CAYLEY GRAPH EXPANDERS AND GROUPS OF FINITE WIDTH

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Abstract. We present new infinite families of expander graphs of vertex degree 4, which is the minimal possible degree for Cayley graph expanders. Our first family defines a tower of coverings (with covering indices equals 2) and our second family is given as Cayley graphs of finite groups with very short presentations with only 2 generators and 4 relations. Both families are based on particular finite quotients of a group G of infinite upper triangular matrices over the ring $\mathrm{M}(3, \mathbb{F}_2)$.

We present explicit vector space bases for the finite abelian quotients of the lower exponent-2 groups of G by upper triangular subgroups and prove a particular 3-periodicity of these quotients.

The pro-2 completion of the group G satisfies the Golod-Shafarevich inequality

$$|R| \geq \frac{|X|^2}{4},$$

it is infinite, not p-adic analytic, contains a free nonabelian subgroup, but not a free pro-p group. We also conjecture that the group G has finite width 3 and finite average width 8/3.

1. Introduction

The first explicit construction of expander graphs was introduced by Margulis [Marg1] and was an application of Kazhdan's property (T). Expanders are simultaneously sparse and highly connected and are not only of theoretical importance but also useful in computer science, e.g., for network designs. An extensive survey of this topic is given in [HLW].

Cartwright and Steger [CS] constructed infinite sequences of groups acting simply transitively on the vertices of buildings of type $\tilde{A}_n$. Using these groups, Lubotzky, Samuels and Vishne [LSV] and, independently, Sarveniazi [Sar] presented explicit constructions of expanders and proved that they are, in fact, (higher dimensional) Ramanujan complexes.

M. Ershov constructed in [Er] for every sufficiently large prime p a group violating the Golod-Shafarevich inequality (as presented in [LS, p. 87]) and having property (T). Since those groups have infinitely many p-quotients, this fact provided new families of expanders in connection with pro-p groups.

In this article, we bring many of these aspects together. In Section 3, we present two new families of Cayley graph expanders of vertex degree 4. One family is given by a tower of coverings with covering indices equals 2, and the other family is given by very short presentations with 2 *generators* and *only* 4 *relations* (see Theorem 1 below). Note that we use the following shorthand notation for higher commutators:

$$[x,\ _n\, y,\ z] := [x, \underbrace{y, \ldots, y}_{n}, z].$$

Theorem 1. *The groups*

$$G_k := \langle\, x_0, x_1 \,|\, r_1, r_2, r_3, [x_1,\ _k\, x_0]\,\rangle,$$

with

$$r_1 = x_1 x_0 x_1 x_0 x_1 x_0 x_1^{-3} x_0^{-3},$$
$$r_2 = x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_1^2 x_0^{-1} x_1 x_0 x_1,$$
$$r_3 = x_1^3 x_0^{-1} x_1 x_0 x_1 x_0^2 x_1^2 x_0 x_1 x_0,$$

are finite and the associated Cayley graphs with respect to the symmetric set $\{x_0^{\pm 1}, x_1^{\pm 1}\}$ *define an infinite family of expanders of vertex degree* 4*, satisfying* $|G_k| \to \infty$*.*

Each expander graph has twice as many edges as it has vertices. To prove that our graphs are expanders indeed, we present them as Cayley graphs of finite quotients of a group G acting cocompactly on an Euclidean building of type $\tilde{A}_2$, like in [LSV] or [Sar], but our graphs are not Ramanujan for sufficiently many vertices.

The group G is given by $\langle x_0, x_1 \mid r_1, r_2, r_3\rangle$. The pro-2 completion $\widehat{G}_2$ of G can be considered as a finitely presented pro-2 group. In Section 4, we discuss particular properties of this pro-2 group. Even though $\widehat{G}_2$ satisfies the Golod-Shafarevich inequality, it is still infinite. Lubotzky [Lub1] has shown that p-adic analytic groups satisfy the Golod-Shafarevich inequality. The group $\widehat{G}_2$, however, is not p-adic analytic. Wilson [Wi] conjectured that discrete, resp., pro-p groups violating the Golod-Shafarevich inequality have free subgroups, resp., free pro-p subgroups of rank two. The first conjecture was later proved in [WZ] and the proof of the second conjecture can be found in [Zel, p. 224]. The group $\widehat{G}_2$ satisfies this inequality, doesn't contain a free pro-2 subgroup, but it nevertheless contains a free subgroup of rank two.

Our considerations are based on a linear representation of the group G by infinite upper unitriangular matrices over the field $\mathbb{F}_2$. It is particularly useful to view these infinite matrices as been built up by diagonals of 3×3 block matrices. Natural normal subgroups H_i are given by infinite upper triangular matrices with vanishing first i upper diagonals. Let

$$G = \lambda_0(G) \geq \lambda_1(G) \geq \cdots$$

denote the lower exponent-2 series of G. Obviously, we have $\lambda_i(G) \leq G \cap H_i$. In Theorem 2 of Section 2, we prove a particular 3-*periodicity* for the abelian quotients $\lambda_i(G)/(\lambda_i(G) \cap H_{i+1})$ and derive *explicit bases* for them. These considerations give useful information about the structure of our families of Cayley graph expanders.

Computer calculations show for the group G the identities $\lambda_i(G) = G \cap H_i$ from $i \geq 1$ onwards and $\lambda_i(G)/\lambda_{i+1}(G) \cong \gamma_i(G)/\gamma_{i+1}(G)$ from $i \geq 2$ onwards, up to the index $i = 100$. Here, $\gamma_i(G)$ denote the lower central series groups of G. If these identities are true for all i-indices, then our group G has *finite width* 3 and finite average width 8/3 (see, e.g., [KLP] for definitions), and the *covering indices* of our expander graphs $\mathcal{G}_i$ are given by the 3-periodic sequence $4, 8, \overline{4, 8, 8}$.

