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Measure Theory and Dynamics

Problems, set 7.

Exercise 19. In this exercise we consider the *one-sided shift map*, which is a very important dynamical system in *Symbolic Dynamics* and which is very similar to the dynamical system $T : S^1 \to S^1$, $T(z) = z^{10}$.

Let X be the set of sequences $\{x_j\}_{j \geq 0}$ with $x_j \in \{0, 1\}$. The shift operator $\sigma : X \to X$ is defined by

$$\sigma(\{x_j\}) = \{y_j\}$$

with $y_j = x_{j+1}$ for $j \geq 0$. Next, we define a probability measure μ on X: The subsets

$$C(i_0, \dots, i_l) := \{\{x_j\} \in X \mid x_0 = i_0, x_1 = i_1, \dots, x_l = i_l\}$$

are called "cylinder set" and we define

$$\mu(C(i_0, \dots, i_l)) = \frac{1}{2^{l+1}}.$$

For an arbitrary subset $A \subset X$ we define

$$\mu(A) = \inf\{\sum_{s=1}^{\infty} \mu(C_s) \mid C_s \text{ cylinder sets and } A \subset \bigcup_{s=1}^{\infty} C_s\}.$$

(The cylinder sets play a similar role in the system (X, σ) as the sets $A_{k,j}$ in the system (S^1, T).)

(a) Write $\sigma^{-1}(C(i_0, i_1, \dots, i_l))$ as a disjoint union of two cylinder sets. Conclude that

$$\mu(\sigma^{-1}(C)) = \mu(C)$$

for every cylinder set C.

(b) Show for two cylinder sets $C = C(i_0, i_1, \dots, i_l)$, $C' = C(j_0, j_1, \dots, i_s)$:

$$\lim_{n \to \infty} \mu(\sigma^{-n}(C) \cap C') = \mu(C)\mu(C').$$

Additional remarks to Exercise 19: Exercise (a) indicates that the one-sided shift system is measure preserving and (b) indicates that it is mixing with respect to the measure μ. The importance of the shift system stems from the fact that many complicated dynamical systems can be identified via a so-called *Markov Partition* with a subsystem of a shift system. This identification reduces the study of the original system to the study of a purely symbolic system and allows often to draw stunning conclusions about the original system. Two recommendable books about Symbolic Dynamics are

D. Lind and B. Marcus: *An introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, 1995

B. P. Kitchens: *Symbolic Dynamics*, Springer, 1998

Exercise 20. We know that not all forward orbits $\mathcal{O}^+(z) \subset S^1$ of $T(z) = z^{10}$ are uniformly distributed with respect to the Lebesgue measure λ, since T has fixed points. In this exercise, however, we will prove that the forward orbits $\mathcal{O}^+(z)$ of almost all $z \in S^1$ are uniformly distributed with respect to λ.

(a) Let $C(S^1)$ denote the vector space of continuous functions $f : S^1 \to \mathbb{C}$. The norm on $C(S^1)$ is defined by

$$\|f\|_\infty := \max_{z \in S^1} |f(z)|.$$

Show that there are countably many functions $f_1, f_2, \dots \in C(S^1)$ which are dense in $C(S^1)$ w.r.t. $\|\cdot\|_\infty$. For the proof use the following version of the Stone-Weierstraß Approximation Theorem: For every $f \in C(S^1)$ and every $\epsilon > 0$ there exists a function

$$p(z) = \sum_{j=-N}^{N} a_j z^j$$

(where N is a finite number and $a_j \in \mathbb{C}$ are appropriately chosen) such that

$$\|f - p\|_\infty < \epsilon.$$

Argue, why it can be assumed that all the coefficients a_j are in $\mathbb{Q} + i\mathbb{Q} \subset \mathbb{C}$.

(b) Use the fact that $T(z) = z^{10}$ is ergodic and Birkhoff's Ergodic Theorem in order to prove the following fact: There is a subset $X_0 \subset S^1$ of full Lebesgue measure such that we have for every $z \in X_0$ and every continuous function $f : S^1 \to \mathbb{C}$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j z) = \int_{S^1} f \, d\lambda,$$

i.e., almost all forward orbits are uniformly distributed with respect to the Lebesgue measure.

Final remark: The best rehearsal of the course is to go over the exercises and to think them over in combination with reading the corresponding course material. Whenever you have solved some of the exercises or you got stuck at a certain point, you are always welcome to contact me in order to get some feedback or some help.

Merry Christmas and Happy New Year!!!