Solitons

September 29, 2006

- General references: Drazin and Johnson - this course organized a bit differently
- http://www.maths.dur.ac.uk/~dma0saa/Solitons
- First term: general ideas about solitons+ some specific methods (Backlund transforms and Hirota method
- Second term: A powerful general technique called inverse scattering. Developed in late 1960’s

1 Introduction

1.1 What is a soliton?

‘Experimental’ discovery by John Scott-Russell in 1834 on the Forth Clyde Canal. “A rounded smooth well-defined heap of water”, going along at about 8mph. Different heights \( \rightarrow \) different velocities.

**Definition:** A soliton is a solution to a p.d.e. which is

1. localized,

2. keeps its localized form over time

3. is preserved under interactions with other solitons
Nowadays can use the relevant equations for water waves and solve numerically to show these properties, as above. These were plotted using the program in the appendix A in mathematica. The Korteweg de Vries equation is the one we need (extension from the simplest linearized equation which is valid for infinitessimally small displacements)

\[ u_t + 6uu_x + u_{xxx} = 0 \]  

(1)

It is instructive to see what solitons are ‘made of’ by looking at what happens if we remove certain terms from the KdV equation. The first option is to eliminate the non-linear term;

\[ u_t + u_{xxx} = 0 \]  

(2)

The result is **dispersion** (waves spread and energy disperses);

![Dispersion](image)

The second option is to remove the dispersive term and leave only non-linearity

\[ u_t + 6uu_x = 0 \]

The result is **breaking** (waves energy concentrates until it becomes singular);

![Breaking](image)

A soliton is a stable balance of these two concentrating and dispersing effects. Not all non-linear p.d.e.s admit such solutions. Those that do are called “**integrable**”. Property 3 usually implies that there is 1 space and one time dimension.
1.2 Relevance of solitons

- Applied maths - water waves, optical fibres etc.
- Particle physics - models of elementary particles like protons, neutrons etc.
- “Pure” mathematics; particularly since the most recent discoveries, studies of solitons involves many different and developing areas

1.3 Relevance to inverse scattering problems

Strictly speaking the word Soliton (coined by Kruskal and Zabusky in 1965) refers to situations where the soliton forms part of a more complicated evolution involving maybe two or more solitons plus possibly plain wave components. (A single isolated wave is called a solitary wave although I won’t bother making a distinction.) If we begin with an initial configuration that is not a solitary wave then soon the different components begin to emerge. For example consider the evolution of the following initial gaussian form of $u$ on a periodic interval;

![Graphs of initial and final configurations](image)

The initially the wave radiates plane waves and the soliton content emerges. Looking at the final configuration (where the soliton has gone once round the periodic interval) it is clear that the soliton is a persistent component. There could have been two or more solitons emerging depending on the initial configuration. The decomposition of functions into these components will be treated in the 2nd term.
2 Waves, Dispersion and Dissipation

The usual situation for a localized water wave is that it spreads out as it travels - it disperses. This prevents it from being a soliton. Need to understand this phenomenon first. Consider some examples;

1. Wave equation (linear)
   \[ \frac{1}{v^2} u_{tt} - u_{xx} = 0 \]  

   D’Alembert’s solution is \( u(x,t) = f(x- vt) + g(x + vt) \). Superposition of left and right moving waves - trivially shape preserving

2. First order linear equation
   \[ \frac{1}{v} u_t + u_x = 0 \]  

   Now \( u = f(x- vt) \) and again all waves move with fixed speed \( v \). No dispersion.

3. Klein-Gordon equation
   \[ \frac{1}{v^2} u_{tt} - u_{xx} + m^2 u = 0 \]  

   Try the solution \( u = e^{i(kx - \omega t)} \). Get a dispersion relation, \( \omega = v\sqrt{k^2 + m^2} \). \( k \) is the wave-number and \( \omega \) is the frequency. The velocity of the wave crests is \( c = \omega/k \) so that different wave-numbers move at different speeds.

   If we make a lump by superposition of different wavelengths,
   \[ u(x,t) = \int_{-\infty}^{\infty} f(k)e^{i(kx - \omega t)}dk, \]  

   then it will disperse.

   • **Phase-velocity;** \( c(k) = \omega(k)/k \); so for the above example have \( c(k) = \frac{v\sqrt{k^2 + m^2}}{k} \).