The group G is a subgroup of a group Γ belonging to a class of groups $\Gamma_{\mathcal{T}}$, which were constructed in [CMSZ] and are related to particular triangle presentations $\mathcal{T}$ of special finite projective planes of prime power order q. We expect that analogous *finite width properties* hold also for these groups $\Gamma_{\mathcal{T}}$ (see Conjecture 2 in Section 4). Again, this conjecture has been checked by computer for many i-indices for the prime powers $q = 2, 4, 5, 7, 9, 11$.

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2. Commutator schemes of the group G

Our group G is a fundamental group of a simplicial complex $\mathcal{K}$, consisting of 14 triangles, and defined by the labeling scheme in Figure 1. Straightforward calculations give the following presentation of this group: $G = \langle\, x_0, x_1 \,|\, r_1, r_2, r_3 \,\rangle$ with

$$\begin{aligned} r_1 &= x_1x_0x_1x_0x_1x_0x_1^{-3}x_0^{-3}, \\ r_2 &= x_1x_0^{-1}x_1^{-1}x_0^{-3}x_1^2x_0^{-1}x_1x_0x_1, \qquad (1) \\ r_3 &= x_1^3x_0^{-1}x_1x_0x_1x_0^2x_1^2x_0x_1x_0. \end{aligned}$$

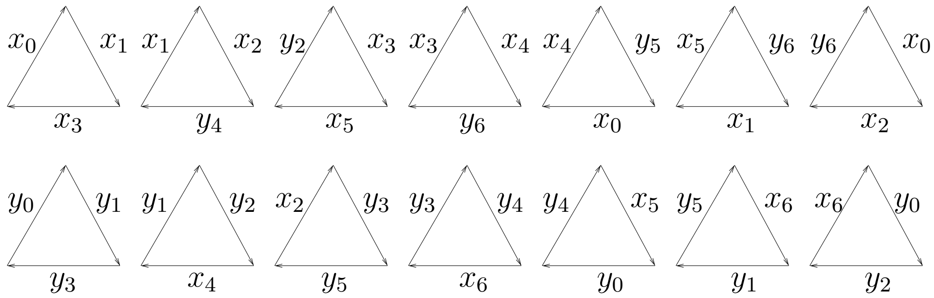

FIGURE 1. Labeling scheme for our simplicial complex

This complex is a 2-fold cover of a classifying space of the group

$$\Gamma := \langle x_0, \dots, x_6 | x_i x_{i+1} x_{i+3} \text{ for } i = 0, 1, \dots, 6 \rangle, \tag{2}$$

where $i, i+1, i+3$ are taken mod 7. The group Γ belongs to the family of groups introduced in [CMSZ, §4].

Proposition 2.1. *The group Γ in (2) is generated by x_0, x_1, x_2 and the subgroup G generated by x_0, x_1 is an index two normal subgroup of Γ.*

Even though $[\Gamma : G] = 2$ follows from covering arguments, we give a different proof for this fact as well.

Proof: The relations imply that $x_3 = (x_0x_1)^{-1}$, $x_4 = (x_1x_2)^{-1}$, $x_5 = x_0x_1x_2^{-1}$, $x_6 = (x_0x_2)^{-1}$, so Γ is generated by x_0, x_1, x_2. The diagrams in Figure 2 show the validity of the three relations $x_2x_1x_2 = x_0^{-1}x_1^{-1}x_0^{-1}$, $x_2x_0^{-1}x_2 = x_1^{-1}x_0x_1$ and ${x_2}^2 = x_0^{-1}x_1x_0x_1$ in Γ.

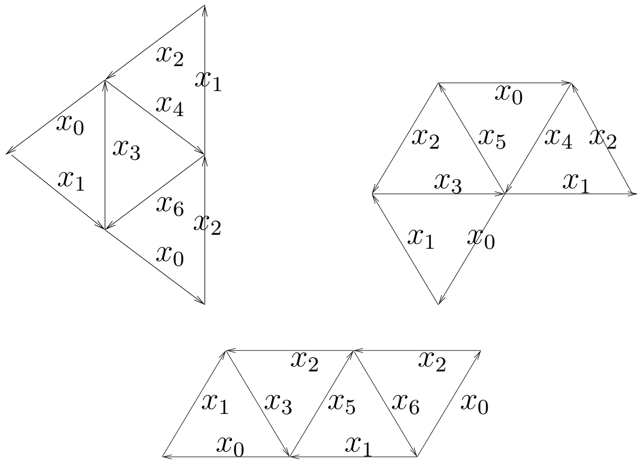

FIGURE 2. Diagram for relations in G_2

It remains to prove that every element of Γ can be written as w or wx_2, where w is a word in x_0, x_1. By relation ${x_2}^2 = x_0^{-1}x_1x_0x_1$, it suffices to prove that any reduced word x_2w with w only containing x_0, x_1 and being of length n can be rewritten as $w_1x_2w_2$ with w_1, w_2 only containing x_0, x_1 and w_2 being of length $\leq n-1$. Assume that $w = x_0w'$. (The other cases $w = x_0^{-1}w'$, $w = x_1w'$ and $w = x_1^{-1}w'$ are treated similarly.) Then we have, using the above three relations:

$$\begin{aligned} x_2x_0w' &= x_2^2(x_2^{-1}x_0x_2^{-1})x_2w' \\ &= x_2^2(x_1^{-1}x_0^{-1}x_1)x_2w' \\ &= (x_0^{-1}x_1x_0x_1)x_1^{-1}x_0^{-1}x_1x_2w' \\ &= x_0^{-1}x_1^2x_2w' = w_1x_2w_2, \end{aligned}$$

with $w_1 = x_0^{-1}x_1^2$ and $w_2 = w'$. ☐

Now, we employ a particular linear representation of Γ in the matrix group $\mathrm{GL}(9, \mathbb{F}_2(y))$ given in [LSV] (note that the b_i in [LSV, Section 10] correspond to our x_i^{-1}):

$$x_0 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&1&0\\ 0&0&1&0&1&1&0&0&1\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} + \frac{1}{y}\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&1&0&1&0\\ 0&1&0&0&1&1&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&1&0&0&1\\ 0&0&1&0&1&0&0&1&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&1&1\\ 0&1&1&0&0&1&0&1&0 \end{pmatrix},$$

$$x_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&1&1&1\\ 0&0&1&1&1&1&0&1&1\\ 0&0&0&1&0&0&0&1&1\\ 0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} + \frac{1}{y}\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&1&0&1&0&1&1&1\\ 0&1&0&1&1&1&0&1&1\\ 0&1&0&1&1&1&0&1&1\\ 0&0&1&1&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&1&1&1&0&1&1\\ 0&0&1&1&0&1&1&0&0\\ 0&1&0&1&1&1&0&1&1 \end{pmatrix}.$$

