Research in Mathematical Analysis – Some Concrete Directions

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Prospects in Mathematics, Durham, 9th January 2009

Outline

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1. Overview
2. Fourier Analysis
   - A case study – the restriction of the Fourier Transform

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and you may or may not have covered some material in the area of PDEs – e.g. Laplace's equation, wave equation – most likely as part of a "methods" course in applied maths.

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DISCLAIMER: The remarks I'll make on this issue are personal views. I'll restrict myself to "Pure" Analysis – including PDE – and I'll not discuss Applied Analysis at all.

Different Styles of Mathematical Analysis

- "Abstract" directions – areas in which the objects of analysis such as Banach spaces, Hilbert spaces and classes of operators acting on them are studied *in their own right and for their own sake.* Typical starting point: "Let X be a Banach space....."; the aim is to understand the *internal structure* of such objects. Some areas of current activity : C^*-algebras, operator algebras, operator spaces; Banach algebras. (The operator algebras group of areas has good connections with Mathematical Physics.)

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- "Mixed" directions – concrete situations where abstract methods are prominent; abstract settings where the analysis is modelled on a previously understood concrete situation – e.g. ergodic theory (shift operators) & dynamical systems; operator theory; "local" theory of Banach spaces – the study of $\mathbb{R}^n$ as a Banach space with particular attention to dependence on n; probabalistic methods.

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For the rest of the talk I'll concentrate on the Real Variables theme within "concrete" directions, beginning with some discussion of some of the ideas currently important in Fourier Analysis, and then we'll see how they link in with other areas.

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The theme throughout is the interplay between specific operators and geometrical considerations, sometimes based on symmetry, and how this interplay is measured using specific spaces adapted to the geometry at hand. Functional analysis and measure theory provide the *language* for this discussion.

Fourier Analysis – basics

For $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx.$$

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- $\widehat{f * g} = \widehat{f}\widehat{g}$ for all suitable f, g
- $\widehat{\partial f/\partial x_j}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$ – smoothness of f implies decay of its Fourier transform

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The restriction paradox

In particular, if $f \in L^p$ with p a little bit larger than 1, all we can expect (via the Hausdorff-Young result) is for its Fourier transform to lie in L^q where $1/p + 1/q = 1$, and thus $\widehat{f}$ is defined in principle only *almost everywhere* on $\mathbb{R}^n$, **not** genuinely pointwise.

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Nevertheless we are about to see that it makes **perfectly good sense** to talk about $\widehat{f}$ restricted to a sphere.

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Then we simply calculate:

$$\int_{\mathbb{S}^{n-1}} |\widehat{f}(x)|^2 d\sigma(x) = \int \overline{\widehat{f}}\,\widehat{f} d\sigma$$
$$= \int \bar{f} f * \sigma^{\vee} \leq \|f\|_p \|f * \sigma^{\vee}\|_q \leq C\|f\|_p^2$$

if p and r are related by $1/p + 1/q = 1$, $1/q = 1/p + 1/r - 1$ and $r > 2n/(n-1)$. Unravelling, this boils down to $1 \leq p < 4n/(3n+1)$, and so for p in this range, $\widehat{f}$ **exists as a member of** $L^2(\mathbb{S}^{n-1})$.

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So we see the following general paradigm emerging:
Curvature of a surface $\implies$ decay of Fourier transform of surface measure $\implies$ boundedness of operators on spaces adapted to the geometry of the surface.

As with any really good piece of mathematics, the argument raises more questions than it answers:

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- Given a curved submanifold, in what precise way does its "curvature" affect matters? Is there an "optimal" choice of measure to put on it to make things work well?

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Example (Jim Wright and co-authors) A new affine isoperimetric inequality for the class of polynomial curves in $\mathbb{R}^n$.

Let $\Gamma : I \to \mathbb{R}^n$ be a curve all of whose components are polynomial. The total affine curvature of Γ is the quantity

$$A(\Gamma) = \int_I \det\left(\Gamma'(t), \Gamma''(t), \dots, \Gamma^{(n)}(t)\right)^{2/n(n+1)} dt.$$

Then there is a constant C depending only on the degree of Γ and the dimension n so that

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Again, more questions arise: What about non-polynomial curves? What about extremals and best constants? What about higher-dimensional surfaces?

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Any odd “methodsy” bits can be easily picked up as you go along.

PDEs in Pure Maths, cont'd

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The main questions which arise become, often, questions of boundedness (or continuity) of certain specific linear or nonlinear operators on certain specific spaces adapted to the problems at hand.

There is a great deal of investment (both money and people) in this area curently in the UK – average academic job prospects for a good PhD graduate in theoretical PDE are somewhat better than those in maths more generally.

Laplace's equation

Example 1. Laplace's equation on "rough" domains. Let $G \subseteq \mathbb{R}^n$ be a domain whose boundary is not presumed to be smooth. So it can have edges, corners, even possibly a fractal-like structure. For many reasons it's important to understand the equation

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When G is the unit ball or the upper-half-space [with the extra condition that u vanish at ∞ thrown in], this is classical Fourier analysis, and the unique solution is obtained by integrating f against the Poisson kernel.

