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Contours and Hyperbolic Geometry II

Solutions, set 3.

Exercise 14. We have $g_1(\infty) = \frac{2}{3}$, so ∞ is not a fixed point of g_1. The equation $g_1(z) = z$ leads to

$$3z^2 - 3z - 5 = 0,$$

i.e., the fixed points are

$$z_0 = \frac{3+\sqrt{69}}{6}, \quad w = \frac{3-\sqrt{69}}{6}.$$

A Möbius transformation $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying $h(z_0) = 0$, $h(w) = \infty$ is, e.g.,

$$h(z) = \frac{z - z_0}{z - w},$$

and the multiplier is

$$\alpha = h \circ g_1 \circ h^{-1}(1) = h \circ g_1(\infty) = h(\frac{2}{3}) = \frac{\frac{2}{3} - \frac{3+\sqrt{69}}{6}}{\frac{2}{3} - \frac{3-\sqrt{69}}{6}} = \frac{1-\sqrt{69}}{1+\sqrt{69}}.$$

Of course, another order of the fixed points would lead to the multiplier

$$\frac{1}{\alpha} = \frac{1+\sqrt{69}}{1-\sqrt{69}}$$

Since $|\alpha| \neq 1$, g_1 is **loxodromic**.

Since $g_2(\infty) = \infty$, one of the fixed points is $z_0 = \infty$. The equation $g_2(w) = w$ lead to $w = -1$, so we can choose

$$h(z) = \frac{1}{z+1}.$$

The multiplier is

$$\alpha = h \circ g_2 \circ h^{-1}(1) = h \circ g_2(0) = h(6) = \frac{1}{7}$$

or, $\frac{1}{\alpha} = 7$. Since $|\alpha| \neq 1$, g_2 is **loxodromic**.

Since $J(\infty) = 0$, ∞ is not a fixed point of J. $J(z) = z$ leads to $z^2 = 1$, i.e., to the fixed points $z_0 = 1$ and $w = -1$. So we may choose

$$h(z) = \frac{z-1}{z+1}$$

and conclude that

$$\alpha = h \circ g_2 \circ h^{-1}(1) = h \circ J(\infty) = h(0) = -1.$$

Since $|\alpha| = 1$, J is **elliptic**.

We have $g_3(z) = \frac{z+\sqrt{3}}{-\sqrt{3}z+1}$. Since $g_3(\infty) = -\frac{1}{\sqrt{3}}$, ∞ is not a fixed point of g_3. The equation $g_3(z) = z$ leads to $z^2 + 1 = 0$, i.e., the fixed points are $z_0 = i$, $w = -i$. Choosing

$$h(z) = \frac{z-i}{z+i},$$

we obtain

$$\alpha = h \circ g_3 \circ h^{-1}(1) = h \circ g_3(\infty) = h(-\frac{1}{\sqrt{3}}) = \frac{-1-\sqrt{3}i}{-1+\sqrt{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Of course, the multiplier could also be

$$\frac{1}{\alpha} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Since $|\alpha| = 1$, g_3 is **elliptic**.

Exercise 15. $c(t) = \cos^3(t) + \sin^3(t)i$ implies that

$$\dot{c}(t) = 3 \sin t \cos t(-\cos(t) + \sin(t)i).$$

So we obtain

$$\|\dot{c}(t)\|_0 = |\dot{c}(t)| = 3|\sin t \cos t| = \frac{3}{2}|\sin(2t)|,$$

and the length is given by

$$
\begin{aligned}
L(c) &= \int_0^{2\pi} \|\dot{c}(t)\|_0 dt \\
&= \frac{3}{2}\left(\int_0^{\pi/2} \sin(2t)dt - \int_{\pi/2}^{\pi} \sin(2t)dt + \int_{\pi}^{\frac{3\pi}{2}} \sin(2t)dt - \int_{\frac{3\pi}{2}}^{2\pi} \sin(2t)dt\right) \\
&= \frac{3}{2}\cdot 4 \cdot \int_0^{\pi/2} \sin(2t)dt = 6.
\end{aligned}
$$

Exercise 16. (a) The matrix representations of f_1 and f_2 are

$$
f_1(z) = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} =: Az, \quad f_2(z) = \begin{pmatrix} 3 & -12 \\ 2 & -7 \end{pmatrix} =: Bz.
$$

Since $\det(A) = 4 > 0$ and $\det(B) = 3 > 0$ both transformations preserve $\mathbb{H}$.