We have $x_0 = A_0 + \frac{1}{y}A_1$ and $x_1 = B_0 + \frac{1}{y}B_1$ with 9×9 matrices $A_0, A_1, B_0, B_1 \in \mathrm{M}(9, \mathbb{F}_2)$, and their inverses x_0^{-1}, x_1^{-1} are of the same form. Therefore, an arbitrary group element $x \in G$ is of the form

$$x = C_0 + \sum_{j=1}^{k} \frac{1}{y^j} C_j,$$

which we identify with the (finite band) upper triangular infinite Toeplitz matrix

$$x = \begin{pmatrix} C_0 & C_1 & C_2 & \dots & C_k & 0 & 0 & \dots \\ 0 & C_0 & C_1 & \dots & C_{k-1} & C_k & 0 & \ddots \\ 0 & 0 & C_0 & \dots & C_{k-2} & C_{k-1} & C_k & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3}$$

where each C_i is a matrix in $\mathrm{M}(9, \mathbb{F}_2)$. One checks that multiplication of elements in $\mathrm{GL}(9, \mathbb{F}_2[1/y])$ and of the corresponding infinite matrices is consistent.

For a more detailed analysis it is useful to rewrite the matrix representation (3) of an arbitrary element $x \in G$ with the help of 3×3 matrices. This requires some notation.

Let $\mathcal{S}$ denote the vector space of all 3×9 matrices over $\mathbb{F}_2$, i.e, every $a \in \mathcal{S}$ is of the form

$$a = \begin{pmatrix} a(1) & a(2) & a(3) \end{pmatrix}$$

with 3×3-matrices $a(i)$ over $\mathbb{F}_2$. Let $\mathbf{0}$ denote the zero matrix in $\mathcal{S}$. For every (finite or infinite) sequence $a_1, a_2, \dots \in \mathcal{S}$ let $M(a_1, a_2, \dots)$ denote the following infinite matrix: All lower diagonals of size 3×3 are zero, the main diagonal of size 3×3 consists only of identity matrices and the i-th upper diagonal of size 3×3 has the 3-periodic entries $a_i(1), a_i(2), a_i(3), a_i(1), a_i(2), a_i(3), \dots$ for $i \geq 1$. In the case of a finite sequence $a_1, \dots, a_p$, all diagonals above the p-th diagonal of size 3×3 are also chosen to be zero. The set of all such infinite matrices has an obvious group structure and is denoted by H. For $i \geq 0$, let $M_i(a_1, a_2, \dots)$ denote the matrices $M(\underbrace{0, 0, \dots, 0}_{i\,\text{times}}, a_1, a_2, \dots)$ and let H_i denote the normal subgroup of H consisting of all those matrices. Let

$$G = \lambda_0(G) \geq \lambda_1(G) \geq \lambda_2(G) \geq \cdots$$

be the lower exponent-2 series of G, i.e., $\lambda_{i+1}(G) = [\lambda_i(G), G]\lambda_i(G)^2$ for $i \geq 0$. Note that $\lambda_i(G) \leq G \cap H_i$. With this notation, we have

$$x_0 = M(a_1, \dots, a_5), \quad a_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \tag{4}$$

$$x_1 = M(b_1, \dots, b_5), \quad b_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5}$$

Theorem 2. *We have the following* 3*-periodicity for the abelian quotients* $\lambda_i(G)/(\lambda_i(G)\cap H_{i+1})$ *for* $i \geq 2$*:*

$$[\lambda_i(G) : (\lambda_i(G)\cap H_{i+1})] = \begin{cases} 8, & \text{if } i \equiv 0,1 \mod 3, \\ 4, & \text{if } i \equiv 2 \mod 3, \end{cases}$$

and a basis of $\lambda_i(G)/(\lambda_i(G)\cap H_{i+1})$ *(as vector space over* $\mathbb{F}_2$*) is given by*

$$\begin{aligned} \{[x_1,\ _i\ x_0], [x_1,\ _{i-2}\ x_0, x_1, x_0], [x_1,\ _{i-2}\ x_0, x_1, x_1]\}, &\quad \text{if } i \equiv 1 \mod 3, \\ \{[x_1,\ _i\ x_0], [x_1,\ _{i-1}\ x_0, x_1]\}, &\quad \text{if } i \equiv 2 \mod 3, \\ \{[x_1,\ _i\ x_0], [x_1,\ _{i-1}\ x_0, x_1], [x_1,\ _{i-2}\ x_0, x_1, x_1]\}, &\quad \text{if } i \equiv 0 \mod 3, \end{aligned}$$

where each commutator $[x_i, x_j, \dots]$ *above is an abbreviation for the left coset* $[x_i, x_j, \dots](\lambda_i(G)\cap H_{i+1})$.

Remark 2.2. *To complete the picture for* $i = 0, 1$*, we have*

$$[G : (G\cap H_1)] = 4 \quad \text{and} \quad [\lambda_1(G) : (\lambda_1(G)\cap H_2)] = 8$$

with basis $\{x_0, x_1\}$ *in the first case and* $\{x_0^2, x_1^2, [x_1, x_0]\}$ *in the second case. Note that the periodicity of the quotients* $\lambda_i(G)/(\lambda_i(G)\cap H_{i+1})$ *starts at* $i = 2$.

Before we start with the proof of Theorem 2, let us state an immediate consequence.