Even in this case, it is not immediately clear in what sense the solution $u(x)$ converges to the boundary data f as x moves towards the boundary as f is only defined almost everywhere. To handle this we need "maximal" functions.

Laplace's equation

Example 1. Laplace's equation on "rough" domains. Let $G \subseteq \mathbb{R}^n$ be a domain whose boundary is not presumed to be smooth. So it can have edges, corners, even possibly a fractal-like structure. For many reasons it's important to understand the equation

$$\triangle u = 0 \text{ on } G$$

with boundary data $f \in L^p(\partial G)$ for some $1 < p < \infty$.

In the general case there are many issues: what is the measure to be used on ∂G? (There are at least two possible natural candidiates). Can we "construct" a Poisson kernel and/or a Green's function? In what sense does the resulting Poisson integral actually solve the problem? Are solutions unique? Do we get almost-everywhere convergence of the solution back to the boundary data?

Martin Dindos and his group work on questions like these.....

NLS

Example 2. Nonlinear Schrödinger equation.

The linear Schrödinger equation for $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ is

$$\triangle u = i\partial u/\partial t$$

with initial data $u(x, 0) = f(x)$.

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We can (in principle) write down the solution to this equation:

$$u(x, t) = f * K_t(x) \text{ where } K_t(x) = t^{-n/2} e^{2\pi i|x|^2/t}.$$

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The nonlinear Schrödinger equation introduces a nonlinear function $h(u)$ (which we may take to be essentially a monomial) and asks to solve

$$\triangle u - i\partial u/\partial t = h(u), \text{ with } u(x, 0) = f.$$

NLS, cont'd

Typically, the analysis of NLS falls into two parts:

- Finding a complete metric spaces of functions and a map between them so that the solution is a fixed point of this map – nonlinear analysis

NLS, cont'd

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Questions like this are investigated by Nikolaos Bournaveas and Pieter Blue.

NLS and restriction

Amazing link: the Schrödinger solution operator

$$f \mapsto u(x,t) = f * K_t(x) := S_t f(x)$$

is PRECISELY the adjoint of the restriction operator for the paraboloid applied to $\widehat{f}$.

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That is, if we define $\mathcal{R}$ to be the restriction map taking functions on $\mathbb{R}^{n+1}$ to functions on $\mathbb{R}^n$ given by

$$(\mathcal{R}g)(x) = \widehat{g}(x, |x|^2/2)$$

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This means that all of the theory developed for the (Fourier Analytic) restriction phenomenon is immediately applicable to problems in nonlinear PDE! In the PDE literature these are called "Strichartz estimates".

Outline

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Kakeya sets

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It turns out that if we could completely solve the restriction problem for the Fourier transform, the conjecture would follow by the following route:

Restriction implies Kakeya

- Suppose $\mathcal{R} : L^{2n/(n+1)}(\mathbb{R}^n) \rightarrow L^{2n/(n+1)}(\mathbb{S}^{n-1})$ boundedly. (A slight lie here.) By duality, $\mathcal{R}^* : L^{2n/(n-1)}(\mathbb{S}^{n-1}) \rightarrow L^{2n/(n-1)}(\mathbb{R}^n)$.

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- Apply this to a well-chosen family of examples, and then average over the family, yielding

$$\int_{\mathbb{B}} \left(\sum_{T \in \mathcal{T}} \alpha_T \chi_T \right)^{n/(n-1)} \leq C_n (\log N) N^{-(n-1)} \sum_T \alpha_T^{n/(n-1)}$$

whenever $\mathcal{T}$ is a family of rectangles of sides $1/N \times 1/N \times \cdots \times 1$, for a large parameter N, with one in each of the essentially N^{n-1} different directions.

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- This implies

$$\frac{|\cup_{T \in \mathcal{T}} T|}{\sum_{T \in \mathcal{T}} |T|} \geq \frac{C_n}{(\log N)^{n-1}}$$

which is a quantitiative version of the claim on the dimension of a Kakeya set.

$n = 2$ and maximal functions

The previous inequality

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has two noteworthy features:

- When $n = 2$, the exponent $n/(n-1)$ is just 2, and one can simply multiply out to prove it, using one's knowledge of the area of the intersection of two rectangles, i.e. a parallelogram!

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- When $n = 2$, the exponent $n/(n-1)$ is just 2, and one can simply multiply out to prove it, using one's knowledge of the area of the intersection of two rectangles, i.e. a parallelogram!
- In general, the inequality has a dual form expressed in terms of *maximal functions*: let

$$M_N f(x) = \sup_{x \in T} \frac{1}{|T|} \int_T f$$

where the sup is taken over the family of all $1/N \times 1/N \times \cdots \times 1$ rectangles T passing through x. Then it's equivalent via duality to

$$\|M_N f\|_n \leq C_n (\log N)^{(n-1)/n} \|f\|_n.$$

Higher dimensions?