   • **Group-velocity;** \( c_g(k) = \frac{d\omega}{dk} \). This is roughly the speed the lump \( u(x,t) \) moves at while it disperses. Proof a bit messy - consider an example in a minute.

   NB : for KG equation, have \( c > v \) but \( c_g = \frac{k}{\sqrt{k^2 + m^2}} < v \). In QM the KG equation can describe a particle moving at speed \( c_g \). The speed of light is \( v \), so even though \( c > v \) relativity is OK because the signal is moving at \( c_g \).

   NB2 : lump with varying \( k \)-uncertainty principle. In words, smaller particles (\( \delta x \)) need larger average \( k \), and so \( \delta k \) is bigger.
2.1 An example of group velocity; the Gaussian wavepacket

\[ f(k) = e^{-a^2(k-k_0)^2} \]  

(7)

Let \( u = Re(z) \) where

\[ z(x, t) = \int_{-\infty}^{\infty} e^{-a^2(k-k_0)^2} e^{i(kx-\omega t)} dk \]

Most of \( u \) comes from \( k \approx k_0 \), so expand \( \omega(k) \) near \( k_0 \).

\[ \omega(k) \approx \omega(k_0) + \frac{d\omega}{dk}(k-k_0) + \ldots \]

\[ = \omega(k_0) + c_g(k-k_0) + \ldots \]

so that

\[ z(x, t) \approx \int_{-\infty}^{\infty} e^{-a^2(k-k_0)^2} e^{i(kx-t(\omega(k_0)+c_g(k-k_0)))} \]

\[ = e^{i(\bar{k}x-\omega(k_0)t)} \int_{-\infty}^{\infty} e^{-a^2k^2} e^{i(kx-c_gt)} \]

\[ = e^{i(\bar{k}x-\omega(k_0)t)} \int_{-\infty}^{\infty} e^{-a^2k^2} e^{i(kx-c_gt)} \]

\[ = e^{i(\bar{k}x-\omega(k_0)t)} e^{-\frac{1}{4a^2}(x-c_gt)^2} \int_{-\infty}^{\infty} e^{-a^2(k-k_0)^2} \]

\[ = \sqrt{\frac{\pi}{a}} e^{i(\bar{k}x-\omega(k_0)t)} e^{-\frac{1}{4a^2}(x-c_gt)^2} \]

The first factor is a ‘travelling’ plane wave part. The second piece is an envelope centred on \( x - c_g t \), with width \( 2a \). To see the dispersion need to expand to higher order in \( k - k_0 \). An example numerical solution is shown above for the KG equation at time \( t = 0 \) and later time (the dispersion, which must go like \( d^2\omega/dk^2 \), is clearly visible).

5
To see the dissipation expand $\omega(k)$ to higher order:

$$\omega(k) \approx \omega(k) + \frac{d\omega}{dk}(k - \bar{k}) + \omega''(k - \bar{k})^2 + \ldots$$

$$= \omega(k) + c_g(k - \bar{k}) + \chi(k - \bar{k})^2$$

where $\chi = \omega''(\bar{k})$. Now

$$z(x, t) = \int_{-\infty}^{\infty} e^{-(a^2 + i\chi t)(k - \bar{k})^2} e^{i(kx - t(\omega(k) + c_g(k - \bar{k})))} dk \quad (8)$$

so we can simply replace $a$ with

$$b(t) = \sqrt{a^2 + i\chi t} \quad (9)$$

and repeat previous steps to arrive at

$$z(x, t) = \sqrt{\pi} b e^{i(kx - \omega(k)t)} e^{-\frac{1}{4\chi t^2}(x - c_g t)^2} \quad (10)$$

To see the dissipation we can use an **envelope function** \(|z|\) noting that

$$|z| \geq \text{Re}(z) = u \quad (11)$$

it is not difficult to show

$$|z| = \sqrt{\frac{\pi}{|b|}} e^{i(kx - \omega(k)t)} e^{-\frac{1}{4\chi t^2}(x - c_g t)^2} \quad (12)$$

where

$$|b| = a \left(1 + \frac{\chi^2 t^2}{a^4}\right)^{\frac{1}{2}} \quad (13)$$

So now the envelope function is a gaussian hump of width $2|b|$ that spreads out over time.

### 2.2 Examples with dissipation

If $\omega$ is not real we get **dissipation**.

