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Contours and Hyperbolic Geometry II

Solutions, set 6.

Exercise 28. A general Möbius transformation $g \in \text{Möb}^+(\mathbb{D})$ is of the type

$$g(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a|^2 - |b|^2 = 1.$$

In order to achieve $g(z_1) = 0$ we have to choose $b = -az_1$. The condition $|a|^2 - |b|^2 = 1$ yields $|a|^2 = \frac{1}{1-|z_1|^2}$, so the choice

$$a = \frac{1}{\sqrt{1-|z_1|^2}} e^{i\alpha}, \quad b = -az_1$$

satisfies $|a|^2 - |b|^2 = 1$ and $g(z_1) = 0$. It remains to prove that we can choose the angle $\alpha \in [0, \pi)$ of a in such a way that $g(z_2)$ lies on the positive real axis. We have

$$g(z_2) = \frac{az_2 - az_1}{\overline{az_1}z_2 - \bar{a}} = \frac{a}{\bar{a}} \frac{z_2 - z_1}{\bar{z}_1 z_2 - 1} = e^{2i\alpha} \frac{z_2 - z_1}{\bar{z}_1 z_2 - 1}.$$

Note that $\frac{z_2-z_1}{\bar{z}_1 z_2 - 1}$ is a non-zero complex number. Let $re^{i\gamma}$, $r > 0$, $\gamma \in (0, 2\pi]$ be the polar representation of this number. Choosing $\alpha = \pi - \frac{\gamma}{2}$, we conclude that $g(z_2) \in (0, \infty)$, and since g maps $\mathbb{D}$ into itself, we even obtain $g(z_2) = r \in (0, 1)$.

Exercise 29. Let

$$g(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a|^2 - |b|^2 = 1.$$

We first calculate $g(z_1) - g(z_2)$:

$$\begin{aligned} g(z_1) - g(z_2) &= \frac{az_1+b}{\bar{b}z_1+\bar{a}} - \frac{az_2+b}{\bar{b}z_2+\bar{a}} = \frac{(az_1+b)(\bar{b}z_2+\bar{a}) - (az_2+b)(\bar{b}z_1+\bar{a})}{(\bar{b}z_1+\bar{a})(\bar{b}z_2+\bar{a})} \\ &= \frac{|a|^2 z_1 + |b|^2 z_2 - |a|^2 z_2 - |b|^2 z_1}{(\bar{b}z_1+\bar{a})(\bar{b}z_2+\bar{a})} = \frac{z_1 - z_2}{(\bar{b}z_1+\bar{a})(\bar{b}z_2+\bar{a})}. \end{aligned}$$

Next, let us look at $1 - |g(z)|^2$:

$$1 - |g(z)|^2 = 1 - \frac{|az+b|^2}{|\bar{b}z + \bar{a}|^2} = \frac{|\bar{b}z+\bar{a}|^2 - |az+b|^2}{|\bar{b}z+\bar{a}|^2}$$
$$= \frac{(\bar{b}z+\bar{a})(b\bar{z}+a) - (az+b)(\bar{a}\bar{z}+\bar{b})}{|\bar{b}z+\bar{a}|^2} = \frac{(|a|^2-|b|^2)(1-|z|^2)}{|\bar{b}z+\bar{a}|^2} = \frac{1-|z|^2}{|\bar{b}z+\bar{a}|^2}.$$

Putting things together we conclude that

$$\begin{aligned}\Phi(g(z_1), g(z_2)) &= \frac{|g(z_1)-g(z_2)|^2}{(1-|g(z_1)|^2)(1-|g(z_2)|^2)} \\ &= \frac{|z_1-z_2|^2}{|\bar{b}z_1+\bar{a}|^2\,|\bar{b}z_2+\bar{a}|^2}\frac{|\bar{b}z_1+\bar{a}|^2}{1-|z_1|^2}\frac{|\bar{b}z_2+\bar{a}|^2}{1-|z_2|^2} \\ &= \frac{|z_1-z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} = \Phi(z_1,z_2).\end{aligned}$$