When $n \geq 3$ one cannot simply multiply out. Partial progress has been made by various authors. Recently, with Bennett and Tao, we considered a *multilinear* variant of the main inequality and proved it "up to end points" using a novel heat-flow method. The main "geometric" interpretation of our results is as follows:

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Consider a family $\mathcal{L}$ of M lines in $\mathbb{R}^n$. Define a *joint* to be an intersection of n lines in $\mathcal{L}$ lying in no affine hyperplane. We say a joint is *transverse* if the parallepiped formed using unit vectors in the directions of the lines has volume bounded below.

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Last month, Guth and Katz disposed of the word "transverse" and the "+". They used totally unrelated methods – topology, algebraic geometry, cohomology, commutative diagrams, building on work of Gromov. These have further implications for "pure" Geometric Measure Theory which have yet to be explored.....

Outline

Anti-Kakeya sets

Kakeya sets contain entire line segments in each of a large set of directions. An **Anti-Kakeya set** is one which contains only a *small* amount of mass in any line or tube. So such sets are **small** and it's natural to ask how "large" such sets may be.

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Not just an idle curiosity – Mizohata–Takeuchi conjecture:

$$(*) \int_{\mathbb{R}^n} |\mathcal{R}^* g(x)|^2 w(x) dx \leq C_n \sup_T w(T) \int_{\mathbb{S}^{n-1}} |g|^2 d\sigma,$$

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To test (*) we thus need good examples of w for which

$$\sup_T w(T) << \|w\|_{(n+1)/2},$$

i.e. **whose mass in any tube is small compared with total mass.**

A Challenge

In this spirit, consider an $N \times N$ array of black and white unit squares. How many squares can be coloured black to that no strip of width 1 meets more than two of them?

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The true orders N^{α} in these and similar problems are unknown.

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This notion arises in Fourier Analysis when considering higher-dimensional analogues of Riemann's localisation theorem (stating that convergence of a Fourier series at a given point is dictated entirely by the values of the function near that point).

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Question 1. Do there exist non-tube-null sets of each dimension greater than $n-1$?

Question 2. Is the radial outer 2 quarters Cantor set based on [1,2] tube-null?

Question 3. Do there exist tube-null Kakeya sets? Is *every* Kakeya set tube-null?

Outline

Given the long history of interplay between Fourier Analysis and Number Theory it's hardly surprising that this persists to the modern day. So I'll just sloganise here:

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- There are certain conjectures of Bonami, Garrigos and Seeger concerning variants of the L^2 restriction result which look likely to have a solution in terms of number-theoretic phenomena such as the number of representations of integers as sums of three squares.

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- There are certain conjectures of Bonami, Garrigos and Seeger concerning variants of the L^2 restriction result which look likely to have a solution in terms of number-theoretic phenomena such as the number of representations of integers as sums of three squares.
- Jim Wright is developing a programme of heuristics linking results for sublevel sets, oscillatory integrals and averaging opertors in Fourier Analysis to their number-theoretic counterparts.

Outline

In each of the three Fourier Analytic case studies, we've seen how a well-chosen question or a crucial observation leads to an entire research programme revealing a rich seam of mathematical ideas, replete with general philosophies, myriad variants and (most importantly) powerful links with other areas of mathematics. Exactly the same holds for theoretical PDEs.

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This (in my opinion) is the hallmark of an area which is exciting and promising for PhD students with a taste for concrete analysis to go into.

Outline

Some places to consider doing a PhD in.....

Fourier Analysis: Edinburgh, Birmingham, Cambridge (esp. additive combinatorics & “quadratic” Fourier Analysis), Glasgow

Rigorous real-variable PDE: Edinburgh, Heriot-Watt, Warwick, Bath, Oxford, Cambridge, Imperial

GMT (and associated combinatorics): St Andrews, (Edinburgh), Warwick, UCL, Open U.

Spectral theory of PDE: KCL, Cardiff, Bristol, UCL, Imperial.

Local theory of Banach spaces and associated probabalistic analysis: UCL

“Abstract” analysis: Leeds, Newcastle, Queen’s Belfast (Banach Algebras) Aberdeen, Glasgow, Lancaster (C*-algebras etc.)

THIS LIST IS FAR FROM EXHAUSTIVE!

Further reading and contacts

The following books give an introduction to Fourier Analysis at the PhD level:

J. Duoandikoetxea, Fourier Analysis (American Math. Soc. Graduate Studies in Mathematics)

T. Wolff, Lectures on Harmonic Analysis (Amer. Math. Soc. University Lecture Series)

www.maths.ed.ac.uk/research/show/group/4

email: A.Carbery@ed.ac.uk

Thanks for your attention –

And good luck with your choices!

I'm happy to talk to any of you and try to answer any questions informally throughout the rest of the meeting.