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Contours and Hyperbolic Geometry II

Solutions, set 7.

Exercise 34. A ball $B_\rho \subset \mathbb{D}$ of hyperbolic radius $\rho > 0$ with centre 0 is given by

$$B_\rho = \{z \in \mathbb{D} \mid |z| \le \tanh \frac{\rho}{2}\}.$$

We conclude that

$$A(\rho) = \mathrm{area}_{\mathbb{D}}(B_\rho) = \int_0^{2\pi} \int_0^{\tanh \frac{\rho}{2}} \frac{4r}{(1-r^2)^2} dr d\alpha = 2\pi \frac{4}{1-r^2}\Big|_{r=0}^{r=\tanh\frac{\rho}{2}}$$

$$= 4\pi \frac{\tanh^2 \frac{\rho}{2}}{1-\tanh^2\frac{\rho}{2}} = 4\pi \sinh^2 \frac{\rho}{2}.$$

We know from Corollary 9.4 that $L(\rho) = 2\pi \sinh \rho$. Differentiation of $A(\rho)$ yields, on the other hand,

$$A'(\rho) = 4\pi \sinh \frac{\rho}{2} \cosh \frac{\rho}{2} = 2\pi \sinh \rho = L(\rho).$$

A similar relation holds in the Euclidean case: $A(r) = \pi r^2$ and $L(r) = A'(r) = 2\pi r$. Moreover, we have

$$\lim_{\rho\to\infty} \frac{L(\rho)}{A(\rho)} = \lim_{\rho\to\infty} \frac{e^\rho - e^{-\rho}}{(e^{\rho/2} - e^{-\rho/2})^2} = \lim_{\rho\to\infty} \frac{e^\rho - e^{-\rho}}{e^\rho - 2 + e^{-\rho}} = 1,$$

whereas, in the Euclidean case

$$\lim_{r\to\infty} \frac{L(r)}{A(r)} = \lim_{r\to\infty} \frac{2\pi r}{\pi r^2} = \lim_{r\to\infty} \frac{2}{r} = 0.$$

Exercise 35. We denote the sides of Δ by a, b, c and the corresponding angles at the opposite sides by α, β, γ. Let $a = b = \ln 2$ and $c = \ln 3$. Then $\alpha = \beta$. Using hyperbolic law of cosines I, we conclude that

$$\frac{5}{4} = \cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha = \frac{5}{4} \cdot \frac{5}{3} - \frac{3}{4} \cdot \frac{4}{3} \cos \alpha,$$

so $\cos\alpha = \frac{5}{6}$, $\alpha = \beta = \arccos\frac{5}{6}$. Similarly,

$$\frac{5}{3} = \cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma = \frac{5}{4}\cdot\frac{5}{4} - \frac{3}{4}\cdot\frac{3}{4}\cos\gamma,$$

so $\cos\gamma = -\frac{5}{27}$, $\gamma = \arccos -\frac{5}{27}$.

Exercise 36. (a) We have

$$\lim_{t\to\infty} d_{\mathbb{H}}(z_1(t), z_2(t)) = \lim_{t\to\infty} \frac{|z_1(t) - z_2(t)|}{2\sqrt{\mathrm{Im}(z_1(t))\mathrm{Im}(z_2(t))}} = \lim_{t\to\infty}\frac{1}{2t} = 0.$$

(b) We have

$$\lim_{t\to\infty} d_{\mathbb{H}}(w_1(t), w_2(t)) = \lim_{t\to\infty} \frac{|w_1(t) - w_2(t)|}{2\sqrt{\mathrm{Im}(w_1(t))\mathrm{Im}(w_2(t))}} = \lim_{t\to\infty}\frac{2t}{2} = \infty.$$

(c) We have $w(t) = -\frac{1}{z(t)} = \frac{-t+i}{1+t^2}$ and

$$|w(t) - \frac{i}{2}|^2 = |-\frac{t}{1+t^2} + \frac{1-t^2}{2(1+t^2)}i|^2 = \frac{4t^2 + (1-t^2)^2}{4(1+t^2)^2} = \frac{(1+t^2)^2}{4(1+t^2)^2} = \frac{1}{4}.$$

