Ergodic Theory: Going with the (Geodesic) Flow

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The word "Ergodic"

The curious word *ergodic* was introduced by Boltzmann, and had its origin in two greek words:

ergon = work or energy

(Perhaps eflecting the fact that ergodic theorists are all hard working?) and

hodos = path or way

However, Boltzmann's knowledge of greek etymology may have been less than perfect ...

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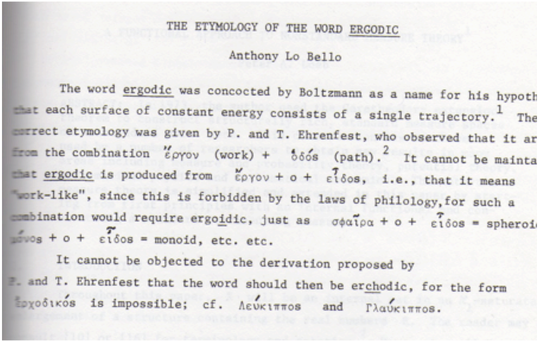
Thus perhaps we should be studying **Erchodic Theory**?



THE ETYMOLOGY OF THE WORD ERGODIC

Anthony Lo Bello

The word ergodic was concocted by Boltzmann as a name for his hypoth that each surface of constant energy consists of a single trajectory.[1] The correct etymology was given by P. and T. Ehrenfest, who observed that it ar from the combination ἔργον (work) + ὁδός (path).[2] It cannot be mainta that ergodic is produced from ἔργον + ο + εἶδος, i.e., that it means "work-like", since this is forbidden by the laws of philology, for such a combination would require ergoidic, just as σφαῖρα + ο + εἶδος = spheroi μόνος + ο + εἶδος = monoid, etc. etc.

It cannot be objected to the derivation proposed by P. and T. Ehrenfest that the word should then be erchodic, for the form ἐρχοδικός is impossible; cf. Λεύκιππος and Γλαύκιππος.

Physical Motivation

The original use of the expression "ergodic" relates to Boltzmann's ergodic hypothesis for systems of gas particles in a box.

For a large system of interacting particles in a equilibrium, the time averages are hoped to be close to the ensemble average, i.e., If we have a measurement (a function f, say) on the phase space M of all possible configurations then the average over the evolution of the system with time is equal to the average over all possible configurations.

Of course if there are approximately 10^{23} (Avogadro's constant) particles then the phase space is 6×10^{23} ...

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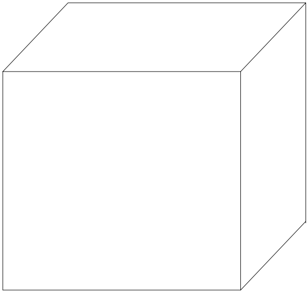
Particles in a box

Each of the 10^{23} particles has coordinates (x_1, x_2, x_3) and velocity vectors (v_1, v_2, v_3) contributing six dimensions to the configuration (phase) space.



The container for the particles.

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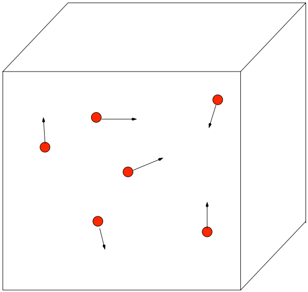

The configuration at time $t = 0$.

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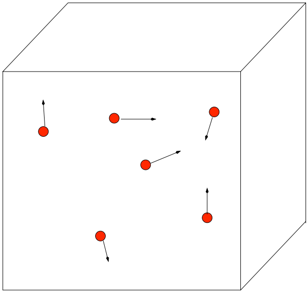

The configuration at time $t = t_1$.

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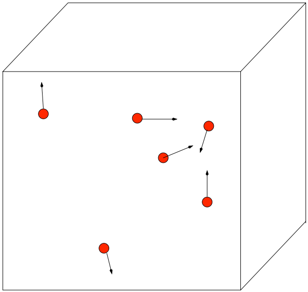

The configuration at time $t = t_2$.

Birkhoff ergodic theorem

In a more general setting, we let v denote some initial position (configuration) in the space M.
We then let $\phi_t(v)$ denote the new position (configuration) after time t.
Clearly

- $\phi_0(v) = v$ (i.e., nothing has moved after time zero); and
- $\phi_{t_1+t_2}(v) = \phi_{t_1}(\phi_{t_2}(v))$ (i.e., the position after time $t_1 + t_2$ starting from v is the same as the position after time t_1 after starting from the position $\phi_{t_2}(v)$ after time t_2).