- First order dissipation;

  $$\frac{1}{v} u_t + u_x + \alpha u = 0 \quad (14)$$

  where $\alpha$ is real. Here $\omega(k) = v(k - i\alpha)$ so not real. The plane wave solutions are

  $$u = e^{ik(x-vt)} e^{-\alpha t} \quad (15)$$

  If $\alpha > 0$ the amplitude decays to zero as $t \to \infty$(dissipation). No dispersion here.
• The heat equation

\[ u_t - \alpha u_{xx} = 0 \]  

(16)

Here have \( \omega(k) = -i\alpha k^2 \) and a solution is

\[ u = e^{ikx} e^{-\alpha k^2 t} \]  

(17)

Again dissipation.

### 2.3 Summary

• Various linear equations describing waves in 1-dimension

• dispersion relation \( \omega(k) \).

• If \( \omega \) complex have can have dispersion or dissipation.

• Both generally break up waves.

• A different behaviour if \( k \) is imaginary is \textit{evanescence}. The wave dies with distance (common in QM tunneling phenomena).

• Solitons are not of this type and they don’t disperse - non-linearity is crucial ingredient.
3 Travelling waves

In this section we’ll find some travelling wave solutions such as the KdV soliton. Travelling waves are solutions of the form \( u(x,t) = f(x - vt) \). i.e. some fixed profile travelling at constant speed \( v \). If the solution is localized it’s part way to being a soliton. The thing we have to check is the preserved form under interaction (property 3). So solutions of \( u_{tt} - u_{xx} = 0 \) are trivial in the sense that there is no interaction.

3.1 KdV soliton

\[
\begin{align*}
\frac{df}{f \sqrt{v - 2f}} &= \pm (x - x_0 - vt) \\
\int \frac{df}{f \sqrt{v - 2f}} &= \pm (x - x_0 - vt)
\end{align*}
\]

Note I have reabsorbed the integration constant into the definition of \( x \), so that wave begins at \( x = x_0 \) when \( t = 0 \).

Try a substitution \( f = \frac{1}{2} v \sech^2 \theta \) so that \( df = -v \sech^2 \theta \tanh \theta d\theta \) (since \( d(\cosh^{-1}) = -\cosh^{-1} \tanh \)) so that

\[
\begin{align*}
\frac{df}{f \sqrt{v - 2f}} &= - \frac{v \sech^2 \theta \tanh \theta d\theta}{\frac{1}{2} v \sech^2 \theta \sqrt{v} \tanh \theta} \\
&= - \frac{2}{\sqrt{v}} d\theta
\end{align*}
\]

where I used \((1 - \sech^2 = \tanh^2)\). Hence \(- \frac{2}{\sqrt{v}} \int d\theta = -2 \sqrt{v} = \pm (x - x_0 - vt)\) and

\( u(x,t) = f(x - vt) = \frac{v}{2} \sech^2 \left( \frac{\sqrt{v}}{2} (x - x_0 - vt) \right) \) (25)

The \( \pm \) is irrelevant but the velocity is positive, \( v \geq 0 \). The waves have
Three examples are shown below with speeds $v = 1, 2, 3$.

<table>
<thead>
<tr>
<th>speed</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>height</td>
<td>$\frac{v}{2}$</td>
</tr>
<tr>
<td>width</td>
<td>$\frac{1}{\sqrt{v}}$</td>
</tr>
</tbody>
</table>

**NB.** Solutions for non-zero $A, B$ involve Jacobi theta functions. (see D+J chapter 2). Actually the figures on the first page are only good upto $x = \pm 40$ rather than $x = \pm \infty$ (computers can’t handle the latter very well) so they are really Jacobi theta functions but close to the real soliton. (I used periodic boundary conditions at $|x| = 40$, so these really are periodic functions. This is more obvious in the dispersive diagrams where the small ripples reappear to the right of the bounding interval.)

**NB2.** In the limit of small $u$ amplitude the KdV equation linearizes and you expect standard trig functions. The above periodic theta functions approximate to the expected trig functions in this limit.
3.2 Solitons in the Sine-Gordon equation

\[ u_{xx} - u_{tt} = \sin u \]  (26)

Again substituting \( u = f(x - vt) \) and using \( u_{xx} = f'' \) and \( u_{tt} = v^2 f'' \) we get

\[ (1 - v^2) f'' = \sin f \]  (27)

or more succinctly

\[ f'' = \gamma^2 \sin f \]  (28)

where \( \gamma = 1/\sqrt{1 - v^2} \). (This looks relativistic which is not a coincidence. The SG eqn has a relativistic form. The speed of light = 1.) Now multiply by \( f' \) and integrating (using \( f' \sin f = -(\cos f)' \)) gives

\[ (f')^2 = A - 2\gamma^2 \cos f \]  (29)

or

\[ f' = \pm \sqrt{A - 2\gamma^2 \cos f} . \]  (30)

