Norbert Peyerimhoff, University of Durham 27/10/2005

Measure Theory and Dynamics

Problems, set 2.

Please hand in your solutions on **Thursday, 3 November**, in the lecture.

Exercise 4. Let $X = \{1, 2, 3\}$. Write down the power set 2^X. How many elements has the power set of a set with n elements. Which of the two sets

$$\mathcal{A}_1 := \{\emptyset, \{1\}, \{2, 3\}, X\}$$

and

$$\mathcal{A}_2 := \{\emptyset, \{1\}, \{2\}, \{3\}, X\}$$

are σ-algebras? Explain why. Let the measure μ be defined as

$$\mu(\emptyset) = \mu(\{3\}) = 0, \mu(\{1\}) = \mu(\{2\}) = 1, \mu(\{1, 2\}) = 3,$$
$$\mu(\{1, 3\}) = \mu(\{2, 3\} = 1, \mu(X) = 3.$$

You don't have to check that μ is a measure. But find out the σ-algebra of μ-measurable sets.

Exercise 5. Let $X = [0, 1]$, λ be the Lebesgue measure on X and μ defined by

$$\mu(A) = \#\left(A \cap \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}\right),$$

where $\#S$ is the cardinality of the set S (number of elements).

(a) Show: μ is a measure.

(b) What is $\mu(\,[1/4, 1)\,), \lambda([0, 1] \cap \mathbb{Q}), \mu([0, 1] \cap \mathbb{Q})$? Explain your answers.

(c) Calculate (again with a short explanation) for $f = 2\cdot\chi_{[0,1/3)} + 3\cdot\chi_{(1/2,3/4]}$ and $g(x) = x$ the values for the following integrals:

$$\int_X f\, d\mu, \quad \int_X f\, d\lambda, \quad \int_X g\, d\mu, \quad \int_X g\, d\lambda.$$

Exercise 6. We consider the following four sets:

$$\begin{aligned}
\mathcal{F}_0 &= \text{set of all open subsets of } \mathbb{R}, \\
\mathcal{F}_1 &= \text{set of all open intervals of } \mathbb{R}, \\
\mathcal{F}_2 &= \{(-\infty, a) \mid a \in \mathbb{R}\}, \\
\mathcal{F}_3 &= \{(-\infty, a] \mid a \in \mathbb{R}\}.
\end{aligned}$$

Note that $\mathcal{F}_1$ contains also unbounded open intervals like (∞, a) or $\mathbb{R} = (\infty, \infty)$. Explain why the σ-algebras generated by each of the four sets coincide.

Assume that $(X, \mathcal{A})$ is a measurable space. Explain with the help of the above result why $f : X \to \mathbb{R}$ is measurable iff

$$f^{-1}((-\infty, a]) \in \mathcal{A} \quad \forall\, a \in \mathbb{R}.$$