Similarly as for the transformations in $\mathrm{Möb}^+(\mathbb{H})$, it can be shown that the transformations in $\mathrm{Möb}^+(\mathbb{D})$ preserve hyperbolic lengths of curves and hyperbolic distances in the Poincaré unit disc. We like to prove

$$\sinh^2(\frac{1}{2}d_{\mathbb{D}}(z_1,z_2)) = \Phi(z_1,z_2).$$

This equation is obvious in the case $z_1 = z_2$ (then both expressions are zero). Assume that $z_1 \neq z_2$. By Exercise 28, there is a $g \in \mathrm{Möb}^+(\mathbb{D})$ with $g(z_1) = 0$ and $g(z_2) = r$, $0 < r < 1$. Using $r = \tanh(\frac{1}{2}d_{\mathbb{D}}(0,r))$, we conclude that

$$\begin{aligned}\Phi(0,r) &= \frac{r^2}{1-r^2} = \frac{\tanh^2(\frac{1}{2}d_{\mathbb{D}}(0,r))}{1-\tanh^2(\frac{1}{2}d_{\mathbb{D}}(0,r))} \\ &= \frac{\sinh^2(\frac{1}{2}d_{\mathbb{D}}(0,r))}{\cosh^2(\frac{1}{2}d_{\mathbb{D}}(0,r)) - \sinh^2(\frac{1}{2}d_{\mathbb{D}}(0,r))} = \sinh^2(\frac{1}{2}d_{\mathbb{D}}(0,r)).\end{aligned}$$

This finally leads to

$$\begin{aligned}\Phi(z_1,z_2) &= \Phi(g(z_1),g(z_2)) = \Phi(0,r) = \sinh^2(\frac{1}{2}d_{\mathbb{D}}(0,r)) \\ &= \sinh^2(\frac{1}{2}d_{\mathbb{D}}(g(z_1),g(z_2))) = \sinh^2(\frac{1}{2}d_{\mathbb{D}}(z_1,z_2)),\end{aligned}$$

where we also used the above mentioned fact that g is a hyperbolic isometry.

Exercise 30.

(a) If $z \in l_{x_0}$ is the point of minimal distance between l_{x_0} and i then one might guess that the geodesic arc connecting z and i meets l_{x_0} perpendicularly. If this is true the point z could be constructed by drawing a circle with centre x_0 through i: z would then be the intersection point of this circle with l_{x_0}. This immediately would lead to $z = x_0 + i\sqrt{1+x_0^2}$.

(b) The distance formula yields for $z(t) = x_0 + ti$:

$$\sinh^2(\frac{1}{2}d_{\mathbb{H}}(i, z(t))) = \frac{|x_0 + i(t-1)|^2}{4t} = \frac{x_0^2 + (t-1)^2}{4t} = \frac{1}{4}t - \frac{1}{2} + \frac{x_0^2+1}{4t}. \quad (1)$$

Since $x \mapsto \sinh^2(\frac{x}{2})$ is monotone increasing in $[0, \infty)$, we only have to find the minimum of the function

$$f(t) = \frac{1}{4}t - \frac{1}{2} + \frac{x_0^2+1}{4t}.$$

Differentiating yields

$$f'(t) = \frac{1}{4} - \frac{x_0^2+1}{4t^2} = \frac{t^2 - (1+x_0^2)}{4t^2}.$$

$f'(t) = 0$ leads to $t = \sqrt{1+x_0^2} > 0$ and $f''(t) = \frac{x_0^2+1}{2t^3} > 0$ for $t > 0$ shows that this is a local minimum. So we have

$$\inf_{z \in l_{x_0}} d_{\mathbb{H}}(i, z) = d_{\mathbb{H}}(i, x_0 + i\sqrt{1+x_0^2}).$$

(c) Using (1) we obtain

$$\sinh^2(\frac{1}{2}d) = \frac{x_0^2 + (\sqrt{1+x_0^2} - 1)^2}{4\sqrt{1+x_0^2}} = \frac{1 + x_0^2 - \sqrt{1+x_0^2}}{2\sqrt{1+x_0^2}} = \frac{1}{2}\sqrt{1+x_0^2} - \frac{1}{2}.$$

This implies

$$\begin{aligned} \frac{(e^{d/2} - e^{-d/2})^2}{4} &= \frac{1}{2}\sqrt{1+x_0^2} - \frac{1}{2}, \\ e^d - 2 + e^{-d} &= 2\sqrt{1+x_0^2} - 2, \\ e^{2d} - 2\sqrt{1+x_0^2}e^d + 1 &= 0. \end{aligned}$$