At \( x = \pm \infty \) we expect \( f'' = 0 \) since we are looking for an isolated lump of finite extent, so

\[ f(\pm \infty) = 2\pi n \]  (31)

where \( n \in \mathbb{Z} \). This also tells us from \( f'(\pm \infty) = 0 \) that

\[ A = 2\gamma^2 . \]  (32)

Integrating eq.30 we have

\[
\int \frac{df}{\sqrt{1 - \cos f}} = \pm \sqrt{2}\gamma(x - vt)
\]

\[
\sqrt{2} \log \tan(f/4) = \pm \sqrt{2}\gamma(x - vt)
\]  (33)

So that

\[ u(x, t) = 4 \tan^{-1} \left( e^{\pm \gamma(x-x_0-vt)} \right) . \]  (34)

This is called a kink or antikink. Three kinks are shown below with speeds \( v = 0.5, 0.9, 0.99 \) (getting steeper respectively). (Note the step is through \( 2\pi \)).
3.3 Aside: Connection with particle kinematics

There is a fantastically useful way to picture soliton solutions. Go back to eq.21 with $A = B = 0$ and rearrange a bit;

$$\frac{1}{2}(f')^2 + U(f) = 0$$ (35)

where

$$U(f) = -\frac{v}{2}f^2 + f^3$$ (36)

This equation is identical to that of a particle of mass 1 rolling in the potential $U(f)$ with $\xi$ playing the roll of time (the first term is the Kinetic energy the second term the potential). The potential is shown below;

Consider the $\xi \to -\infty$ limit. Here we require $f' = f = 0$ so that the total energy is zero. The particle begins stationary at $f = 0$ at point A. After sitting for a semi-infinite period of time it rolls down the slope and by conservation of energy it reaches point B, and then rolls back again to A and comes to rest at $\xi \to +\infty$. B corresponds to the top of the soliton since $f_{\text{max}} = v/2$ since $U(f_{\text{max}}) = 0$ simply by conservation of energy. This method is extremely useful for visualizing solitons for more general functions.

For the Sine Gordon equation the potential is

$$U(f) = \gamma^2(\cos f - 1).$$ (37)

The difference in this case is that the particle rolls from one peak to another as $\xi$ goes from $-\infty$ to $+\infty$.

**NB:** The $U(f)$ here is the negative of the potential energy $V$ in the equations of motion; this is because we are letting $x$ play the roll of time and the usual equation of motion is of the form $u_{tt} - u_{xx} - V'(u) = 0$.
3.4 Physical model for the SG soliton

Consider the physical set-up ...

consisting of masses \( m \) separated by elastic of length \( a \) on pendula of length \( l \). If \( \theta_n \) is the angle of the \( n \)th pendulum then by resolving forces and acceleration in radial direction (exercise) we get

\[
ml^2 \ddot{\theta}_n = -mgl \sin \theta_n + \frac{k}{a} (\theta_{n+1} - \theta_n) + \frac{k}{a} (\theta_{n-1} - \theta_n)
\]  

(38)

where \( k \) is a coefficient of force in the elastic (linear by Young - related to Young’s modulus of elasticity). Now let our \( x \) direction be given by \( na = x \) so that if \( a \) is small can approximate

\[
\frac{k}{a} (\theta_{n-1} + \theta_{n+1} - 2\theta_n) = ka \left( \frac{\theta_{n+1} - \theta_n}{a} - \frac{\theta_n - \theta_{n-1}}{a} \right)
\approx ka \theta''(x)
\]  

(39)

So eqn of motion becomes

\[
ml^2 \ddot{\theta} = -mgl \sin \theta + ka \theta''
\]  

(40)

Now let \( a \to 0 \) and \( k \to \infty \) such that \( ka \to ml^2 \) is finite. Then

\[
\theta_{tt} - \theta_{xx} = -\frac{g}{l} \sin \theta
\]  

(41)

which is the SG equation.