Thus $\phi : M \to M$ is a flow.
Birkhoff's Ergodic theorem from 1931 (when it applies) says that for a typical $v \in M$ the orbits are evenly distributed:

$$\underbrace{\frac{1}{T}\int_0^T f(\phi_t v)dt}_{\text{average along orbit}} \to \underbrace{\int fdm}_{\text{average over space}} \quad \text{as } T \to \infty$$

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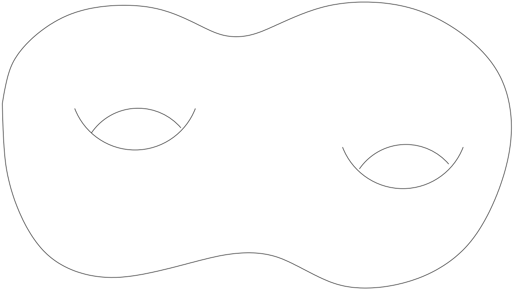
Geodesic flow on a surface

Another classical example of a flow is that of the geodesic flow on a surface. This has a phase space which is only three dimensional and consists of all tangent vectors to V of unit length.



A two dimensional surface V.

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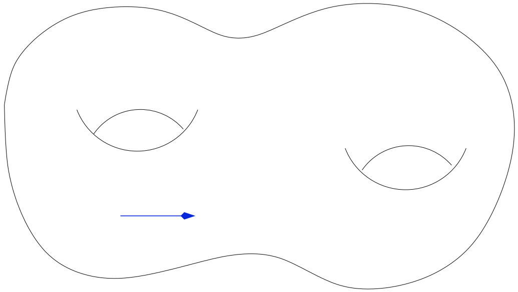

A (blue) unit tangent vector v

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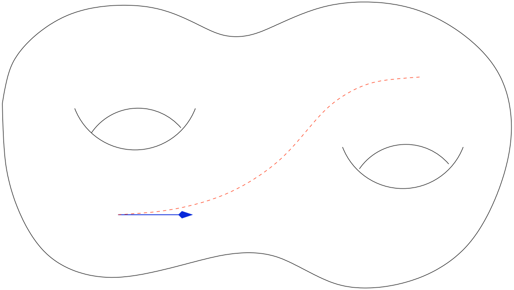

The (red) unit speed geodesic starting at v at time $t = 0$.

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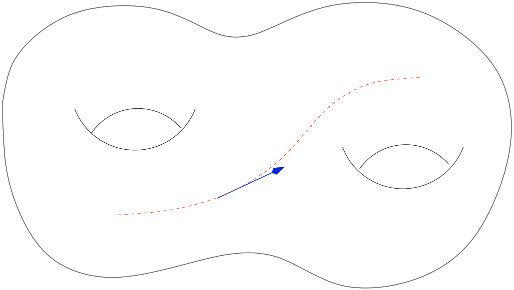

The (blue) unit tangent vector tangent to the geodesic at time t_1.

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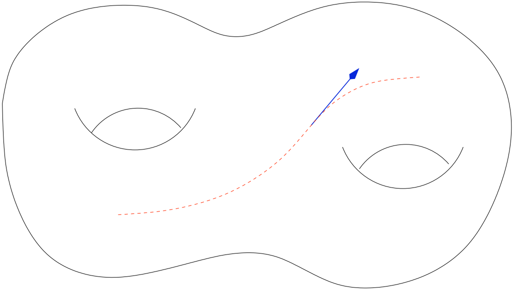

The (blue) unit tangent vector tangent to the geodesic at time t_2.

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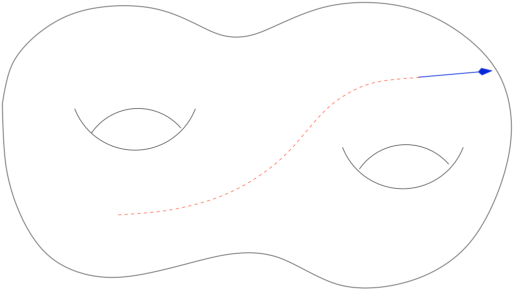

The (blue) unit tangent vector tangent to the geodesic at time t_3.

Negative curvature

We will mainly consider orientable surfaces with negative curvature: For each $x \in V$

$$\kappa(x) := 12 \lim_{r \to 0} \frac{\pi r^2 - \text{Area}(B(x,r))}{\pi r^4} < 0$$

- By a theorem of Poincaré the surface V must be a doughnut with $g \geq 2$ holes (i.e., g is the *genus*); and
- By a theorem of Hilbert such surfaces cannot be embedded isometrically in $\mathbb{R}^3$ (so the figures are just impressionistic).

