Dr. Norbert Peyerimhoff, Durham University 01/03/2006

Contours and Hyperbolic Geometry II

Problems, set 6.

Exercise 28. Let $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$ be arbitrary. Show that there is a Möbius transformation $g \in \text{Möb}^+(\mathbb{D})$, such that

$$g(z_1) = 0, \quad g(z_2) = r,$$

with a real number $0 < r < 1$.

Exercise 29. For $z_1, z_2 \in \mathbb{D}$, let

$$\Phi(z_1, z_2) = \frac{|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}.$$

Show that we have, for every $g \in \text{Möb}^+(\mathbb{D})$,

$$\Phi(g(z_1), g(z_2)) = \Phi(z_1, z_2).$$

Using Exercise 28 and

$$r = \tanh(\frac{1}{2} d_{\mathbb{D}}(0, r))$$

from Proposition 9.3, show that

$$\sinh^2(\frac{1}{2} d_{\mathbb{D}}(z_1, z_2)) = \frac{|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2)^2)}.$$

Exercise 30. For $x_0 \in \mathbb{R}$ let $l_{x_0} = x_0 + i(0, \infty) \subset \mathbb{H}$ be the corresponding (vertical) geodesic. The distance between the point $i \in \mathbb{H}$ and l_{x_0} is then defined as

$$d_{\mathbb{H}}(i, l_{x_0}) = \inf_{z \in l_{x_0}} d_{\mathbb{H}}(i, z).$$

In this exercise we will prove that

$$d_{\mathbb{H}}(i, l_{x_0}) = \ln(|x_0| + \sqrt{1 + x_0^2}). \tag{1}$$

Obviously, in the case $x_0 = 0$ we have $d_{\mathbb{H}}(i, l_{x_0}) = 0$ and there is nothing to prove. By symmetry, we can henceforth assume that $x_0 > 0$. In the following steps of the proof you can use the formula

$$\sinh^2(\frac{1}{2}d_{\mathbb{H}}(z_1, z_2)) = \frac{|z_1 - z_2|^2}{4\mathrm{Im}(z_1)\mathrm{Im}(z_2)}.$$

(a) Draw a picture of the upper half plane with the points i and the line l_{x_0}. Use your intuition and guess a geometric construction which might exhibit the point on l_{x_0} of minimal distance to i.

(b) Let $z(t) := x_0 + ti$. Show that

$$\inf_{z \in l_{x_0}} d_{\mathbb{H}}(i, z) = \inf_{t \in (0,\infty)} d_{\mathbb{H}}(i, z(t)) = d_{\mathbb{H}}(i, x_0 + \sqrt{1 + x_0^2}\, i).$$

Does this fit with your proposal for (a)?

(c) Let $d := d_{\mathbb{H}}(i, x_0 + \sqrt{1 + x_0^2}\, i)$. Show that

$$\sinh^2(\frac{1}{2}d) = \frac{1}{2}\sqrt{1 + x_0^2} - \frac{1}{2}.$$

Conclude that

$$e^{2d} - 2\sqrt{1 + x_0^2}e^d + 1 = 0,$$

and, finally, prove (1).

(d) What is the distance between i and the lines l_{x_0} defined by $x_0 = \pm 1$?

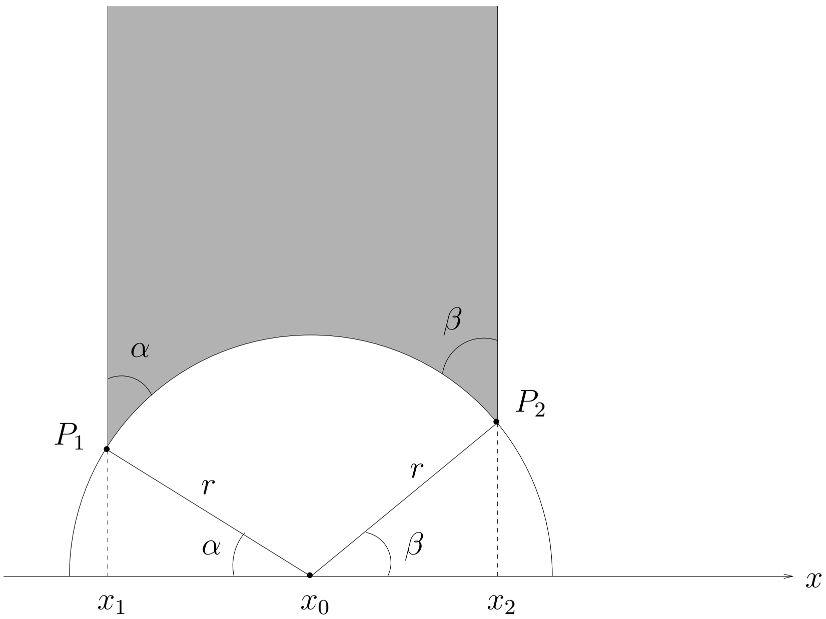
Exercise 31. Let $g : \mathbb{H} \to \mathbb{H}$, $g(z) = az + b$, with $a > 0, b \in \mathbb{R}$. Show that

$$\mathrm{area}_{\mathbb{H}}(g(A)) = \mathrm{area}_{\mathbb{H}}(A).$$

Exercise 32. In this exercise we derive the formula

$$\mathrm{area}_{\mathbb{H}}(\Delta P_1 P_2 P_3) = \pi - \alpha - \beta$$

for a hyperbolic triangle $\Delta P_1 P_2 P_3$ with an ideal vertex P_3 and interior angles α, β. Look at the following picture and denote the "ideal" point at infinity by P_3.



Show that

$$\mathrm{area}_{\mathbb{H}}(\Delta P_1 P_2 P_3) = \arcsin(\frac{x_0 - x_1}{r}) + \arcsin(\frac{x_2 - x_0}{r}).$$

(You may use $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.) Conclude from this with $\arcsin(x) + \arccos(x) = \pi/2$ that

$$\mathrm{area}_{\mathbb{H}}(\Delta P_1 P_2 P_3) = \pi - \alpha - \beta.$$

Exercise 33. We know that a hyperbolic triangle with interior angles $\alpha, \beta, \gamma \in [0, \pi)$ has hyperbolic area $\pi - \alpha - \beta - \gamma$. Use this fact in order to prove that a hyperbolic n-gon Π with interior angles $\alpha_1, \ldots, \alpha_n$ has hyperbolic area

$$\mathrm{area}_{\mathbb{H}}(\Pi) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$