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Measure Theory and Dynamics

Problems, set 1.

Exercise 1.

a) The maps $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by

$$\phi_t(v) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $\{\phi_t : \mathbb{R}^3 \to \mathbb{R}^3 \mid t \in \mathbb{R}\}$ is a continuous dynamical system.

b) For every real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

one defines e^A (similarly as in the scalar case) by the series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

It can be proved that this series always converges. Find a matrix A such that

$$e^{tA} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

c) Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Calculate all entries of the matrix e^{tA}. Check that the maps

$$\phi_t(v) = e^{tA} v$$

on $\mathbb{R}^2$ define a continuous dynamical system. Describe the orbits $\phi_{\mathbb{R}}(v)$ geometrically.

Additional remarks to Exercise 1: The well-known identity

$$e^{AB} = e^A e^B$$

holds no longer in this generality for matrices. However, this identity is still true if the two matrices commute, i.e.,

$$AB = BA.$$

Many continuous dynamical systems are described by a differential equation. The orbits $f(t) = e^{tA}v$ are solutions of the linear differential equation with constant coefficients

$$\dot{f}(t) = Af(t).$$

Exercise 2.

a) Check that the Lebesgue measure λ_n on $\mathbb{R}^n$, defined in the lectures, actually satisfies σ-subadditivity.

b) Let X be an arbitrary space, $x \in X$ and δ_x be the corresponding Dirac measure. Determine the σ-algebra of δ_x-measurable subsets of X.

c) Let $I \subset \mathbb{R}$ be a compact interval and λ_1 the Lebesgue measure on $\mathbb{R}$. Convince yourself that I is a λ_1-measurable set.

Exercise 3.

a) Show that countable unions of sets of measure zero have, again, measure zero.

b) Let μ be a probability measure on X. Show that countable intersections of sets of measure one in X have, again, measure one.

c) Let $X \subset \mathbb{R}$ denote the set of all real numbers x in the interval $[0, 1)$ which have a decimal expansion

$$x = 0.a_1a_2a_3\ldots$$

such that none of the digits a_j is equal to 0. Determine the Lebesgue measure $\lambda_1(X)$.