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Contours and Hyperbolic Geometry II

Problems, set 5.

Next week there are **NO LECTURES** in Hyperbolic Geometry. You are encouraged to download the solutions of the first four problem collections (at http://www.maths.dur.ac.uk/ dma0np/hypgeom0506/hypgeom.html). Have a look at these solutions with respect to the homework assignments and the discussions in the problem classes. If you have difficulties in understanding particular topics, please discuss them with me (or, even better, with fellow students).

Exercise 24. Let $L_{\mathbb{D}}(\gamma)$, respectively, $L_{\mathbb{H}}(\gamma)$ denote the (hyperbolic) length of a curve γ in the model $\mathbb{D}$, respectively, $\mathbb{H}$. Let $\varphi : \mathbb{H} \to \mathbb{D}$ denote the Möbius transformation

$$\varphi(z) = \frac{z-i}{z+i}.$$

The goal of this exercise is to prove

$$L_{\mathbb{D}}(\varphi \circ \gamma) = L_{\mathbb{H}}(\gamma)$$

for every curve $\gamma : [a,b] \to \mathbb{H}$.

(a) Let $\| \cdot \|_{w,\mathbb{D}}$, respectively, $\| \cdot \|_{z,\mathbb{H}}$ denote the inner products at w, respectively, z, defined in the models $\mathbb{H}$, respectively, $\mathbb{D}$. Show that, for any $v \in \mathbb{C}$,

$$\|\varphi'(z)v\|_{\varphi(z),\mathbb{D}} = \frac{4|v|}{|z+i|^2 - |z-i|^2}.$$

(b) Let $z = x + iy$. Show that $|z+i|^2 - |z-i|^2 = 4y$ and conclude that

$$\|\varphi'(z)v\|_{\varphi(z),\mathbb{D}} = \|v\|_{z,\mathbb{H}}.$$

(c) Show the above mentioned agreement of the lengths.

Exercise 25. For $\alpha \in [0, 2\pi)$ let

$$f_\alpha(z) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} z.$$

Check that $f_\alpha \in \text{Möb}^+(\mathbb{H})$ and $f_{\alpha+\pi} = f_\alpha$. Show that $f_\alpha(i) = i$, i.e., $i \in \mathbb{H}$ is a fixed point of f_α. Conclude that

$$d_{\mathbb{H}}(i, z) = d_{\mathbb{H}}(i, f_\alpha(z)),$$

for all $\alpha \in [0, \pi)$. For $\rho > 0$, check that $d_{\mathbb{H}}(i, ie^\rho) = \rho$. This observation indicates that the curve $c : [0, \pi) \to \mathbb{H}$, $c(\alpha) := f_\alpha(ie^\rho)$ parametrizes a hyperbolic circle of hyperbolic radius ρ with hyperbolic centre $i \in \mathbb{H}$. (It only remains to show that this parametrization covers the **whole** circle; you don't need to prove this.)

Exercise 26. This exercise leads to the following nice result: *The hyperbolic circle $S_\rho(i) \subset \mathbb{H}$ with hyperbolic radius $\rho > 0$ and hyperbolic centre i coincides with the Euclidean circle with Euclidean radius* $\sinh(\rho)$ *and Euclidean centre* $\cosh(\rho)i$, *in the upper half plane model.* The main tool is the hyperbolic distance formula in Exercise 23.

(a) The Euclidean circle with Euclidean radius $\sinh(\rho)$ and Euclidean centre $\cosh(\rho)i$ can obviously be parametrized by $c : [0, 2\pi) \to \mathbb{H}$,

$$c(t) = \cos t \sinh\rho + (\cosh\rho + \sin t \sinh\rho)i.$$

Show that

$$\frac{|c(t) - i|^2}{4\,\text{Im}(c(t))} = \frac{\cosh(\rho) - 1}{2}.$$

(You may use that $\cosh^2(\rho) = 1 + \sinh^2(\rho)$.)

(b) Check that

$$\sinh^2\left(\frac{x}{2}\right) = \frac{\cosh(x) - 1}{2}.$$

(c) Use (a), (b) and the distance formula to conclude that

$$d_{\mathbb{H}}(i, c(t)) = \rho, \text{ for all } t \in [0, 2\pi).$$

Exercise 27. Provide the missing part for the proof of $\text{Möb}^+(\mathbb{D}) = \text{SU}(1,1)$, namely,

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} \text{SU}(1,1) \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \subset \text{SL}(2, \mathbb{R}).$$