In this case, for $\phi_t : M \to M$:

- Hedlund (1936) showed that there exists a dense orbit (i.e, *transitivity*); and
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The plan

We want to talk about applications of ideas from ergodic theory (and particularly geodesic flows) to three problems:

1. Geometry (Closed geodesics)
2. Mechanics (Dynamics of linkages)
3. Number Theory (Oppenheim Conjecture and Littlewood Problem)

followed by some propaganda for (Pure) Mathematics at Warwick.

Counting closed geodesics

If the vector $v \in M$ returns to itself under the flow (i.e., $\phi_l(v) = v$ for the smallest such $l > 0$) then it corresponds to a closed geodesic γ of length $l = l(\gamma)$ on the surface V.

In fact there are a countable infinity of closed geodesics $(\gamma_n)_{n=1}^{\infty}$ *(one in each free homotopy class = conjugacy class for the fundamental group).*

If we order the geodesics by length $0 < l(\gamma_1) \leq l(\gamma_2) \leq l(\gamma_2) \to \infty$ then we can ask:

Question: How many closed geodesics there are with length less than T, say, and how this behaves as $T \to \infty$?
(Or in terms of orbits for ϕ: How many closed orbits are there for the geodesic flow with least period l less than T?)

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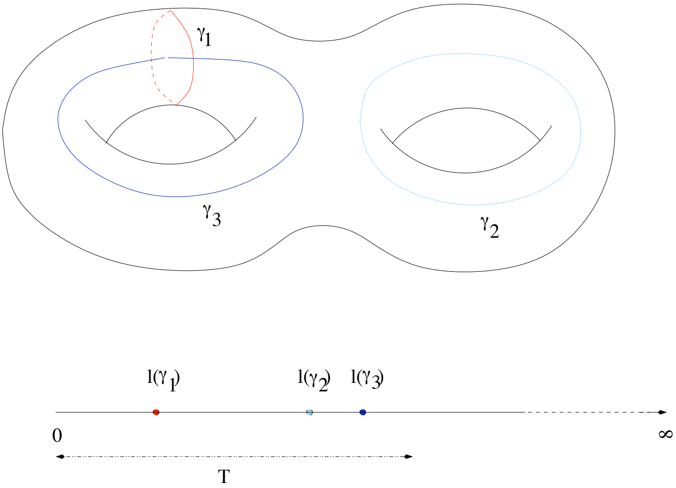
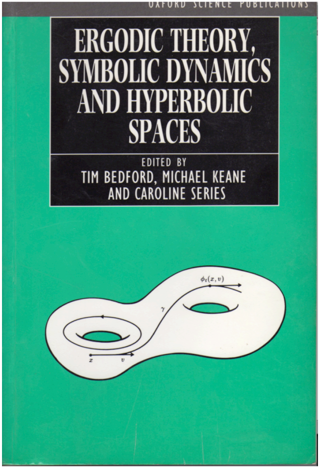
Lengths of closed geodesics



The proceedings of a conference from 1989



Margulis' Theorem. There exists a constant $h > 0$ such that the number $N(T)$ of closed geodesics of length at most T grows like $\frac{e^{hT}}{hT}$ as $T \to \infty$, i.e.,

$$\lim_{T \to +\infty} \frac{N(T)}{e^{hT}/hT} = 1.$$

This was first proved for surfaces of constant negative curvature by Selberg (in 1956) using trace formulae (using representation theory for $SL(2, \mathbb{R})$ and the Laplacian on V).

This was proved for surfaces of variable negative curvature by Margulis (in 1969) using ergodic theory.

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$$(p_n) = (2, 3, 5, 7, 11, 13, 17, 19, \cdots)$$

Primes Number Theorem. The number $P(t)$ of primes less than t grows like $t/\log t$, i.e., $\lim_{t\to\infty} \frac{P(t)}{t/\log t} = 1$

(Then if we set $t = e^{hT}$ then $P(t) \sim \frac{e^{hT}}{hT}$ as $T \to +\infty$).

- In particular, the original proof of the Prime Number Theorem using the Riemann Zeta Function motivates the use of Dynamical Zeta functions.
- The Riemann Hypothesis (or Conjecture) corresponds to better estimates on the number of primes $P(t)$. It remains unproved, although an analogue for closed geodesics on surfaces is known (Selberg-Huber for constant curvature; P.-Sharp for variable curvature).