Corollary 2.3. *We have*

$$[G : (G\cap H_{3i+j})] \geq \begin{cases} 2^2, & \text{if } (i,j) = (0,1), \\ 2^5, & \text{if } (i,j) = (0,2), \\ 2^{8i-1+\mu(j)}, & \text{if } i \geq 1 \text{ and } j \in \{0,1,2\}, \end{cases}$$

where $\mu(0) = 0$*,* $\mu(1) = 3$ *and* $\mu(2) = 6$.

Proof: The estimates follow immediately from Theorem 2 and Remark 2.2 via

$$\begin{aligned} &[G : (G\cap H_k)] \\ &\quad = [G : (G\cap H_1)]\cdot[(G\cap H_1) : (G\cap H_2)]\cdots[(G\cap H_{k-1}) : (G\cap H_k)] \\ &\quad \geq [G : (G\cap H_1)]\cdot[\lambda_1(G) : (\lambda_1(G)\cap H_2)]\cdots[\lambda_{k-1}(G)\cap(\lambda_{k_1}\cap H_k)]. \end{aligned}$$

□

The rest of this section is devoted to the proof of Theorem 2.

Proposition 2.4. *We have*

$$M_k(a,\dots)^2 = M_{2k+1}(c,\dots) \tag{6}$$

with $c \in \mathcal{S}$ satisfying $c(i) = a(i)a(i+k+1)$ and

$$[M_k(a,\dots), M(b,\dots)] = M_{k+1}(c,\dots), \tag{7}$$

with $c \in \mathcal{S}$ defined by $c(i) = a(i)b(i+k+1) - b(i)a(i+1)$ (where the indices $i, i+1, i+k+1$ are taken mod 3).

Since we are working in vector spaces over $\mathbb{F}_2$, there is no difference between the expressions $\alpha + \beta$ and $\alpha - \beta$, but we prefer to use minus signs such that the formulas would also hold in vector spaces over other fields.

Proof: The identity (6) is easily checked. Equation (7) requires considerably more work and is proved in the Appendix. $\square$

Now, we introduce the following elements of $\mathcal{S}$:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Using Proposition 2.4 as well as $x_0 = M(\alpha_1 + \gamma_1, \dots)$ and $x_1 = M(\alpha_1, \dots)$, we obtain the following commutator scheme by straightforward calculations:

Proposition 2.5. *We have for every integer $k \geq 0$:*

$$[M_{3k}(\alpha_1,\dots), x_0] = M_{3k+1}(\alpha_2,\dots), \qquad [M_{3k}(\alpha_1,\dots), x_1] = M_{3k+1}(\mathbf{0},\dots),$$
$$[M_{3k}(\beta_1,\dots), x_0] = M_{3k+1}(\beta_2+\gamma_2,\dots), \quad [M_{3k}(\beta_1,\dots), x_1] = M_{3k+1}(\beta_2,\dots),$$
$$[M_{3k}(\gamma_1,\dots), x_0] = M_{3k+1}(\alpha_2,\dots), \qquad [M_{3k}(\gamma_1,\dots), x_1] = M_{3k+1}(\alpha_2,\dots),$$

$$[M_{3k+1}(\alpha_2,\dots),x_0] = M_{3k+2}(\alpha_3,\dots), \quad [M_{3k+1}(\alpha_2,\dots),x_1] = M_{3k+2}(\beta_3,\dots),$$
$$[M_{3k+1}(\beta_2,\dots),x_0] = M_{3k+2}(\mathbf{0},\dots), \quad [M_{3k+1}(\beta_2,\dots),x_1] = M_{3k+2}(\alpha_3,\dots),$$
$$[M_{3k+1}(\gamma_2,\dots),x_0] = M_{3k+2}(\beta_3,\dots), \quad [M_{3k+1}(\gamma_2,\dots),x_1] = M_{3k+1}(\mathbf{0},\dots),$$

$$[M_{3k+2}(\alpha_3,\dots),x_0] = M_{3k+3}(\alpha_1,\dots), \quad [M_{3k+2}(\alpha_3,\dots),x_1] = M_{3k+3}(\beta_1,\dots),$$
$$[M_{3k+2}(\beta_3,\dots),x_0] = M_{3k+3}(\beta_1,\dots), \quad [M_{3k+2}(\beta_3,\dots),x_1] = M_{3k+3}(\gamma_1,\dots).$$

Note that this proposition can be used to calculate the k-th upper diagonal $a \in \mathcal{S}$ of every k-fold commutator $[x_{i_1},\dots,x_{i_k}] = M_k(a,\dots)$ with $i_1,\dots,i_k \in \{0,1\}$.

To simplify notation, let $\Lambda_i := \lambda_i(G)$ and $L_{i+1} := \lambda_i(G) \cap H_{i+1}$ and $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3 \subset \mathcal{S}$ denote the subspaces spanned by $\{\alpha_1,\beta_1,\gamma_1\}$, $\{\alpha_2,\beta_2,\gamma_2\}$ and $\{\alpha_3,\beta_3\}$, respectively. Proposition 2.4 yields

(8)
$$x_0^2 = M_1(\beta_2,\dots), \quad x_1^2 = M_1(\gamma_2,\dots), \quad \text{and } [x_1,x_0] = M_1(\alpha_2,\dots).$$

Since $M_1(a,\dots)\cdot M_1(b,\dots) = M_1(a+b,\dots)$ and $M_1(a,\dots)^{-1} = M_1(-a,\dots)$, we conclude from (8) that Λ_1/L_2 is spanned by $\alpha_2,\beta_2,\gamma_2 \in \mathcal{S}$ (under the identification $a \mapsto M_1(a)L_2$). $\alpha_2,\beta_2,\gamma_2$ are linear independent and, therefore, Λ_1/L_2 is 3-dimensional and isomorphic to $\mathcal{S}_2$.

Propositions 2.4 and 2.5 are the key ingredients for the proof of Theorem 2, which we carry out by induction.