This shows that $w(t)$ lies on the Euclidean circle of radius $\frac{1}{2}$ about $\frac{i}{2}$. We also have

$$\lim_{t\to\pm\infty} w(t) = \lim_{t\to\pm\infty}\frac{-t+i}{1+t^2} = 0.$$

Since $J(z) = -\frac{1}{z}$ is an isometry of $\mathbb{H}$, we conclude with (b):

$$\lim_{t\to\infty} d_{\mathbb{H}}(w(-t), w(t)) = \lim_{t\to\infty} d_{\mathbb{H}}(J(i-t), J(i+t)) = \lim_{t\to\infty} d_{\mathbb{H}}(i-t, i+t) = \infty.$$

Exercise 37. By connecting the centre of the pentagon Π with the five vertices, we see that the pentagon is made up of five congruent triangles. Each triangle is isosceles with base l and two equal base angles $\frac{\pi}{4}$ and two equal sides of length d meeting in an angle of $\frac{2\pi}{5}$. The hyperbolic law of cosines II yields

$$\cos\frac{2\pi}{5} = -\cos\frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\sin\frac{\pi}{4}\cosh l.$$

Solving for $\cosh l$, we obtain

$$\cosh l = \frac{\cos\frac{2\pi}{5} + \cos^2\frac{\pi}{4}}{\sin^2\frac{\pi}{4}} = \frac{\frac{\sqrt{5}-1}{4} + \left(\frac{1}{\sqrt{2}}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{5}+1}{2}.$$

Similarly, we obtain the following formula for d, by the hyperbolic law of cosines II,

$$\cos\frac{\pi}{4} = -\cos\frac{\pi}{4}\cos\frac{2\pi}{5} + \sin\frac{\pi}{4}\sin\frac{2\pi}{5}\cosh d.$$

Note that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ implies

$$\sin^2\frac{2\pi}{5} = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{5+\sqrt{5}}{8}.$$

This implies that

$$\cosh d = \frac{\cos\frac{\pi}{4} + \cos\frac{\pi}{4}\cos 2\pi 5}{\sin\frac{\pi}{4}\sin\frac{2\pi}{5}} = \frac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}} = \frac{3+\sqrt{5}}{\sqrt{2}\sqrt{5+\sqrt{5}}}.$$

Exercise 38. The straight Euclidean line L connecting P_0 with a point $P = (X, Y, Z) \in \mathbb{W}$ is given by

$$L = \{(0,0,-1) + t((X,Y,Z) - (0,0,-1)) \mid t \in \mathbb{R}\}.$$

In order to calculate the intersection of L with the (X,Y)-plane, we have to equate the last component of the points of L with zero:

$$-1 + t(Z+1) = 0,$$

which yields $t = \frac{1}{1+Z}$. So we obtain

$$\Phi(P) = (0,0,-1) + \frac{1}{1+Z}((X,Y,Z) - (0,0,-1)) = (\frac{X}{1+Z}, \frac{Y}{1+Z}, 0),$$

or, using complex numbers,

$$\Phi(X,Y,Z) = \frac{X}{1+Z} + \frac{Y}{1+Z}i.$$

To calculate $\Phi^{-1}(x+yi)=(X,Y,Z)$, we use the identities

$$x=\frac{X}{1+Z} \quad y=\frac{Y}{1+Z}$$

and conclude from $X^2+Y^2=Z^2-1$ that

$$x^2+y^2=\frac{X^2+Y^2}{(1+Z)^2}=\frac{Z^2-1}{(1+Z)^2}=\frac{Z-1}{Z+1}.$$

This implies

$$Z=\frac{1+x^2+y^2}{1-x^2-y^2}$$

and

$$1+Z=\frac{2}{1-x^2-y^2},$$

and we end up with

$$\begin{aligned}\Phi^{-1}(x+yi)=(X,Y,Z)&=((1+Z)x,(1+Z)y,Z)\\&=(\frac{2x}{1-x^2-y^2},\frac{2y}{1-x^2-y^2},\frac{1+x^2+y^2}{1-x^2-y^2}).\end{aligned}$$