Norbert Peyerimhoff, University of Durham 02/11/2005

Measure Theory and Dynamics

Problems, set 3.

Exercise 7. The symmetric difference of two subsets $A, B \subset X$ is defined as

$$A \Delta B = (A \backslash B) \cup (B \backslash A),$$

and describes how much the sets A, B differ. We have $A \Delta B = \emptyset$ if and only if $A = B$.

Show that

$$A \Delta B \subset (A \Delta C) \cup (C \Delta B),$$

which can be considered as some type of a "triangle inequality". Conclude for arbitrary $A_1, A_2, \dots, A_n$ that

$$A_1 \Delta A_n \subset \bigcup_{j=1}^{n-1} A_j \Delta A_{j+1}.$$

Finally, let $\mu : 2^X \to [0, \infty]$ be a measure and $\mathcal{A}$ denote the σ-algebra of μ-measurable sets. Let $A, B \in \mathcal{A}$. Show that $\mu(A \Delta B) = 0$ implies

$$\mu(A) = \mu(B).$$

This fact was used in the proof of Proposition 1.24 in the lecture.

Exercise 8. Let $X = \{1, 2, 3, \dots, 10\}$ and $T : X \to X$ be the permutation

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 9 & 4 & 5 & 1 & 2 & 7 & 10 & 6 & 8 \end{pmatrix}$$

We consider the discrete dynamical system $\{T^n : X \to X \mid n \in \mathbb{Z}\}$. Find all T-invariant subsets of X. We consider the following three counting measures

$$\begin{aligned} \mu_1(A) &:= \#(A \cap \{1, 3, 4\}), \\ \mu_2(A) &:= \#(A \cap \{2, 6, 9\}), \\ \mu_3(A) &:= \#(A \cap \{7, 8, 10\}). \end{aligned}$$

Check whether T is ergodic with respect to μ_1, μ_2 and μ_3.

Exercise 9. *(The Middle-Third Cantor set as a T-invariant subset of $[0,1]$)*
Define $T : [0,1] \to \mathbb{R}$ as

$$T(x) = \begin{cases} 3x, & \text{if } 0 \leq x \leq 1/2, \\ 3 - 3x, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Draw the graph of T. Let

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= T^{-1}([0,1]) = [0,1/3] \cup [2/3,1], \\ C_2 &= T^{-2}([0,1]) = T^{-1}(C_1), \\ C_3 &= T^{-3}([0,1]) = T^{-1}(C_2), \\ &\vdots \end{aligned}$$

Write down explicitely C_2 and C_3. Show that

$$C_j = \{x \in [0,1] \mid x, Tx, \ldots, T^j(x) \in [0,1]\}$$

and conclude that $C_{j+1} \subset C_j$. Make a guess, how one obtains C_{j+1} from C_j geometrically. The **Middle-Third Cantor set** C is then defined as

$$C := \bigcap_{j=0}^{\infty} C_j.$$

Show that C is T-invariant. Calculate $\lambda(C)$ where λ is the Lebesgue measure on $[0,1]$.

Additional remark to Exercise 9: The Cantor set is not countable, since it contains all real numbers of the form

$$\sum_{j=1}^{\infty} \frac{a_j}{3^j},$$

with all coefficients $a_j \in \{0,2\}$. But C has Lebesgue measure zero. Topologically, C is a closed, totally disconnected and perfect subset of $[0,1]$. It is a self similar set, since

$$C = \frac{1}{3}C \ \cup \ (\frac{1}{3} + \frac{1}{3}C) \ \cup \ (\frac{2}{3} + \frac{1}{3}C),$$

where $a + bC := \{a + bx \mid x \in C\}$.