Proof of Theorem 2: We already know that every element in Λ_1 is of the form $M_1(a,\dots)$ with $a \in \mathcal{S}_2$. This is the begin of the induction.

Assume that we already know that every element in Λ_{3k+1} is of the form $M_{3k+1}(a,\dots)$ with $a \in \mathcal{S}_2$ for some $k \geq 0$. Using Propositions 2.4 and 2.5, we conclude that every element in Λ_{3k+2} is of the form $M_{3k+2}(a,\dots)$ with $a \in \mathcal{S}_3$. Proposition 2.5 yields also that we have

$$\begin{aligned}[][x_1,\ {}_{3k-1}\ x_0] &= M_{3k+2}(\alpha_3,\dots),\\ [x_1,\ {}_{3k-2}\ x_0,x_1] &= M_{3k+2}(\beta_3,\dots).\end{aligned}$$

Since α_3,β_3 span $\mathcal{S}_3$, we see that Λ_{3k+2}/L_{3k+3} is spanned by $[x_1,\ {}_{3k-1}\ x_0]L_{3k+3}$ and $[x_1,\ {}_{3k-2}\ x_0,x_1]L_{3k+3}$.

Repeating this reasoning twice, we obtain that Λ_{3k+3} and Λ_{3k+4} contain only elements of the form $M_{3k+3}(a,\dots)$ and $M_{3k+4}(b,\dots)$ with $a \in \mathcal{S}_1$ and $b \in \mathcal{S}_2$, and that the commutators given in the theorem are bases of the quotients Λ_{3k+3}/L_{3k+4} and Λ_{3k+4}/L_{3k+5}. This completes the induction step $k \to k+1$ and thus the proof of the theorem. $\square$

3. Explicit construction of expanders

The simplicial complex $\mathcal{K}$, introduced at the beginning of Section 2, consists of 2 vertices and 14 triangular faces. The link of each vertex is isomorphic to an incidence graph of a finite projective plane of order 2 (see [V] for more details of constructing polyhedra and determining their links). Using [BS] (see also [Pa] or [Z]) we conclude that the fundamental group G of $\mathcal{K}$ has Kazhdan property (T). We also like to mention that the Kazhdan constants of the groups $\Gamma_{\mathcal{T}}$ of Desarguesian projective planes were *exactly* calculated in [CMS].

We choose the symmetric generating set $S := \{x_0^{\pm 1}, x_1^{\pm 1}\}$ of the group G. As explained, e.g., in [Lub2, Prop. 3.3.1 and Prop. 4.2.4], Kazhdan property (T) of G implies that the Cayley graphs of all quotients of finite index normal subgroups of G (with respect to the set S) have a uniform positive lower bound on their combinatorial Cheeger constants as well as on the smallest positive eigenvalue of the combinatorial Laplacian (spectral gap). Thus, any sequence of normal subgroups with finite indices converging to infinity yields a family of expanders. We choose the normal subgroups $N_i = G \cap H_i$. Note that $[H : H_i]$ is a power of two, since the quotient H/H_i can be identified with the vector space $\mathcal{S}^i$ over $\mathbb{F}_2$ via the map

$$M(a_1, a_2, \dots)H_i \mapsto (a_1, a_2, \dots, a_i) \in \mathcal{S}^i. \tag{9}$$

This implies that the groups N_i have finite indices in G which are, again, powers of 2. We know from Corollary 2.3 that these indices converge to infinity, so the corresponding Cayley graphs $\mathcal{G}_i$ are expanders.

Let us now have a closer look at the explicit matrix models of the quotients G/N_i, obtained via the identification (9). This identification induces a nonabelian group structure on the space $\mathcal{S}^i$. In fact, the identity element is $\mathbf{0}^i \in \mathcal{S}^i$ and we have

$$\begin{aligned} (a_1, a_2, \dots, a_i) \cdot (b_1, b_2, \dots, b_i) &= (c_1, c_2, \dots, c_i), \\ (a_1, a_2, \dots, a_i)^{-1} &= (d_1, d_2, \dots, d_i) \end{aligned}$$

with

$$c_j(k) = a_j(k) + b_j(k) + \sum_{s=1}^{j-1} a_s(k) b_{j-s}(k+s), \tag{10}$$

where $k, k+s$ are taken mod 3. For the coefficients $d_j(k)$, we obtain the recursion formulas $d_1(k) = -a_1(k)$ and

$$d_j(k) = -a_j(k) - \sum_{s=1}^{j-1} a_s(k) d_{j-s}(k+s). \tag{11}$$

The identification (9) induces an embedding $G/N_i \hookrightarrow \mathcal{S}^i$ and we denote the image of the group G/N_i in $\mathcal{S}^i$ by K_i. K_i is generated by the images of $x_0 N_i$ and $x_1 N_i$, which we denote by v_0 and v_1. Hence, we have $v_0 = (a_1, \dots, a_5, \mathbf{0}, \dots) \in \mathcal{S}^i$ and $v_1 = (b_1, \dots, b_5, \mathbf{0}, \dots) \in \mathcal{S}^i$ with $a_1, \dots, b_5$ defined in (4) and (5). (If $i < 5$, we set $v_0 = (a_1, \dots, a_i)$ and $v_1 = (b_1, \dots, b_i)$.) The expanders $\mathcal{G}_i$ are the Cayley graphs of K_i with respect to $\{v_0^{\pm 1}, v_1^{\pm 1}\}$. They are all regular graphs with vertex degree 4 and

$$\dots \mathcal{G}_i \to \mathcal{G}_{i-1} \to \dots \mathcal{G}_1 \to \mathcal{G}_0 \tag{12}$$

is a tower of coverings. The covering indices of (12) are powers of 2, since $[G : N_i]$ are powers of 2. $\mathcal{G}_0$ and $\mathcal{G}_1$ are illustrated in Figure 3.

