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Contours and Hyperbolic Geometry II

Solutions, set 5.

Exercise 24.
(a) We have

$$\varphi'(z) = \frac{(z+i)-(z-i)}{(z+i)^2} = \frac{2i}{(z+i)^2}.$$

So, for any $v \in \mathbb{C}$:

$$|\varphi'(z)v| = \frac{2|v|}{|z+i|^2}.$$

Using the definition of $\|\cdot\|_{z,\mathbb{D}}$ we conclude that

$$\|\varphi'(z)v\|_{\varphi(z),\mathbb{D}} = \frac{|\varphi'(z)v|}{1-|\varphi(z)|^2} = \frac{4|v|}{|z+i|^2}\frac{1}{1-|\frac{z-i}{z+i}|^2} = \frac{4|v|}{|z+i|^2-|z-i|^2}.$$

(b) We have

$$|z+i|^2 - |z-i|^2 = x^2 + (y+1)^2 - (x^2 + (y-1)^2) = 4y,$$

and so

$$\|\varphi'(z)v\|_{\varphi(z),\mathbb{D}} = \frac{4|v|}{4y} = \frac{|v|}{\mathrm{Im}(z)} = \|v\|_{z,\mathbb{H}}.$$

(c) We have, using the chain rule,

$$L_{\mathbb{D}}(\varphi\circ\gamma) = \int_a^b \|\varphi'(\gamma(t))\cdot\dot\gamma(t)\|_{\varphi(\gamma(t)),\mathbb{D}}\,dt = \int_a^b \|\dot\gamma(t)\|_{\gamma(t),\mathbb{H}}\,dt = L_{\mathbb{H}}(\gamma).$$

Exercise 25. The entries of $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ are obviously real and the determinant is $\cos^2(\alpha)+\sin^2(\alpha)=1$. So $f_\alpha \in \text{Möb}^+(\mathbb{H})$. Moreover,

$$f_{\alpha+\pi}(z) = \frac{\cos(\alpha+\pi)z - \sin(\alpha+\pi)}{\sin(\alpha+\pi)z+\cos(\alpha+\pi)} = \frac{-\cos(\alpha)z+\sin(\alpha)}{-\sin(\alpha)z-\cos(\alpha)}$$
$$= \frac{\cos(\alpha)z-\sin(\alpha)}{\sin(\alpha)z+\cos(\alpha)} = f_\alpha(z).$$

The following calculation shows that i is a fixed point of f_α:

$$\begin{aligned} f_\alpha(i) &= \frac{\cos(\alpha)i - \sin(\alpha)}{\sin(\alpha)i + \cos(\alpha)} \\ &\qquad = \frac{(i\cos\alpha - \sin\alpha)(\cos\alpha - i\sin\alpha)}{\sin^2\alpha + \cos^2\alpha} = (\cos^2\alpha + \sin^2\alpha)i = i. \end{aligned}$$

Since $f_\alpha \in \text{Möb}^+(\mathbb{H})$ is a hyperbolic isometry, we conclude that

$$d_{\mathbb{H}}(i,z) = d_{\mathbb{H}}(f_\alpha(i), f_\alpha(z)) = d_{\mathbb{H}}(i, f_\alpha(z)).$$

Using $d_{\mathbb{H}}(ai, bi) = |\ln \frac{b}{a}|$, we conclude that

$$d_{\mathbb{H}}(i, e^\rho i) = \ln \frac{e^\rho}{1} = \rho > 0.$$

Exercise 26.

(a) We have

$$\begin{aligned} \frac{|c(t)-i|^2}{4\,\mathrm{Im}(c(t))} &= \frac{(\cosh\rho + \sin t\sinh\rho - 1)^2 + \cos^2 t\sinh^2\rho}{4(\cosh\rho + \sin t\sinh\rho)} \\ &= \frac{\cosh^2\rho + \sinh^2\rho + 1 + 2\sin t\cosh\rho\sinh\rho - 2\cosh\rho - 2\sin t\sinh\rho}{4(\cosh\rho + \sin t\sinh\rho)} \\ &= \frac{\cosh^2\rho + \cosh\rho\sin t\sinh\rho - \cosh\rho - \sin t\sinh\rho}{2(\cosh\rho + \sin t\sinh\rho)} = \frac{\cosh\rho - 1}{2}. \end{aligned}$$

(b) We have

$$\sinh^2(\frac{x}{2}) = \frac{(e^{x/2} - e^{-x/2})^2}{4} = \frac{e^x + e^{-x} - 2}{4} = \frac{\cosh(x) - 1}{2}.$$

(c) Using the distance formula we obtain

$$\sinh^2(\frac{1}{2}d_{\mathbb{H}}(i, c(t))) = \frac{|c(t)-i|^2}{4\,\mathrm{Im}(c(t))\,\mathrm{Im}(i)} = \frac{\cosh(\rho) - 1}{2} = \sinh^2(\frac{\rho}{2}).$$

Since $f(x) = \sinh^2(x)$ satisfies $f(0) = 0$ and $f'(t) = 2\sinh(t)\cosh(t) > 0$ on $(0,\infty)$, $f : [0,\infty) \to [0,\infty)$ is a injective map. So we end up with

$$d_{\mathbb{H}}(i, c(t)) = \rho.$$

Exercise 27. Let $A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$. Then

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha + \beta & -i\alpha + i\beta \\ \bar{\beta} + \bar{\alpha} & -i\bar{\beta} + i\bar{\alpha} \end{pmatrix}$$
$$= -\frac{i}{2} \begin{pmatrix} i(\alpha + \bar{\alpha} + \beta + \bar{\beta}) & \alpha - \beta + \bar{\beta} - \bar{\alpha} \\ -\alpha - \beta + \bar{\beta} + \bar{\alpha} & i\alpha - i\beta - i\bar{\beta} + i\bar{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\alpha+\beta+\overline{\alpha+\beta}}{2} & \frac{\beta-\alpha-\overline{\beta-\alpha}}{2} i \\ \frac{\alpha+\beta-\overline{\alpha+\beta}}{2} i & \frac{\alpha-\beta+\overline{\alpha-\beta}}{2} \end{pmatrix}.$$

Obviously, all entries of the matrix are real. So we only have to check the determinant:

$$\det \begin{pmatrix} \frac{\alpha+\beta+\overline{\alpha+\beta}}{2} & \frac{\beta-\alpha-\overline{\beta-\alpha}}{2} i \\ \frac{\alpha+\beta-\overline{\alpha+\beta}}{2} i & \frac{\alpha-\beta+\overline{\alpha-\beta}}{2} \end{pmatrix}$$
$$= \frac{(\alpha + \bar{\alpha} + \beta + \bar{\beta})(\alpha + \bar{\alpha} - (\beta + \bar{\beta})}{4} + \frac{(\beta - \bar{\beta} + \alpha - \bar{\alpha})(\beta - \bar{\beta} - (\alpha - \bar{\alpha})}{4}$$
$$= \frac{(\alpha + \bar{\alpha})^2 - (\beta + \bar{\beta})^2 + (\beta - \bar{\beta})^2 - (\alpha - \bar{\alpha})^2}{4} = \frac{4\alpha\bar{\alpha} - 4\beta\bar{\beta}}{4} = |\alpha|^2 - |\beta|^2 = 1.$$

So we conclude that

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} A \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$