The system has zero energy (is in the “vacuum”) when \( \theta = 0 \forall x \). Two other possibilities; small oscillations as shown in the figure above and “kinks” as below where \( \theta(-\infty) = 0 \) and \( \theta(+\infty) = 2\pi \)
In terms of $\theta$ this is

The one twist configuration can’t be deformed back into the “vacuum” in a continuous way (unlike the small wave configuration which can). This is a *topological* property (see next lecture).
3.5 A topological defect that is not a soliton: $\phi^4$ theory

Consider a one-dimensional field theory (like the SG theory) but with the equation of motion

$$u_{tt} - u_{xx} = \lambda u(a^2 - u^2) \tag{42}$$

This (c.f. appendix B.4) is equivalent to a field theory in a potential

$$P(u) = \frac{\lambda}{4}(u^2 - a^2)^2$$

known as the $\phi^4$ potential (conventionally $\phi$ is used for scalar field theories).

Again substituting $u = f(x - vt)$ and using $u_{xx} = f''$ and $u_{tt} = v^2 f''$ we get

$$f'' = \gamma^2 \lambda f(f^2 - a^2) \tag{43}$$

and then integrating up gives eq.458 in the appendix. Taking the square root of eq.458 we get

$$f' = \pm \gamma \sqrt{\frac{\lambda}{2}}(f^2 - a^2). \tag{44}$$

Again at $\xi = \pm \infty$ we expect both $f'$ and $f'' = 0$ so that the integration constant has been set to zero and

$$f(\pm \infty) = \pm a. \tag{45}$$

The potential looks like

so the particle kinematic equivalent of the would-be soliton is a particle stationary at $-a$ at $\xi = -\infty$ rolling to $+a$ as $\xi \to +\infty$ in the inverted potential. Integrating we have

$$\int \frac{df}{(a^2 - f^2)} = \pm \sqrt{\frac{\lambda}{2}}\gamma \xi$$

$$u = \pm a \tanh \left( \sqrt{\frac{\lambda}{2}} a \gamma (x - vt) \right) \tag{46}$$

A stationary example (with $\lambda = 2$ and $a = 1$) is shown below. Again the theory is relativistic so the solution depends on $\gamma(x - vt)$. 
Why is this not a soliton? Because it doesn’t satisfy invariance under interaction. Below I show before and after shots of two interactions that I did numerically; the first is between two SG solitons and the second interaction between two of the $\phi^4$ would-be solitons. The latter looks almost like a soliton collision, but you can just make out small oscillations. The $\phi^4$ collision is slightly “inelastic” with some kinetic energy of the walls being transferred to radiation (oscillations of the scalar field).
4 Topological lumps and the Bogomolny bound

Take SG as example
\[ u_{tt} - u_{xx} + \sin u = 0 \]  
(Note that we can always remove \( g/l \) by rescaling \( t \) and \( x \)).

\[ KE = T = \int_{-\infty}^{\infty} \frac{1}{2} (u_t)^2 \, dx \]
\[ PE = V = \int_{-\infty}^{\infty} \frac{1}{2} (u_x)^2 + (1 - \cos u) \, dx \]  

(48)

Note that in terms of Lagrange density
\[ L = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 - (1 - \cos u) = T - V \]  
so that the Euler Lagrange eqn
\[ \frac{d}{dt} \frac{\partial L}{\partial u_t} + \frac{d}{dx} \frac{\partial L}{\partial u_x} - \frac{\partial L}{\partial u} = 0 \]  

indeed give the SG equation eq.47. The form of \( V \) (i.e. the “1”) comes from the requirement of finite energy so that \( V \to 0 \) at \( x \to \infty \) where \( u \to 2\pi n \).

(The pendula must all be pointing down at infinity).

- Physically the absolute number \( n \) is irrelevant - a shift \( u(x) \to u(x) + 2\pi k \) doesn’t change any of the equations in particular \( V(u) \). What is important is the difference which measures the number of kinks

\[ u(+\infty) - u(-\infty) = 2\pi (m - n) \]  

This number is invariant under a finite \( 2\pi k \) shift. The number \( n - m \) is topological because unchanged under continuous changes to \( u \). e.g.

\[ u(x) \to u(x) + f(x) \]

changes \( n - m \) only if \( f \) has non-zero \( n - m \).

- If \( n - m \neq 0 \) then \( u \) can never dissipate to zero.

- Localized non-dissipative solutions that arise this way are topological lumps. So the SG kinks are topological lumps and also solitons - (i.e. still have to prove property 3).