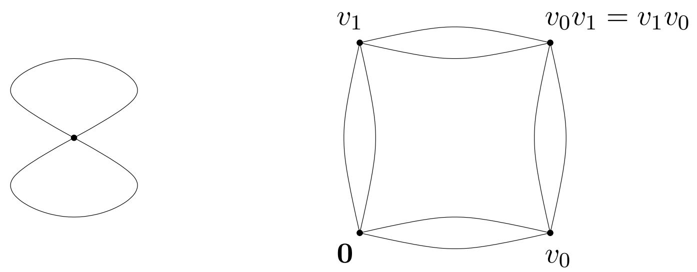

FIGURE 3. The graphs $\mathcal{G}_0$ and $\mathcal{G}_1$

Let us briefly explain how to construct $\mathcal{G}_2$, a regular graph with $2^5 = 32$ vertices. Let $v_0 = (a_1, a_2)$ and $v_1 = (b_1, b_2)$ be the images of x_0 and x_1 in the group $K_2 \subset \mathcal{S}^2$ and $w_1 = (\mathbf{0}, \alpha_2)$, $w_2 = (\mathbf{0}, \beta_2)$ and $w_3 = (\mathbf{0}, \gamma_2)$. Note that $\mathcal{S}^2$ has both a vector space structure and a nonabelian multiplicative group structure (given by (10) and (11)). The elements of K_2 are given by

$$K_2 := \bigoplus_{v \in \{\mathbf{0}^2, v_0, v_1, v_0 \cdot v_1\}} v + \mathbb{F}_2 w_1 + \mathbb{F}_2 w_2 + \mathbb{F}_2 w_3 \subset \mathcal{S}^2,$$

and the center of K_2 is generated by w_1, w_2, w_3. Thus we have $v \cdot w_i = w_i \cdot v = v + w_i$ for all $v \in K_2$. Using $v_0^2 = w_2, v_1^2 = w_3, [v_1, v_0] = w_1$ (which we compute with (10) and (11) or we conclude it from (8)), we obtain

$$v_1 \cdot v_0 = v_0 \cdot v_1 + w_1, \quad v_0^{-1} = v_0 + w_2, \quad \text{and} \quad v_1^{-1} = v_1 + w_3. \tag{13}$$

The vertices of $\mathcal{G}_2$ are the elements of K_2 and the neighbours of a vertex $v \in \mathcal{G}_2$ are the vertices $v \cdot v_0^{\pm 1}$ and $v \cdot v_1^{\pm 1}$. These neighbours can all be calculated with the help of (13). The graph $\mathcal{G}_2$ is illustrated in Figure 4 (where we use the abbreviations w_{ij} and w_{ijk} for $w_i + w_j$ and $w_i + w_j + w_k$).

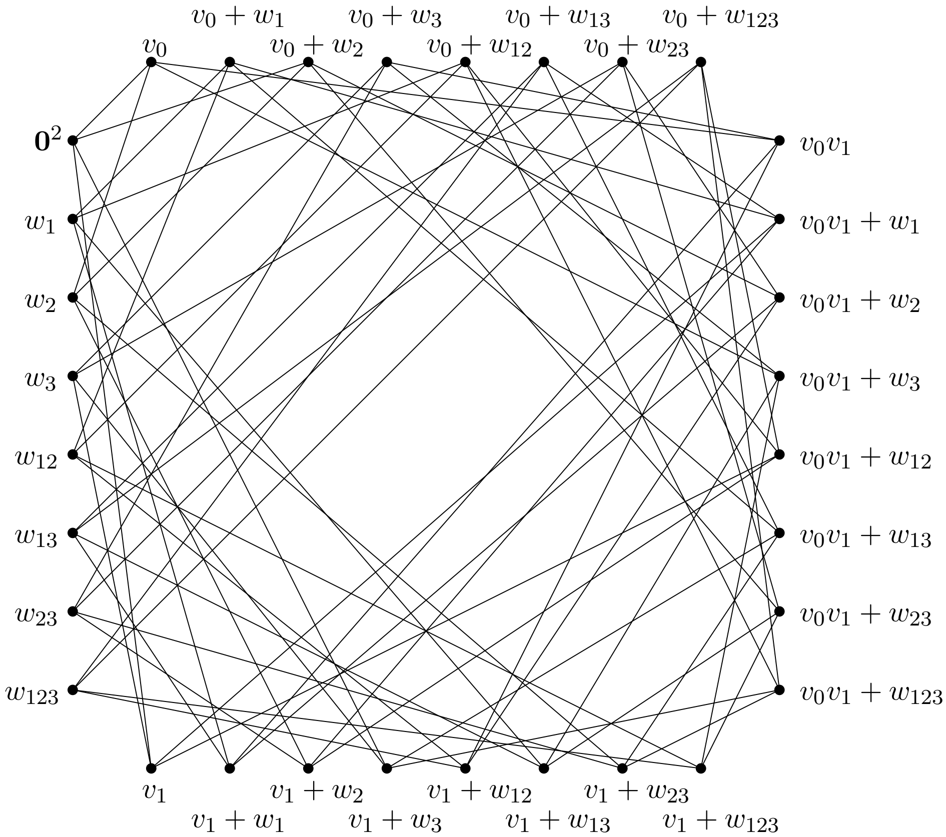

FIGURE 4. The graph $\mathcal{G}_2$

Computer calculations with MAGMA show that the graph $\mathcal{G}_5$ with $2^{13} = 8192$ vertices is the first graph which is *not Ramanujan.* Note also that we can fill into the tower of coverings (12) new intermediate covering graphs in order to obtain a new tower of coverings

$$\dots \widetilde{\mathcal{G}}_i \to \widetilde{\mathcal{G}}_{i-1} \to \dots \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_0,$$

where the covering indices of two subsequent graphs *are exactly* 2. This follows easily from the fact that every finite 2-group has a normal index 2 subgroup. The graphs $\widetilde{\mathcal{G}}_i$ are still expanders with the same positive lower bound on their spectral gaps (since every finite covering of a graph $\mathcal{G}$ inherits all the Laplace eigenvalues of $\mathcal{G}$). This "completed" tower of 2-fold coverings fits well to results of Bilu and Linial [BL]. They present a construction of 2-fold towers of covering graphs with nearly optimal spectral gap. They also conjecture, based on extensive numerical tests, that *every Ramanujan graph has a* 2*-fold covering which is again Ramanujan.* If their conjecture is true, there should be a different continuation of the sequence $\mathcal{G}_4 \to \mathcal{G}_3 \to \cdots \to \mathcal{G}_0$ by Ramanujan graphs.