(b) Since $f_1(\infty) = 2$, ∞ is not a fixed point of f_1. The equation $f_1(z) = z$ leads to $z^2 = 0$, so f_1 has only one fixed point $z = 0$. So f_1 is **parabolic**.

$f_2(\infty) = \frac{3}{2}$ shows that ∞ is not a fixed point of f_2. The equation $f_2(z) = z$ leads to

$$
0 = z^2 - 5z + 6 = (z-2)(z-3),
$$

so f_2 has two fixed points and is, therefore, not parabolic.

(c) We only have to consider f_2: Let h be a Möbius transformation which maps $z_0 = 2$ to 0 and $w = 3$ to ∞, e.g.,

$$
h(z) = \frac{z-2}{z-3}.
$$

Then we obtain the multiplier α by

$$
\alpha = h \circ f_2 \circ h^{-1}(1) = h \circ f_2(\infty) = h(\frac{3}{2}) = \frac{\frac{3}{2}-2}{\frac{3}{2}-3} = \frac{3-4}{3-6} = \frac{1}{3}.
$$

Of course, if one would have chosen another order of the fixed points, the multiplier would have been $\frac{1}{\alpha} = 3$. Since $|\alpha| \neq 1$, f_2 is **loxodromic**.

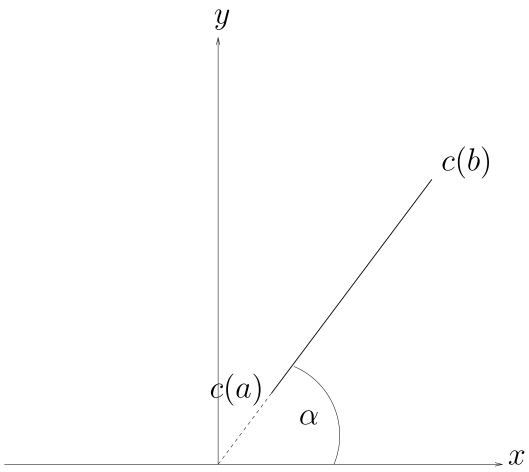
Exercise 17. Since $\dot{c}(t) = (1 - \cos(t)) + \sin(t)i$, we obtain

$$
\|\dot{c}(t)\|^2 = 2 - 2\cos t = 2 - 2\cos(\frac{t}{2} + \frac{t}{2}) = 4\sin^2(\frac{t}{2}).
$$

So we have $\|\dot{c}(t)\| = 2\sin(\frac{t}{2})$ for $0 \leq t \leq \pi$ and, consequently,

$$\int_0^{2\pi} \|\dot{c}(t)\|\ dt = 4\int_0^{\pi} \sin s\ ds = 8.$$

Exercise 18. The curve looks as follows:



We have $\dot{c}(t) = e^{i\alpha}$ and so $\|\dot{c}(t)\|_{c(t)} = \frac{1}{t\sin(\alpha)}$. This implies

$$L(c) = \int_a^b \|\dot{c}(t)\|_{c(t)}\, dt = \frac{1}{\sin\alpha}\int_a^b \frac{dt}{t} = \frac{1}{\sin(\alpha)}\ \ln\frac{b}{a}.$$

Exercise 19.

(a) We have

$$f'(x) = \frac{1+\tan^2(\frac{x}{2})}{2\tan(\frac{x}{2})} = \frac{\cos(\frac{x}{2})}{2\sin(\frac{x}{2})} + \frac{\sin(\frac{x}{2})}{2\cos(\frac{x}{2})} = \frac{1}{2\sin(\frac{x}{2})\cos(\frac{x}{2})} = \frac{2}{\sin(x)}.$$

(b) $\dot{c}(t) = ire^{it}$ implies

$$\|\dot{c}(t)\|_{c(t)} = \frac{r}{r\sin t}.$$

So the hyperbolic length of c is

$$L(c) = \int_\alpha^\beta \|\dot{c}(t)\|_{c(t)}\, dt = \int_\alpha^\beta \frac{dt}{\sin t} = \ln(\tan(\frac{\beta}{2})) - \ln(\tan(\frac{\alpha}{2})).$$

$\alpha \to 0$ implies $\tan(\frac{\alpha}{2}) \to 0$ and $\ln(\tan(\frac{\alpha}{2})) \to -\infty$, so $L(c) \to \infty$ in this case. $\beta \to \pi$ implies $\tan(\frac{\beta}{2}) \to \infty$ and $\ln(\tan(\frac{\beta}{2})) \to \infty$, so $L(c) \to \infty$ also in this case. The curve looks as follows: