Norbert Peyerimhoff, University of Durham 05/12/2005

Measure Theory and Dynamics

Problems, set 6.

Exercise 16. This exercise treats the notion "compactness".

(a) Find an open cover of (a, b) which does not admit a finite subcover. Consequently, (a, b) is not compact (in the sense of Heine-Borel).

(b) Let $I \subset \mathbb{R}$. A function $f : I \to \mathbb{R}$ is continuous if, for every $x \in I$ and every $\epsilon > 0$ there exists a $\delta_x > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \forall\, y \in I, |y - x| < \delta_x.$$

A function is called "uniformly continuous" on I, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in I$ we have

$$|f(x) - f(y)| < \epsilon \quad \forall\, y \in I, |y - x| < \delta.$$

The strenghtening in "uniform continuity" is the fact that the choice of $\delta > 0$ does not depend on the choice of reference point x in which we study continuity and can be uniformly chosen for all $x \in I$. Show: If $I \subset \mathbb{R}$ is compact then every continuous function $f : I \to \mathbb{R}$ is uniformly continuous. (The proof is a nice application of the Heine-Borel cover-property of compact sets.)

(c) In this course we often identified $[0, 1)$ with S^1. But $[0, 1) \subset \mathbb{R}$ with the metric $d_1(x, y) = |y - x|$ is not compact and $(S^1 \subset \mathbb{C}, d)$ is compact. What becomes different if we choose the metric

$$d_2(x, y) = \min\{|y - x|, 1 - |y - x|\}$$

on $[0, 1)$?

Exercise 17. Let $\mu : 2^X \to [0, 1]$ be a probability measure with corresponding σ-algebra $\mathcal{A} \subset 2^X$ of μ-measurable sets. Show that the following properties are equivalent:

(a) $T : X \to X$ is ergodic.

(b) For all $A \in \mathcal{A}$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_A(T^j x) \to \mu(A) \quad \text{almost everywhere.}$$

Exercise 18. We consider the continued fraction map $T : [0,1) \to [0,1)$ defined as follows:

$$T(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if } x \in (0,1), \\ 0 & \text{if } x = 0, \end{cases}$$

where $[z]$ denotes the largest integer less or equal to z.

(a) Let $x = \sqrt{2} - 1 \in [0,1)$. Calculate x, Tx, T^2x, T^3x, T^4x.

(b) Let $x = \sqrt{3} - 1 \in [0,1)$. Calculate x, Tx, T^2x, T^3x, T^4x.

(c) Let $I = (0, a) \subset [0,1)$. Show that

$$T^{-1}(I) = \bigcup_{k\in\mathbb{N}} \left(\frac{1}{k+a}, \frac{1}{k}\right).$$

(d) C.F. Gauß introduced the measure

$$\mu((a,b)) = \int_a^b \frac{dt}{1+t}$$

on the set $[0,1)$. It turns out that the continued fraction map is measure preserving w.r.t. μ. We check this for sets $I = (0,a) \subset [0,1)$. Show that

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{N}\left(\frac{1}{k+a}, \frac{1}{k}\right)\right) &= \sum_{k=1}^{N} \ln(\frac{k+1}{k}) - \sum_{k=1}^{N} \ln(\frac{k+a+1}{k+a}) \\ &= \ln\left(\frac{N+1}{N+a+1}\right) + \ln(1+a). \end{aligned}$$

Conclude that

$$\mu(T^{-1}(I)) = \ln(1+a) = \mu(I).$$

Additional remarks to Exercise 18: The continued fraction map is closely related to number theory, in particular diophantine approximation. An irrational $x \in (0,1)$ is called a "quadratic irrational", if it is a root of a quadratic equation $ax^2 + bx + c = 0$ with integer coefficients a, b, c. By a Theorem of Lagrange, quadratic irrationals are precisely those initial points $x \in [0,1)$ such that the forward orbit $\mathcal{O}^+(x)$ of the continued fraction map becomes periodic. Compare this statement with (a) and (b) of this exercise.