The group G can also be used to obtain another family of Cayley graph expanders of minimal vertex degree 4, presented in Theorem 1, given by finite groups with two generators and *only four relations.*

Proof of Theorem 1: We conclude from Proposition 2.5 that $[x_1, {}_k\, x_0] = M_k(\alpha_{k+1}, \dots)$ (taking k in the index of α_{k+1} mod 3), and hence this commutator never represents the identity in G. By the normal subgroup theorem (see, e.g., [Marg2]), we know that the group G is just infinite. Consequently, all the groups G_k are finite. The corresponding Cayley graphs are expanders because G has Kazdhan property (T). Since $[x_1, {}_k\, x_0] \in G \cap H_k = N_k$, we conclude from Corollary 2.3 that

$$|G_{3i+j}| \geq [G : N_{3i+j}] \geq 2^{8i-1+\mu(j)}.$$

□

Remark 3.1. *Expanders are increasing families of finite graphs with a uniform positive lower bound on their* combinatorial Cheeger constants *(or* edge expansion ratios*). The combinatorial Cheeger constants for infinite regular tessellations $\mathcal{G}_{p,q}$ of the hyperbolic plane (i.e., every vertex is of degree p and every face is a q-gon) was exactly calculated in* [HJL] *and* [HiShi]*. Lower bounds for more general planar tessellations (in terms of combinatorial curvature) were derived in* [KP]*. It would be interesting to derive similar results for Euclidean and hyperbolic buildings and more general non-planar simplicial complexes. Note, however, that there is no simple relation between the Cheeger constants of infinite graphs and their finite quotients.*

4. Further properties of the group G and its pro-2 completion

Let us now have a closer look at the pro-2 completion $\widehat{G}_2$ of our group G. Since $N_i = G \cap H_i$ are finite index normal subgroups of G with $[G : N_i]$ equals powers of 2 and $\cap_i N_i = \{e\}$, G can be considered as a dense subgroup of $\widehat{G}_2$. Moreover, by [Lub1, Lemma 2.1], $\widehat{G}_2$ has a minimal pro-2 presentation given by $\langle x_0, x_1 | r_1, r_2, r_3 \rangle$, with the relations r_1, r_2, r_3 defined in (1). Consequently, every minimal presentation $\langle X | R \rangle$ of $\widehat{G}_2$ satisfies the *Golod-Shafarevich inequality*

$$|R| \geq \frac{|X|^2}{4} = 1,$$

since $\widehat{G}_2$ is not free. (Golod-Shafarevich theorem implies that every presentation of a finite p-group with minimal number of generators satisfies this inequality.) Further properties of the group $\widehat{G}_2$ are given in the following theorem.

Theorem 3. *The pro-2 completion $\widehat{G}_2$ of G satisfies the Golod-Shafarevich inequality even though it is infinite. $\widehat{G}_2$ is not 2-adic analytic and doesn't contain a free pro-2 subgroup, but it does contain a free subgroup of rank two.*

Proof. The statement concerning the Golod-Shafarevich inequality was already discussed before. Note that $\widehat{G}_2 \cap H_i$ is an infinite sequence of subgroups of finite index. Next we show that $\widehat{G}_2$ is not 2-adic analytic: We conclude from (6) that

$$\widehat{G}_2^{2^k} \subset \widehat{G}_2 \cap H_{2^k-1}.$$

Corollary 2.3 implies that $[G : N_n] \geq 2^{2n}$. Consequently, we have

$$[\widehat{G}_2 : \overline{\widehat{G}_2^{2^k}}] \geq [G : N_{2^k-1}] \geq 2^{(2^k)}.$$

By [DdSMS, Thm 3.16], $\widehat{G}_2$ cannot be of finite rank and therefore not 2-adic analytic.

The presentation of Γ given in (2) satisfies the conditions $C(3)$ and $T(6)$. (In fact, Γ is isomorphic to the group $G_3 = \langle x | r_3 \rangle$ in [EH, Example 3.3].) Thus we conclude with [EH] that Γ (and, therefore, also G and $\widehat{G}_2$) contains a free subgroup of rank two. On the other hand, $\widehat{G}_2$ cannot contain a free pro-2 subgroup since it is a linear group over a local field (see [BaL]). $\square$

Finally, let us state our conjectures which are based on MAGMA-computer calculations.

Conjecture 1. *Let $\lambda_i(G)$ and $\gamma_i(G)$ denote the groups in the lower exponent-2 series and the lower central series of G. Then we have*

$$\lambda_i(G) = G \cap H_i \text{ for } i \geq 1,$$

and

$$\lambda_i(G)/\lambda_{i+1}(G) \cong \gamma_i(G)/\gamma_{i+1}(G) \text{ for } i \geq 2.$$

If Conjecture 1 is true then Theorem 2 is still valid if we replace $\lambda_i(G)$ and $\lambda_i(G) \cap H_{i+1}$ by $\gamma_i(G)$ and $\gamma_{i+1}(G)$, respectively, and, consequently, the group Γ is of finite width 3 and of finite average width $(3+3+2)/3 = 8/3$. Moreover, the covering indices of our tower of expander graphs $\mathcal{G}_i$ are given by the periodic sequence $4, 8, \overline{4, 8, 8}$.

