z.$$

Exercise 15. Find the length of the curve $c : [0, 2\pi] \to \mathbb{C}$,

$$c(t) = \cos^3(t) + \sin^3(t)i.$$

The following picture illustrates the shape of the curve:



Exercise 16. Consider the following Möbius transformations:

$$f_1(z) = \frac{2z}{z+2} \quad \text{and } f_2(z) = \frac{3z-12}{2z-7}.$$

(a) Decide for both transformations whether they preserve $\mathbb{H}$.

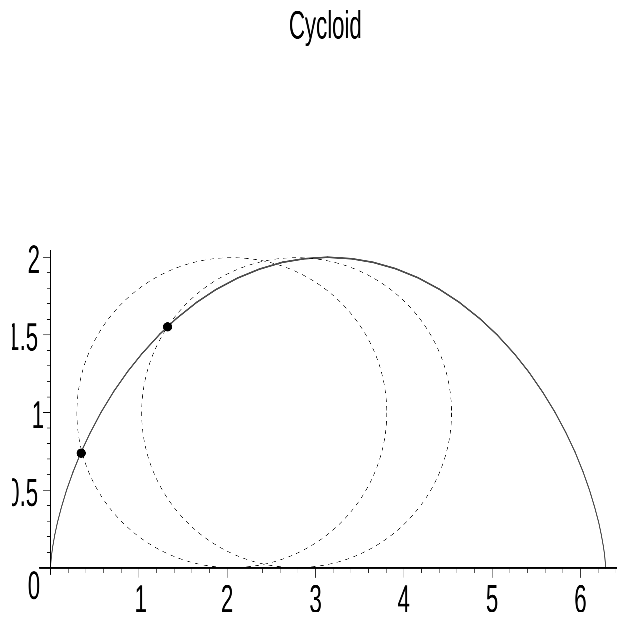
(b) Determine the fixed points of $f_1(z)$ and $f_2(z)$ in $\overline{\mathbb{C}}$ and decide for both transformations whether they are parabolic or not.

(c) For every non-parabolic transformation find its multiplier and decide whether the transformation is elliptic or loxodromic.

Exercise 17. A circular disk of radius one rolls without slipping in the complex plane along the real axis. The curve obtained by the trace of a point of the circumference of the disk is called a *cycloid* (see figure). One can check that the explicit description of this curve $c : [0, 2\pi] \to \mathbb{C}$ is given by

$$c(t) = (t - \sin(t)) + (1 - \cos(t))i.$$

Calculate the length of this curve.



Exercise 18. Let $0 < a < b$ and $\alpha \in (0, \pi)$. Draw the curve $c : [a, b] \to \mathbb{H}$, $c(t) = te^{i\alpha}$ and calculate the (hyperbolic) length of this curve.

Exercise 19.

(a) Let $f(x) = \ln(\tan(\frac{x}{2}))$. Show that

$$f'(x) = \frac{1}{\sin x}.$$

(b) Let $r > 0$, $0 < \alpha < \beta < \pi$ and $c : [\alpha, \beta] \to \mathbb{H}$, $c(t) = re^{it}$. Draw the curve c and find its (hyperbolic) length. How does this length behave as $\alpha \to 0$ or $\beta \to \pi$?