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Application to Mechanics: Linkages

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Assume that

- Some of the ends are anchored to fixed points in the plane, about which they pivot;
- Some of the ends are connected to other ends, but are not fixed in the plane;

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Easy example

Consider a linkage consisting of two rods.
The first rod has one end anchored at the origin, say, in the plane, about which it pivots. The other end moves freely in the plane.

One end of the second rod is attached to the (moving) end of the first rod, and other end moves freely in the plane.

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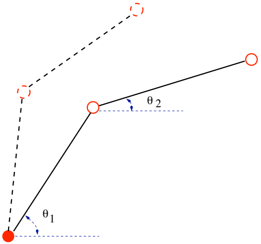
Simple linkage



The position of the linkage at any time is completely described by two angles $\theta_1, \theta_2 \in [0, 2\pi)$, which each of the arms makes to the horizontal, say. Thus this space corresponds to a two dimensional torus.

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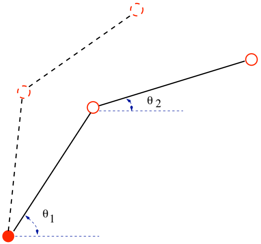

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Triple Linkage (or Three legged spider)

Consider three rods each having one end attached to the vertices of an equilateral triangle in the plane: $(0,0)$, $(1,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Consider three more rods, each of which has one end attached to the (free) end of one of the first three rods. Moreover, the other ends of each of the latter three rods are attached to each other.

Let us assume that the first three legs are relatively short compared with the other three legs. Then the space of linkage positions is equivalent to a surface of genus three (i.e., three holed doughnut).

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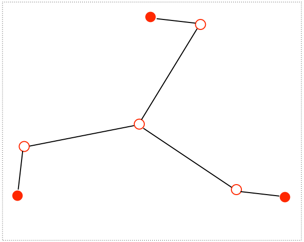

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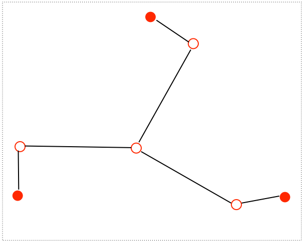

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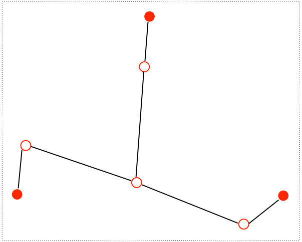

Hamiltonian Dynamics

In the general case, assume that we put masses on some of the movable joints (and that there is no friction, the rods have no inertia, etc). In particular, for any configuration $x \in V$ let us place equal masses m on the movable pivots $y_1, \cdots, y_n \in \mathbb{R}^2$, say.

The dynamics of motion of the linkage is governed by the usual Hamiltonian Dynamics. In particular, its is equivalent to that of the geodesic flow on the embedded surface

$$V \ni x \mapsto (y_1(x), \cdots, y_n(x)) \in \mathbb{R}^{2n}.$$

Aim: Therefore, we would like to change the parameters (lengths of rods) so that the embedded surface $V \subset \mathbb{R}^{2n}$ has negative curvature (or close to it).

In the case of the simple linkage the phase space is a torus, but then by the Gauss Bonnet Theorem $\int \kappa dm = 0$ ... so there is always "as much positive curvature as negative curvature".

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Hunt-Mackay Theorem

The triple linkage is more promising, since the phase space has genus $g = 3$ and so by Gauss Bonnet $\int \kappa dm = -2(g-1) = -4$.

Theorem (Hunt-Mackay). If the first three rods are chosen sufficiently short and the second set of rods are chosen to have lengths close to 1 the curvature of phase space will have negative curvature (except possibly on small islands on positive curvature).

In particular, the associated dynamics will be ergodic (thus typical orbits are uniformly distributed, etc).

Corollary. The number $N(T)$ of closed orbits of length less than T grows like $\frac{e^{hT}}{hT}$ as $T \to +\infty$.

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3.1. Indefinite Quadratic forms

Let $Q(x_1, \cdots, x_n)$ be an indefinite quadratic form ($n \geq 3$) which is not a multiple of one with rational coefficients.

For example, $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - \sqrt{2}x_3^2$.

Margulis Theorem (Oppenheim Conjecture)

$$Q(\mathbb{Z}^n) = \{Q(x_1, \cdots, x_n) : x_1, \cdots, x_n \in \mathbb{Z}\} \subset \mathbb{R}.$$

is dense in the real line.

The result was conjectured by Oppenheim in his PhD thesis in 1929. The result was proved by Margulis in 1986 (using the ergodic theory of flows).

