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Contours and Hyperbolic Geometry II

Problems, set 4.

All exercises and **solutions** of this course will be posted consecutively at the following web-address:

http://www.maths.dur.ac.uk/ dma0np/hypgeom0506/hypgeom.html

Moreover, J. W. Anderson's book "Hyperbolic Geometry" is an excellent background literature for this course.

Exercise 20. As explained in the course, the lengths of curves induce a distance function $d_X : X \times X \to \mathbb{R}$ via

$$d_X(x,y) := \inf_{c \in \Gamma(x,y)} L(c). \tag{1}$$

In the example $X = \mathbb{R}^2 \backslash \{0\}$, the ordinary Euclidean length of curves yields the Euclidean norm as the corresponding distance function, i.e.,

$$d_X(x,y) = \|x - y\|_0.$$

(You don't need to prove this!)

(a) For any two points $x, y \in \mathbb{R}^2 \backslash \{0\}$, does there exists a shortest curve connecting x and y? An intuitive argument is sufficient. (Note that if the answer would be YES, then the infimum in the above formula (1) could be replaced by a minimum.)

(b) Describe (without proof) all geodesics (as sets as well as by parametrized curves) in this example.

Exercise 21. This exercise anticipates the second model of the hyperbolic plane, the so-called *Poincaré Disc Model* $\mathbb{D}$. Let

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\},$$

where $|z|$ is the absolute value of the complex number z. For every $z \in \mathbb{D}$, we define the following inner produce

$$\langle v, w \rangle_z := \frac{4\langle v, w\rangle_0}{(1-|z|^2)^2}, \quad \text{for all } v, w \in \mathbb{C},$$

where $\langle v, w\rangle_0$ denotes the Euclidean inner product of the two vectors $v, w \in \mathbb{C} \cong \mathbb{R}^2$. Use this inner product (the "first fundamental form" of $\mathbb{D}$) together with its induced norm $\|v\|_z$ to calculate the length of the curve

$$c : [0, r] \to \mathbb{D}, \quad c(t) := te^{i\alpha},$$

where $\alpha \in [0, 2\pi)$ is a fixed angle and $0 < r < 1$. (You may need an easy partial fraction decomposition for the integration. If you did it correctly, the result should be

$$L(c) = \ln\left(\frac{1+r}{1-r}\right).)$$

How does the trace of c look like in the set $\mathbb{D}$?

Exercise 22. Let $C \subset \mathbb{H}$ denote the Euclidean semicircle of radius $R > 0$ centered at the real point $x \in \mathbb{R}$.

(a) Check that the following Möbius transformation

$$f(z) = \begin{pmatrix} 1 & \frac{1}{2R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -(x-R) \end{pmatrix} z \tag{2}$$

lies in $\text{Möb}^+(\mathbb{H})$.

(b) Show that f maps C to the positive imaginary axis. *Hint:* Use the representation (2) of f to calculate it as an explicit map. Parametrize C by $c : (0, \pi) \to \mathbb{H}$, $c(t) = x + Re^{it}$. You may use the identities

$$\sin(s) = \frac{e^{is} - e^{-is}}{2i}, \quad \cos(s) = \frac{e^{is} + e^{-is}}{2}.$$

Exercise 23. In this exercise, we derive the following explicit formula for the hyperbolic distance function in $\mathbb{H}$:

$$\sinh\left(\frac{1}{2} d_{\mathbb{H}}(z_1, z_2)\right) = \frac{|z_1 - z_2|}{2\sqrt{\text{Im}(z_1)\text{Im}(z_2)}}. \tag{3}$$

(Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$.)

(a) Let $z_1, z_2 \in \mathbb{H}$ with $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$. Check (3) for this case. You may use the well known hyperbolic distance in this case.

(b) Let $f(z) = \frac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. Using the identity

$$\mathrm{Im}(f(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2},$$

show that both left- and right-hand sides of (3) are invariant under f.

(c) Let $z_1, z_2 \in \mathbb{H}$ with $\mathrm{Re}(z_1) \neq \mathrm{Re}(z_2)$. Then, obviously, the (unique) geodesic through z_1 and z_2 is a Euclidean semicircle (and not a vertical line). Exercise 22 showed that there is always a map $f \in \text{Möb}^+(\mathbb{H})$ such that $f(z_1), f(z_2)$ lie on the imaginary axis. Using this map, conclude the validy of (3) for the points z_1, z_2.