Computer calculations suggest that not only the group Γ is of finite width 3, but also all groups $\Gamma_{\mathcal{T}}$ introduced in [CMSZ, Section 4] and associated to prime powers $q = p^k$ with primes $p \neq 3$ (we exclude $p = 3$ to avoid torsion phenomena). Here we expect the following statements to be true:

Conjecture 2. *Let $\Gamma = \Gamma_{\mathcal{T}}$ be one of the groups introduced in* [CMSZ, Section 4]*, associated to a prime power $q = p^3$ with $p \neq 3$. Then we have the following 3-periodicity for the ranks of the abelian quotients $\gamma_i(\Gamma)/\gamma_{i+1}(\Gamma)$ of the lower central series for $i \geq 2$:*

$$\log_p[\gamma_i(\Gamma) : \gamma_{i+1}(\Gamma)] = \begin{cases} 3, & \text{if } i \equiv 0, 1 \mod 3, \\ 2, & \text{if } i \equiv 2 \mod 3. \end{cases}$$

5. Appendix

This section is devoted to the proof of Proposition 2.4. For any 3×3 matrix $\alpha \in M(3, \mathbb{F}_2)$ and $m, n \in \mathbb{Z}$, we denote by $E_{m,n}(\alpha)$ the infinite matrix, built up by 3×3 matrices, which vanishes everywhere expect for its 3×3 entry at position (m, n), which coincides with α. (m denotes the $3 \times \infty$ row and n denotes the $\infty \times 3$ column.) Moreover, given an infinite matrix A, built up by 3×3 matrices, let $\pi_{m,n}(A)$ denote its 3×3 entry at position (m, n). Obviously, we have $\pi_{m,n}(E_{m,n}(\alpha)) = \alpha$. For simplicity we sometimes denote $\pi_{m,n}(A)$ also by $A_{m,n}$.

Lemma 5.1. *Let $m, n \geq 1$, $\alpha \in M(3, \mathbb{F}_2)$ and $b \in \mathcal{S}$. Then we have*

$$\begin{aligned} \pi_{m,n}(M_0(b, \dots)^{-1} E_{m,n}(\alpha) M_0(b, \dots)) &= \alpha, \\ \pi_{m-1,n}(M_0(b, \dots)^{-1} E_{m,n}(\alpha) M_0(b, \dots)) &= -b(m-1) \cdot \alpha, \\ \pi_{m,n+1}(M_0(b, \dots)^{-1} E_{m,n}(\alpha) M_0(b, \dots)) &= \alpha \cdot b(n), \end{aligned}$$

where we have taken m and n mod 3 at the right hand side. Moreover, we have at all positions (m', n') with $m' > m$ or $n' < n$

$$\pi_{m',n'}(M_0(b, \dots)^{-1} E_{m,n}(\alpha) M_0(b, \dots)) = 0.$$

Proof: A straightforward calculation shows $M_0(b, \dots)^{-1} = M_0(-b, \dots)$. Then we have

$$\pi_{m',n'}(ABC) = \sum_{i,j} A_{m',i} B_{i,j} C_{j,n'},$$

and in particular

$$\pi_{m',n'}(A E_{m,n}(\alpha) C) = A_{m',m} \alpha C_{n,n'}.$$

The lemma follows now immediately from

$$\pi_{m-1,m}(M_0(b, \dots)^{-1}) = -b(m-1), \quad \pi_{n,n+1}(M_0(b, \dots)) = b(n+1),$$

and $\pi_{i,j}(M_0(b,\dots)^{\pm 1}) = 0$ for $j < i$. $\square$

Corollary 5.2. *We have*

$$M_0(b,\dots)^{-1} M_k(a_1,a_2,\dots) M_0(b,\dots) = M_k(a_1,c_2,\dots)$$

with $c_2(i) = a_2(i) - b(i)a_1(i+1) + a_1(i)b(i+k+1)$, *where the indices* $i, i+1, i+k+1$ *are taken mod* 3.

Proof: Note that

$$M_k(a_1,a_2,\dots) = I + \sum_{m=1}^{\infty} \left(\sum_{l=1}^{\infty} E_{m,m+k+l}(a_l(m)) \right),$$

and consequently,

$$M_0(b,\dots)^{-1} M_k(a_1,a_2,\dots) M_0(b,\dots) = I + C,$$

with

$$C = \sum_{m,l=1}^{\infty} M_0(b,\dots)^{-1} E_{m,m+k+l}(a_l(m)) M_0(b,\dots).$$

Lemma 5.1 implies that $I + C$ is of the type $M_k(c_1,c_2,\dots)$. Applying Lemma 5.1 again, we obtain the desired results for the entries $c_1(m)$ and $c_2(m)$ at the positions $(m,m+1)$ and $(m,m+2)$. $\square$

Proof of Proposition 2.4: We distinguish the cases $k=0$ and $k \geq 1$:
Case $k = 0$**:** One easily checks that

$$M_0(a_1,a_2,\dots)^{-1} = M_0(-a_1,d_2,\dots)$$

with $d_2(i) = a_1(i)a_1(i+1) - a_2(i)$ and, using Corollary 5.2,

$$\begin{aligned}
[M_0(a_1,a_2,\dots), M_0(b,\dots)] &= M_0(-a_1,d_2,\dots) M_0(a_1,c_2,\dots) \\
&= M_0(0,e_2,\dots) = M_1(e_2,\dots)
\end{aligned}$$

with $e_2(i) = d_2(i) - a_1(i)a_1(i+1) + c_2(i)$. This yields

$$e_2(i) = c_2(i) = a_2(i) = a_1(i)b(i+1) - b(i)a_1(i+1),$$

finishing this case.
Case $k \geq 1$**:** Now we have

$$M_k(a_1,a_2,\dots)^{-1} = M_k(-a_1,-a_2,\dots)$$

and, using again Corollary 5.2,

$$\begin{aligned}
[M_k(a_1,a_2,\dots), M_0(b,\dots)] &= M_k(-a_1,-a_2,\dots) M_k(a_1,c_2,\dots) \\
&= M_k(0,c_2-a_2,\dots) = M_{k+1}(c_2-a_2,\dots),
\end{aligned}$$

where

$$c_2(i) - a_2(i) = a_1(i)b(i+k+1) - b(i)a_1(i+1).$$

This settles the second case. $\square$

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