This is a quadratic equation in e^d. Since $d \geq 0$, we have $e^d \geq 1$, and therefore, we obtain

$$e^d = \frac{2\sqrt{1+x_0^2} + \sqrt{4(1+x_0^2)-4}}{2} = |x_0| + \sqrt{1+x_0^2}.$$

This leads finally to

$$d = \ln(|x_0| + \sqrt{1+x_0^2}).$$

(d) Choosing $x_0 = \pm 1$ we obtain the distance $\ln(1+\sqrt{2})$.

Exercise 31. We have $g(x,y) = (ax+b, ay)$. The Jacobian of g is

$$Dg(x,y) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Let $f(x,y) = \frac{1}{y^2}$. The transformation rule yields

$$\begin{aligned} \mathrm{area}_{\mathbb{H}}(g(A)) &= \int_{g(A)} f(x,y)\,dxdy = \int_A f(g(x,y))\,|\det Dg(x,y)|\,dxdy \\ &= \int_A \frac{1}{a^2y^2}a^2\,dxdy = \int_A \frac{1}{y^2}\,dxdy = \mathrm{area}_{\mathbb{H}}(A). \end{aligned}$$

Exercise 32. The formula for the hyperbolic area yields

$$\begin{aligned} \mathrm{area}_{\mathbb{H}}(\Delta P_1P_2P_3) &= \int_{x_1}^{x_2}\int_{\sqrt{r^2-(x-x_0)^2}}^{\infty} \frac{dy}{y^2}dx = \int_{x_1}^{x_2} \frac{1}{y}\Big|_{\infty}^{\sqrt{r^2-(x-x_0)^2}} dx \\ &= \int_{x_1}^{x_2} \frac{1}{\sqrt{r^2-(x-x_0)^2}}dx = \frac{1}{r}\int_{x_1}^{x_2} \frac{1}{\sqrt{1-\left(\frac{x-x_0}{r}\right)^2}}\,dx \\ &= \frac{1}{r}\int_{x_1-x_0}^{x_2-x_0} \frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}\,dx = \int_{\frac{x_1-x_0}{r}}^{\frac{x_2-x_0}{r}} \frac{dy}{\sqrt{1-y^2}} \end{aligned}$$

$$= \arcsin\left(\frac{x_2-x_0}{r}\right) - \arcsin\left(\frac{x_1-x_0}{2}\right) = \arcsin\left(\frac{x_2-x_0}{r}\right) + \arcsin\left(\frac{x_0-x_1}{2}\right).$$

The picture implies that $\cos\alpha = \frac{x_0-x_1}{r}$. Since $\alpha \in (0,\pi/2)$, we have

$$\alpha = \arccos\left(\frac{x_0-x_1}{r}\right).$$

Similarly, we conclude that

$$\beta = \arccos\left(\frac{x_2 - x_0}{r}\right).$$

Using $\arcsin(x) + \arccos(x) = \pi/2$ we finally obtain the desired formula:

$$\begin{aligned}\mathrm{area}_{\mathbb{H}}(\Delta P_1 P_2 P_3) &= \arcsin\left(\frac{x_2 - x_0}{r}\right) + \arcsin\left(\frac{x_0 - x_1}{2}\right) \\ &= \pi - \arccos\left(\frac{x_2 - x_0}{r}\right) - \arccos\left(\frac{x_0 - x_1}{2}\right) = \pi - \alpha - \beta.\end{aligned}$$

Exercise 33. Choosing a point P of the n-gon Π (as in the picture below), we can decompose the n-gon into $n-2$ triangles. The interior angles of these triangles add up to the interior angles of the n-gon. The hyperbolic area of the n-gon is the sum of the hyperbolic areas of the $n-2$ triangles, so we conclude that

$$\begin{aligned}&\mathrm{area}_{\mathbb{H}}(\Pi) = \text{sum of areas of the } n-2 \text{ triangles} \\ &= (n-2)\pi - \text{sum of interior angles of the triangles} = (n-2)\pi - \sum \alpha_i.\end{aligned}$$