Dr. Norbert Peyerimhoff, Durham University 08/03/2006

Contours and Hyperbolic Geometry II

Problems, set 7.

Exercise 34. Calculate the hyperbolic area $A(\rho)$ of a ball in $\mathbb{D}$ of hyperbolic radius $\rho > 0$. You can assume, w.l.o.g., that the centre of the ball is at the origin of $\mathbb{D}$. If $L(\rho)$ denoted the hyperbolic length of a circle of hyperbolic radius $\rho > 0$, is it still true that $L(\rho) = A'(\rho)$, as in the Euclidean case? What can be said about $\lim_{\rho\to\infty} \frac{L(\rho)}{A(\rho)}$ in the Euclidean and the hyperbolic case?

Exercise 35. Let $\Delta \subset \mathbb{H}$ be an isosceles triangle with hyperbolic side lengths $\ln 2, \ln 2$ and $\ln 3$. Calculate the interior angles of Δ.

Exercise 36. In this exercise we discuss some distance properties which might be counterintuitive from the Euclidean perspective.

(a) Let $z_1(t) = it$ and $z_2(t) = 1 + it$, $t \in \mathbb{R}$. Use the distance formula and determine $\lim_{t\to\infty} d_{\mathbb{H}}(z_1(t), z_2(t))$.

(b) Let $w_1(t) = i - t$ and $w_2(t) = i + t$, $t \in \mathbb{R}$. Use the distance formula and determine $\lim_{t\to\infty} d_{\mathbb{H}}(w_1(t), w_2(t))$.

(c) Let $z(t) = i + t$, $t \in \mathbb{R}$ and $w(t) = -\frac{1}{z(t)}$. Show that $w(t)$ lies on the Euclidean circle of radius $\frac{1}{2}$ about $\frac{i}{2}$. Show that

$$\lim_{t\to\pm\infty} w(t) = 0,$$

but

$$\lim_{t\to\infty} d_{\mathbb{H}}(w(-t), w(t)) = \infty.$$

Exercise 37. Let Π be the regular pentagon (polygon with 5 sides) with interior angles at each vertex equal to $\pi/2$. Let l be the hyperbolic length of its sides and d be the hyperbolic distance between the centre of Π and any of its vertices. Calculate $\cosh l$ and $\cosh d$. You may use that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$.

Exercise 38. As explained in the lecture, the hyperboloid model $\mathbb{W}$ of the hyperbolic plane is given by

$$\mathbb{W} = \{P = (X, Y, Z) \mid b(P, P) = -1, Z > 0\},$$

with

$$b(P_1, P_2) = X_1 X_2 + Y_1 Y_2 - Z_1 Z_2,$$

where $P_j = (X_j, Y_j, Z_j)$. $\mathbb{W}$ inherits the geometry of $\mathbb{D}$ by stereographic projection from $P_0 = (0, 0, -1)$. Here $\mathbb{D}$ is identified with $\{(X, Y, 0) \mid X^2 + Y^2 < 1\}$ via $\mathbb{D} \ni z \mapsto (X, Y, 0) = (\mathrm{Re}(z), \mathrm{Im}(z), 0)$. Calculate the stereographic projection

$$\Phi : \mathbb{W} \to \mathbb{D}$$

and its inverse Φ^{-1} explicitly.