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Measure Theory and Dynamics

Problems, set 5.

Please hand in your solutions to Exercises 11, 12 (of set 4) and Exercises 13, 14 (of this set) on **Thursday, 1 December**, in the lecture.

Exercise 13. As in the lecture, we identify $[0,1)$ with S^1 via the map $x \mapsto e^{2\pi i x}$. Consider the circle map

$$T : [0,1) \to [0,1), \quad T(x) = 10x - [10x],$$

where $[y]$ is the largest integer $\leq y$ ("Gauss bracket").

(a) Let

$$x_1 = 14/135, \quad x_2 = \sqrt{2} - 1, \quad x_3 = 0.047\overline{81142}.$$

Write down the first 10 elements of the forward orbits $\mathcal{O}^+(x_j)$ and decide which of the orbits become periodic and, if yes, determine the period. Find a number $x \in [0,1)$ which satisfies

$$T^4(x) = T^8(x),$$

and consequently also $T^{4+n}(x) = T^{8+n}(x)$ for all $n \geq 0$. Which reals in $[0,1)$ become eventually periodic and which don't?

(b) Let $I = [0.3, 0.35]$. Determine $T^{-1}(I)$.

(c) Show for every interval $I = [a,b]$:

$$\lambda(I) = \lambda(T^{-1}(I)). \tag{1}$$

In fact, this map is measure preserving with respect to the Lebesgue measure λ. (It is not easy to prove this in full generality. But (**??**) gives a strong indication for this fact.)

Exercise 14. Let $C([a,b])$ be the vector space of continuous functions on the finite closed interval $[a,b]$ and

$$\|f\|_\infty := \max_{x \in [a,b]} |f(x)|.$$

(a) Show that $\| \cdot \|_\infty$ satisfies all properties of a norm.

(b) Let $[a, b] = [0, 5]$, $f(x) = -x^2/4 + 2x - 1$ and $g(x) = x^3 - 10x^2 + 27x - 18$. Calculate

$$\|f\|_\infty, \quad \|g\|_\infty.$$

(c) Let $f_n(x) = x^n$. Decide whether f_n is convergent in $C([-1/2, 1/2])$ and in $C([0, 1])$ with respect to the norm $\| \cdot \|_\infty$ and, if yes, write down the limit function.

(d) Assume that a sequence $f_n \in C([a, b])$ converges to a function $f : [a, b] \to \mathbb{R}$ with respect to the norm $\| \cdot \|_\infty$. Show that the limit function f is continuous.

Hints for Part (d): A function $f : [a, b] \to \mathbb{R}$ is continuous at $x \in [a, b]$ if for every $\epsilon > 0$ there exists a δ such that

$$|f(x) - f(y)| < \epsilon \quad \forall\, y \in [a, b], |y - x| < \delta.$$

Use the fact that, for large enough indices n, f_n lies in a $\epsilon/3$-tube about f and that f_n is continuous.

Exercise 15. In this exercise we present a proof of the following fact:

Fact: *For every irrational $\alpha \in [0, 1)$ there are infinitely co-prime pairs (p, q) such that*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

The proof is a powerful application of the Dirichlet *pigeon hole principle* ("If $N+1$ different numbers are distributed into N pairwise disjoint intervals, then there is an interval containing at least two of these numbers.") To prove the above statement it suffices to prove the following: for every $N \geq 1$ there is a pair (p, q) (not necessarily co-prime) with $q \leq N$ such that

$$|\alpha - \frac{p}{q}| < \frac{1}{Nq} \leq \frac{1}{q^2}.$$

Convince yourself first that this statement is sufficient to prove the above fact.

(a) Using the splitting of $[0,1)$ into N disjoint intervals

$$[0,1) = [0, \frac{1}{N}) \cup [\frac{1}{N}, \frac{2}{N}), \ldots \cup [\frac{N-1}{N}, 1),$$

conclude that there are two numbers $1 \leq k < l \leq N+1$ and an integer $p \in \mathbb{Z}$ such that

$$|l\alpha - k\alpha - p| < \frac{1}{N}.$$

(Hint: Consider the numbers $\alpha, 2\alpha - [2\alpha], \ldots, N\alpha - [N\alpha]$.)

(b) Conclude from (a) that there is a $1 \leq q \leq N$ with

$$|\alpha - \frac{p}{q}| < \frac{1}{Nq}.$$