Dr. Norbert Peyerimhoff, Durham University 08/02/2006

# Contours and Hyperbolic Geometry II

**Solutions, set 4.**

**Exercise 20.** (a) For any two points $x, y \in \mathbb{R}^2$, the shortest connecting curve is obviously the Euclidean straight line segment connecting $x$ and $y$. The length of this line segment is, obviously, equals $\|y - x\|_0$. If the point 0 lies on this line segment, then this is not a valid curve in $\mathbb{R}^2 \backslash \{0\}$ and, near 0 the curve has to modified to avoid the origin. It is intuitively clear that such a modified curve has a lenght $> \|y - x\|_0$, but can be constructed to be arbitrarily close to $\|y - x\|_0$. So we conclude

$$d_{\mathbb{R}^2 \backslash \{0\}}(y, x) = \|y - x\|_0,$$

but the answer to the question is **NO**! A shortest connecting curve exists if and only if the Euclidean straigh line segment doesn't contain the origin.

(b) The geodesics as sets are all straight Euclidean lines not passing through the origin. The parametrized geodesics are precisely those curves given by $c(t) = p + tv$ with $p, v \in \mathbb{R}^2$ and $\|v\|_0 = 1$, which satisfy $0 \notin c(\mathbb{R})$.

**Exercise 21.** The induced norm in $\mathbb{D}$ has the form

$$\|v\|_z = \frac{2|v|}{1 - |z|^2}.$$

Let $c(t) = te^{i\alpha}$. Then $\dot{c}(t) = e^{i\alpha}$ and we conclude that

$$\|\dot{c}(t)\|_{c(t)} = \frac{2}{1 - t^2} = \frac{1}{1 - t} + \frac{1}{1 + t}.$$

So we end up with

$$L(c) = \int_0^r \frac{2\,dt}{1 - t^2} = \int_0^r \frac{1}{1 - t} + \frac{1}{1 + t}\,dt = -\ln(1 - t)\Big|_0^r + \ln(1 + t)\Big|_0^r$$
$$= \ln(1 + r) + \ln(1 - r) = \ln\left(\frac{1 + r}{1 - r}\right).$$

The trace of $c$ is a Euclidean ray of Euclidean length $r$ starting at the origin with angle $\alpha$ counterclockwise against the positive real axis.

**Exercise 22.** (a) We have

$$\begin{pmatrix} 1 & \frac{1}{2R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -(x-R) \end{pmatrix} = \begin{pmatrix} \frac{1}{2R} & -1 - \frac{x-R}{2R} \\ 1 & -(x-R) \end{pmatrix}.$$

Obviously, all entries of the matrix on the right hand side are real. Since

$$\det \begin{pmatrix} \frac{1}{2R} & -1 - \frac{x-R}{2R} \\ 1 & -(x-R) \end{pmatrix} = -\frac{x-R}{2R} - (-1 - \frac{x-R}{2R}) = 1,$$

we have

$$\begin{pmatrix} \frac{1}{2R} & -1 - \frac{x-R}{2R} \\ 1 & -(x-R) \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

This implies $f \in \text{Möb}^+(\mathbb{H})$.

(b) The matrix representation (2) of $f$ yields

$$f(z) = -\frac{1}{z-(x-R)} + \frac{1}{2R}.$$

Using $c(t) = x + Re^{it}$ we calculate

$$\begin{aligned} f(c(t)) &= -\frac{1}{x + Re^{it} - (x-R)} + \frac{1}{2R} = -\frac{1}{R(e^{it} + 1} + \frac{1}{2R} = \frac{e^{it}-1}{2R(1+e^{it})} \\ &= \frac{1}{2R}\frac{e^{it/2} - e^{-it/2}}{e^{it/2} + e^{-it/2}} = \frac{i}{2R}\frac{(e^{it/2} - e^{-it/2})/(2i)}{(e^{it/2} + e^{-it/2})/2} = \frac{i}{2R}\frac{\sin(t/2)}{\cos(t/2)} = \frac{i}{2R}\tan(\frac{t}{2}). \end{aligned}$$

As $t$ runs from 0 to $\pi$, the expression $\frac{i}{2R}\tan(\frac{t}{2})$ runs from the origin along the positive imaginary axis to infinity.

**Exercise 23.** (a) If $z_1 = z_2$ there is nothing to check. Let $z_1 = x + ai$ and $z_2 = x + bi$. In this case we have

$$d_{\mathbb{H}}(z_1, z_2) = |\ln b/a|,$$

which implies that

$$\sinh(\frac{1}{2}d_{\mathbb{H}}(z_1, z_2)) = \frac{1}{2}\Big|\exp(\frac{\ln(b/a)}{2}) - \exp(-\frac{\ln(b/a)}{2})\Big| = \frac{1}{2}\Big|\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\Big|.$$

On the other hand we obtain

$$\frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} = \frac{|a-b|}{2\sqrt{ab}} = \frac{1}{2}\left|\frac{a}{\sqrt{ab}} - \frac{b}{\sqrt{ab}}\right|,$$

which shows the validity of the formula in this particular case.

(b) Since $f$ is a hyperbolic isometry, we have $d_{\mathbb{H}}(f(z_1), f(z_2)) = d_{\mathbb{H}}(z_1, z_2)$. This immediately implies invariance of the left hand side under $f$. As for the right-hand side note first that

$$\begin{aligned}|f(z_1) - f(z_2)| = \frac{|(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)|}{|cz_1+d|\,|cz_2+d|} \\ = \frac{|z_1-z_2|}{|cz_1+d|\,|cz_2+d|}.\end{aligned}$$

Using the identity

$$\mathrm{Im}(f(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2},$$

we obtain

$$\begin{aligned}\frac{|f(z_1) - f(z_2)|}{2\sqrt{\mathrm{Im}(f(z_1))\mathrm{Im}(f(z_2))}} = \frac{|z_1-z_2|}{|cz_1+d|\,|cz_2+d|} \cdot \frac{|cz_1+d|\,|cz_2+d|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ = \frac{|z_1-z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}.\end{aligned}$$

(c) We choose an appropriate $f \in \mathrm{M\ddot{o}b}^+(\mathbb{H})$ such that $w_1 := f(z_1), w_2 := f(z_2) \in i(0,\infty)$. This is possible by Exercise 22. Then we obtain with (b):

$$\begin{aligned}\sinh(\frac{1}{2}d_{\mathbb{H}}(z_1,z_2)) = \sinh(\frac{1}{2}d_{\mathbb{H}}(w_1,w_2)) = \frac{|f(z_1)-f(z_2)|}{2\sqrt{\mathrm{Im}(f(z_1))\mathrm{Im}(f(z_2))}} \\ = \frac{|z_1-z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}.\end{aligned}$$