Dr. Norbert Peyerimhoff, Durham University 16/01/2006

Contours and Hyperbolic Geometry II

Solutions, set 1.

Exercise 1. The line L through $S = (0, 0, -1)$ and $(X, Y, Z) \in S^2\backslash\{S\}$ is given by

$$L := \{(-tX, -tY, -1 + t(-1 - Z)) \mid t \in \mathbb{R}\}.$$

To find the intersection point of L with the (X, Y)-plane, we have to equate the last coordinate to zero:

$$-1 + t(-1 - Z) = 0,$$

which implies that $t = \frac{-1}{1+Z}$. So

$$\Phi_S(X, Y, Z) = \left(\frac{X}{1 + Z}, \frac{Y}{1 + Z}, 0\right) = \frac{X - Yi}{1 + Z}.$$

For the calculation of Φ_S^{-1} we assume that

$$\Phi_S^{-1}(u + iv) = (X, Y, Z)$$

and want to solve for X, Y, Z the relations

$$\begin{aligned} u &= \frac{X}{1 + Z}, \\ v &= \frac{-Y}{1 + Z}, \\ X^2 + Y^2 + Z^2 &= 1. \end{aligned}$$

This implies

$$u^2 + v^2 = \frac{X^2 + Y^2}{(1 + Z)^2} = \frac{1 - Z^2}{(1 + Z)^2} = \frac{1 - Z}{1 + Z}.$$

Solving for Z yields

$$Z = -\frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

and

$$1 + Z = \frac{2}{u^2 + v^2 + 1}.$$

So we end up with

$$\Phi_S^{-1}(u + iv) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{-2v}{u^2 + v^2 + 1}, -\frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Exercise 2. We have

$$\begin{aligned} \Phi_S \circ \Phi_N^{-1}(u + iv) &= \Phi_S \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right) \\ &= \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}, 0 \right) = \frac{u - iv}{u^2 + v^2} = \frac{1}{u + iv}. \end{aligned}$$

The second to last equation follows from the identification $\mathbb{C} \cong \mathbb{R}^2 \times 0$, $u + iv \mapsto (u, -v, 0)$. Similarly, we obtain

$$\begin{aligned} \Phi_N \circ \Phi_S^{-1}(u + iv) &= \Phi_N \left(\frac{2u}{u^2 + v^2 + 1}, \frac{-2v}{u^2 + v^2 + 1}, -\frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right) \\ &= \left(\frac{u}{u^2 + v^2}, \frac{-v}{u^2 + v^2}, 0 \right) = \frac{u - iv}{u^2 + v^2} = \frac{1}{u + iv}. \end{aligned}$$

Here, the second to last equation follows from the identification $\mathbb{C} \cong \mathbb{R}^2 \times 0$, $u + iv \mapsto (u, v, 0)$.

Exercise 3.

(a) The preimages of straight lines through the origin plus ∞ are great circles in S^2 which contain both north and south pole.

(b) The images of horizontal circles in S^2 are circles in the (X, Y)-plane centered at the origin.

Exercise 4.

We have

$$\begin{aligned} &\Phi_N(\cos(\alpha)(0, 1, 0) + \sin(\alpha)(\sqrt{1 - h^2}, 0, h)) \\ &\qquad = \left(\frac{\sin(\alpha)\sqrt{1 - h^2}}{1 - \sin(\alpha)h}, \frac{\cos(\alpha)}{1 - \sin(\alpha)h}, 0 \right) = (X, Y, Z). \end{aligned}$$

Let

$$(X_0, Y_0, Z_0) = \left(\frac{1}{2}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right), 0, 0\right).$$

Then we obtain

$$\begin{aligned}
&\|(X,Y,Z) - (X_0,Y_0,Z_0)\|^2 \\
&= \left(\frac{\sin(\alpha)\sqrt{1-h^2}}{1-\sin(\alpha)h} - \frac{1}{2}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right)\right)^2 + \left(\frac{\cos\alpha}{1-\sin(\alpha)h}\right)^2 \\
&= \frac{\sin^2(\alpha)(1-h^2)+\cos^2(\alpha)}{(1-\sin(\alpha)h)^2} - \frac{\sin(\alpha)\sqrt{1-h^2}}{1-\sin(\alpha)h}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right) \\
&\qquad + \frac{1}{4}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right)^2 \\
&= \frac{1-h^2\sin^2(\alpha) - 2h\sin(\alpha)(1-h\sin(\alpha))}{(1-h\sin(\alpha))^2} + \frac{1}{4}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right)^2 \\
&= 1 + \frac{1}{4}\left(\sqrt{\frac{1+h}{1-h}} - \sqrt{\frac{1-h}{1+h}}\right)^2 = \frac{1}{4}\left(\sqrt{\frac{1+h}{1-h}} + \sqrt{\frac{1-h}{1+h}}\right)^2
\end{aligned}$$

Exercise 5.

(a) Since the triangle $P'ON$ is rectangular, the angle $P'NO$ is equal to $\pi/2-\alpha$. The triangle NPO is isosceles since two sides are the radii of S^2, i.e., equals one. So the angles ONP and OPN are equal. So the angle OPN is equal to $\pi/2-\alpha$. Since OP and PX are orthogonal (radius and tangent of the circle), we conclude that the angle NPX is equal to $\pi/2-(\pi/2-alpha) = \alpha$.

(b) Since the triangle PQX has two equal angles, it is isosceles and we have $|XP| = |XQ|$.

(c) Since the lines L and L' are parallel, the triangles NXQ and $NX'P'$ are similar. This implies that

$$\frac{|X'P'|}{|X'N|} = \frac{|XQ|}{|XN|}.$$

Using $|XP| = |XQ|$, we end up with

$$|X'P'| = \frac{|XP| \cdot |X'N|}{|XN|}.$$

(d) Note that the distances $|XP|$ (distance tip of the cone to a point of the circle C), $|XN|$ (distance tip of the cone to the north pole) and $|X'N|$ (distance north pole to the fixed point X') do not depend on the choice of P on S^2. Therefore, (d) shows that $|X'P'| = |X\Phi_N(P)|$ is constant, so $\Phi_N(C)$ is a circle with centre X'.

Exercise 6. The equation of C is given by

$$(x-1)^2 + (y-1)^2 = 9.$$

This yields

$$\begin{aligned} x^2 + y^2 - 2x - 2y - 7 &= 0, \\ |z|^2 - (z + \bar{z}) + i(z - \bar{z}) - 7 &= 0, \\ |z|^2 + (-1+i)z + (-1-i)\bar{z} - 7 &= 0. \end{aligned}$$

Exercise 7. With $z = x + yi$ we derive

$$\begin{aligned} |z|^2 + (-3-i)z + (-3+i)\bar{z} + 6 &= 0, \\ x^2 + y^2 + (-3-i)(x+yi) + (-3+i)(x-yi) + 6 &= 0, \\ x^2 - 6x + y^2 + 2y + 6 &= 0, \\ (x-3)^2 - 9 + (y+1)^2 - 1 + 6 &= 0, \\ (x-3)^2 + (y+1)^2 &= 4. \end{aligned}$$

So we conclude that the centre of C is $3 - i$ and the radius is 2.