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Contours and Hyperbolic Geometry II

Solutions, set 2.

Exercise 8. We have

$$f(z) = \begin{pmatrix} 4 & -i \\ 0 & 7 \end{pmatrix} z, \quad g(z) = \begin{pmatrix} 0 & 2-i \\ -1 & 3 \end{pmatrix} z,$$

and $\det A = 21$, $\det B = 2 - i$, so $A, B \in \mathrm{GL}(2, \mathbb{C})$. This implies that

$$f \circ g(z) = \begin{pmatrix} 4 & -i \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 0 & 2-i \\ -1 & 3 \end{pmatrix} z = \begin{pmatrix} i & 8-7i \\ -7 & 21 \end{pmatrix} z = \frac{iz + (8-7i)}{-7z + 21}$$

and

$$g^{-1}(z) = \frac{1}{2-i} \begin{pmatrix} 3 & i-2 \\ 1 & 0 \end{pmatrix} z = \begin{pmatrix} 3 & i-2 \\ 1 & 0 \end{pmatrix} z = \frac{3z + i - 2}{z}.$$

Exercise 9. We have

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta\gamma & \alpha + \beta\delta \\ \gamma & \delta \end{pmatrix}.$$

So we obtain the equations

$$\beta\gamma = a, \quad \alpha + \beta\delta = b, \quad \gamma = c, \quad \delta = d,$$

which yields $\gamma = c \neq 0$, $\delta = d$, $\beta = \frac{a}{\gamma} = \frac{a}{c}$, and $\alpha = b - \beta\delta = b - \frac{ad}{c} = \frac{bc-ad}{c} \neq 0$. So we have the equation

$$\begin{pmatrix} \frac{bc-ad}{c} & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which translates to

$$f_{\frac{bc-ad}{c}, \frac{a}{c}} \circ J \circ f_{c,d}(z) = \frac{az+b}{cz+d}.$$

Exercise 10.

(a) $f_0(z) = \frac{1}{z-3i}$ satisfies $f_0(3i) = \infty$, but $f_0(i) = \frac{i}{2}$. So

$$f_1(z) = f_0(z) - f_0(i) = \frac{1}{z-3i} - \frac{i}{2} = -\frac{iz+1}{2z-6i}$$

satisfies $f_1(3i) = \infty$, $f_1(i) = 0$. But $f_1(\infty) = -\frac{i}{2}$, so the desired Möbius transformation is

$$f(z) = -\frac{2}{i} f_1(z) = \frac{z-i}{z-3i} = \begin{pmatrix} 1 & -i \\ 1 & -3i \end{pmatrix} z = Az.$$

(b) $g_0(z) = \frac{1}{z-2}$ satisfies $g_0(2) = \infty$, but $g_0(1) = -1$. So

$$g_1(z) = g_0(z) - g_0(1) = \frac{1}{z-2} + 1 = \frac{z-1}{z-2}$$

satisfies $g_1(2) = \infty$, $g_1(1) = 0$. But $g_1(1+i) = \frac{1}{1+i}$, so the desired Möbius transformation is

$$g(z) = (1+i)g_1(z) = \frac{(1+i)z - (1+i)}{z-2} = \begin{pmatrix} 1+i & -(1+i) \\ 1 & -2 \end{pmatrix} z = Bz.$$

(c) The desired Möbius transformation is $f^{-1} \circ g(z)$. So we have to calculate

$$A^{-1}B = \frac{i}{2} \begin{pmatrix} -3i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1+i & -(1+i) \\ 1 & -2 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} -2i+3 & i-3 \\ -i & i-1 \end{pmatrix},$$

and we end up with

$$h(z) = \frac{(-2i+3)z + i - 3}{-iz + i - 1}.$$

Exercise 11. Assume that $h_1(z_j) = h_2(z_j)$ for $j = 1, 2, 3$. Then we have $(h_2)^{-1} \circ h_1(z_j) = z_j$ for $j = 1, 2, 3$. By Theorem 3.6, there is a Möbius transformation $g(z)$ which maps $0, 1, \infty$ to z_1, z_2, z_3, respectively. Then we have

$$g^{-1} \circ (h_2)^{-1} \circ h_1 \circ g(z) = z \quad \text{for } z = 0, 1, \infty.$$

This implies that

$$g^{-1} \circ (h_2)^{-1} \circ h_1 \circ g = \mathrm{id}_{\overline{\mathbb{C}}},$$

and we immediately obtain $h_1 = h_2$.

Exercise 12.

(a) $f_0(z) = \frac{1}{z-i}$ satisfies $f_0(i) = \infty$, but $f_0(2) = \frac{1}{2-i}$. So

$$f_1(z) = f_0(z) - f_0(2) = \frac{1}{z-i} - \frac{1}{2-i} = \frac{-z+2}{(2-i)z - i(2-i)}$$

satisfies $f_1(i) = \infty$, $f_1(2) = 0$. But $f_1(\infty) = -\frac{1}{2-i}$, so the desired Möbius transformation is

$$f(z) = (i-2)f_1(z) = \frac{z-2}{z-i} = \begin{pmatrix} 1 & -2 \\ 1 & -i \end{pmatrix} z = Az.$$

(b) $g_0(z) = \frac{1}{z-1}$ satisfies $g_0(1) = \infty$ and $g_0(\infty) = 0$. But $g_0(0) = -1$, so the desired Möbius transformation is

$$g(z) = \frac{1}{-z+1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} z = Bz.$$

(c) The desired Möbius transformation is $g^2 \circ f(z)$. So we have to calculate

$$B^2 A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & -2 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 0 & 2-i \\ -1 & 2 \end{pmatrix},$$

and we end up with

$$h(z) = \frac{2-i}{2-z}.$$

Exercise 13.

(a) Assume that $c = 0$ and choose $d = 1$, by rescaling. Then we have $f(z) = az+b$ and $f : \mathbb{R} \to \mathbb{R}$. This implies $f(0) = b \in \mathbb{R}$ and $f(1) - b = a \in \mathbb{R}$.

(b) Assume that $c = 1$. Then

$$f(z) = \frac{az+b}{z+d} = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} z$$

and

$$f^{-1}(z) = \frac{1}{ad-b} \begin{pmatrix} d & -b \\ -1 & a \end{pmatrix} z = \frac{dz-b}{-z+a}.$$

This implies that $f(\infty) = a \in \mathbb{R}$ and $f^{-1}(\infty) = -d \in \mathbb{R}$. Finally, we have

$$f(1-d) = a - ad + b \in \mathbb{R},$$

which implies that $b \in \mathbb{R}$.