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3.2. The Littlewood conjecture

Let α be an irrational number. By Dirichet's theorem, there are infinitely many $q_n \in \mathbb{Z}$ such that

$$\|q_n\alpha\| := \inf\{|p - q_n\alpha| : p \in \mathbb{Z}\} \leq \frac{1}{q_n} \to 0$$

Littlewood Conjecture (from 1930) : Given two irrationals α, β we can choose q_n such that

$$q_n\|q_n\alpha\|\|q_n\beta\| \to 0 \tag{1}$$

The conjecture hasn't been fully solved. However ...

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Ergodic Theory

Both of the previous theorems can be proved by reformulating them in terms of the properties of suitable flows on homogeneous spaces. These are generalizations (sort of ...) of the algebraic formulation of geodesic flows:

- Let $SL(2,\mathbb{R})$ be 2×2 matrices of determinant one;
- Let $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in SL(2,\mathbb{R})$ for $t \in \mathbb{R}$; and
- Let $\Gamma \subset SL(2,\mathbb{R})$ be a discrete group with $\mathrm{Vol}(\Gamma \backslash SL(2,\mathbb{R})) < +\infty$.

The flow $\phi_t : \Gamma \backslash SL(2,\mathbb{R}) \to \Gamma \backslash SL(2,\mathbb{R})$ defined by $\phi_t(\Gamma g) = \Gamma g g_t$ is an algebraic version of the geodesic flow on a surface of *constant* negative curvature.

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Being a PhD student ... at Warwick

Usually EPSRC funded students at Warwick have a grant for 3.5 years. Many students follow the traditional PhD plan:

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What should a student expect from a PhD?

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Why choose Warwick?

- The Mathematics Department has strength in a large number of research areas, including: *Algebra, Analysis, Geometry, Dynamical Systems, Number Theory, Probability and Stochastic Processes, Topology, Continuum Mechanics, Computational Mathematics and Mathematical Biology, etc.*
- The number of permanent full time academic staff is 62.
- The number of post doctoral researchers is approximately 25 (EPSRC, Marie Curie EU fellows,etc.).
- Warwick has the largest Doctoral Training Account (i.e., PhD grants) in the UK.
- We have a Taught Course Centre jointly with Oxford, Imperial, Bath and Bristol.

Mathematics Research Centre

Research at Warwick is coordinated by the Mathematics Research Centre:

- Annual symposium: Concentrating on topical research.
- Last 7 years: 113 workshops;
- more than 2750 individual talks and seminars; and
- hosted at least 3600 international visitors.
- The Department is a EU-Marie Curie training site in three areas (Algebraic Geometry, Dynamical Systems and Stochastic Analysis).
- Three Doctoral Training Centres (Complexity Sciences, Systems Biology, and Molecular Organisation and Assembly of Cells)
- The number of graduate students over 100 (including over 70 PhD students).

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Pure mathematics				
Imperial College	3.25	5*	40	22
Warwick	3.15	5	35	32
Oxford	3.10	5*	35	55
Cambridge	3.05	5*	30	55
Bristol	2.95	5	30	35
Edinburgh	2.95	5*	25	31
Heriot-Watt	2.95	n/a	25	10
Bath	2.85	5	25	10
Aberdeen	2.85	5	20	14
King's College	2.80	5	20	13
Manchester	2.75	n/a	20	27
UCL	2.75	5	20	15
Durham	2.70	5	20	15
East Anglia	2.70	5	15	7
Sheffield	2.70	5	15	17
Queen Mary	2.70	5	10	20
Loughborough	2.65	n/a	10	11
Nottingham	2.60	5	15	15
Glasgow	2.60	5	15	16
Exeter	2.60	4	10	5
Leeds	2.60	5	10	23
Leicester	2.60	5	10	10
Birmingham	2.55	5	15	18
York	2.50	5	10	12
Lancaster	2.45	4	10	10
Liverpool	2.45	5	10	15
LSE	2.45	n/a	5	13
Southampton	2.45	5	5	16
Queen's Belfast	2.45	3a	5	8
Newcastle	2.35	5	5	10
Swansea	2.35	5	5	21
London Metropolitan	2.30	n/a	10	4
St Andrews	2.30	5	5	12
Aberystwyth	2.30	3b	5	8
Cardiff	2.30	5	5	30
Kent	2.25	3a	0	6
Open	2.05	4	5	17
Royal Holloway	1.65	5	0	27
National profile	**2.69**	**n/a**	**18**	**685**