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Measure Theory and Dynamics

Problems, set 4.

Exercise 10.

(a) Find all accumulation points of the sequence

$$a_n = \sin(\frac{\pi}{3}n) + (-1)^n.$$

What is $\liminf a_n$ and $\limsup a_n$ in this example?

(b) Let a sequence $\{x_n\}$ be given and

$$x := \lim_{n\to\infty} \inf\{x_k \mid k \geq n\} \in \mathbb{R}.$$

Explain how one can construct a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty} x_{n_j} = x.$$

Exercise 11. This exercise deals with the transformation rule in integration theory, which is a generalization of the substitution law for the Riemann integral.

(a) It was shown in the lectures that the map $T : [0,1) \to [0,1)$, $T(x) = 2x - [2x]$, ($[x]$ Gauss bracket of x, i.e., the largest integer not exceeding x) is measure preserving with respect to the Lebesgue measure λ. Let

$$\varphi(x) = 3\chi_{[0,\frac{1}{3})} - 2\chi_{(\frac{1}{2},\frac{7}{8}]}.$$

Calculate

$$\int_{[0,1)} \varphi d\lambda, \quad \varphi \circ T (\text{this is, again, a step function}), \quad \int_{[0,1)} \varphi \circ T d\lambda.$$

(b) Let $\mu : 2^X \to [0,\infty]$ be a measure, $\mathcal{A}$ be the corresponding σ-algebra of measurable sets and $\varphi = \sum_{j=1}^n c_j \chi_{A_j}$ be a step function on X (i.e., $A_j \in \mathcal{A}$). Let $T : X \to X$ be measure preserving. Calculate the step function $\varphi \circ T$ and show that

$$\int_X \varphi \circ T \, d\mu = \int_X \varphi \, d\mu. \tag{1}$$

c) How can one conclude from (b) that equation (1) holds also for arbitrary integrable functions?

Additional remark to Exercise 11: The general transformation rule works as follows: Let $\mu : 2^X \to [0,\infty]$ be a measure and $T : X \to Y$ be any map. Then Y inherits a image measure μ^T as follows:

$$\mu^T(A) := \mu(T^{-1}(A)) \quad \forall\, A \subset Y.$$

For every μ^T-integrable function φ on Y we have

$$\int_X \varphi \circ T \, d\mu = \int_Y \varphi \, d\mu^T.$$

Exercise 12. Let $\mu : 2^X \to [0,\infty]$ be a measure on X and $\mathcal{A}$ denote the corresponding σ-algebra of measurable subsets. Let $T : X \to X$ be measure preserving and, for every $x \in X$, let

$$\mathcal{O}^+(x) = \{T^j(x) \mid j \in \{0,1,2,\dots\}\}$$

denote the forward orbit of x. Let $A \in \mathcal{A}$ be an almost T-invariant set and

$$A_0 := \{x \in A \mid \mathcal{O}^+(x) \subset A\}.$$

(A_0 consists of all elements of A such that the corresponding forward orbit stays in A.) Show that A_0 is, again, a measurable set and that $\mu(A_0) = \mu(A)$. This means that, for almost all initial points in A, the corresponding forward orbit stays in A.