- The conservation of \( n - m \) is a topogical conservation law
4.1 The Bogomolny bound

The kink solution we found earlier was static. But are such solutions (local) minima of the energy and therefore stable? Bogomolny’s argument show’s they are. In many field theories (e.g. electromagnetism) the Bogomolny bound makes finding topologically non-trivial solutions much easier. For example “complete the square” of the energy for the SG kink

\[ E = T + V = \int_{-\infty}^{\infty} \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + (1 - \cos u) \, dx \]

\[ \geq \int_{-\infty}^{\infty} \frac{1}{2} u_x^2 + (1 - \cos u) \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{2} u_x^2 + 2 \sin^2 \left( \frac{u}{2} \right) \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{2} \left( u_x \pm 2 \sin \left( \frac{u}{2} \right) \right)^2 \, dx \pm 2 \sin \left( \frac{u}{2} \right) u_x \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{1}{2} \left( u_x \pm 2 \sin \left( \frac{u}{2} \right) \right)^2 \, dx \pm 4 \left[ \cos \left( \frac{u}{2} \right) \right]_{-\infty}^{\infty} \]  

(52)

If \( u \) is a single kink then

\[ 4 \left[ \cos \left( \frac{u}{2} \right) \right]_{-\infty}^{\infty} = -8 \]

and we get the **Bogomolny bound** \( E \geq 8 \) with saturation when

\[ u_x - 2 \sin \left( \frac{u}{2} \right) = 0 \]  \hspace{1cm} (53)

Solving this

\[ \int dx = \int \frac{du}{2 \sin \left( \frac{u}{2} \right)} = \log \tan \left( \frac{u}{4} \right) \]  \hspace{1cm} (54)

so that

\[ u = 4 \tan^{-1} e^{x-x_0} \]

Note that this is the earlier solution in the rest frame of the soliton

\[ x - x_0 \rightarrow \gamma(x - x_0 - vt) \]

we asked for the static solution so this makes sense. (Also note there is clearly a problem if there is more than 1 kink - the Bogomolny argument breaks down.)
In fact we can extend this to generic relativistic scalar field theories using Appendix B4.

\[
E = \int T + V \, dx = \int_{-\infty}^{\infty} \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \mathcal{P}(u) \right) \, dx \\
\geq \int_{-\infty}^{\infty} \frac{1}{2} u_x^2 + \mathcal{P} \, dx \\
= \int_{-\infty}^{\infty} \frac{1}{2} (u_x \pm \sqrt{2\mathcal{P}})^2 \mp \sqrt{2\mathcal{P}} u_x \, dx \\
= \int_{-\infty}^{\infty} \frac{1}{2} (u_x \pm \sqrt{2\mathcal{P}})^2 \, dx \mp \left[ \int \sqrt{2\mathcal{P}} \, du \right]_{u(-\infty)}^{u(\infty)} \quad (55)
\]

When saturated, this inequality tells us that

\[
\frac{u_x^2}{2} - \mathcal{P}(u) = 0 \quad (56)
\]

which is simply eq.458 when the wave is static (\(\gamma = 1\)) and \(A = 0\). It also tells us the total energy of this configuration is the Bogomolny bound energy;

\[
E_{kink} = \pm \left[ \int \sqrt{2\mathcal{P}} \, du \right]_{u(-\infty)}^{u(\infty)} \quad (57)
\]

4.2 The energy is localized...

For the static SG kink, the energy density is

\[
\frac{1}{2} u_x^2 + 2 \sin^2 \left( \frac{u}{2} \right) = u_x^2 \\
= 2 \text{sech}^2 (x - x_0) \quad (58)
\]
4.3 Derrick’s Theorem:

So far we have concentrated on solutions in $d = 1$ space dimensions. Is it possible to get static stable lumps (solitonic or not) in 2 or more space dimensions? The answer (for any number of simple scalar fields such as displacement) is no;

Derrick’s Theorem: Let $\phi$ be a set of $n$ scalar fields assembled into an $n$-vector, and let the dynamics be defined by

$$\mathcal{L} = \phi_t, \phi_t - \phi_x, \phi_x - \phi_y, \phi_y \ldots - \mathcal{P}(\phi)$$

Then for $d \geq 2$ the only time independent non-singular solutions of finite energy are the ground states.

Proof: Define

$$K = \frac{1}{2} \int d^d \mathcal{x} (\nabla \phi)^2$$
$$P = \int d^d \mathcal{x} \mathcal{P}(\phi)$$

(Note I am avoiding using $T$ and $V$ since $V$ includes the $(\partial_x \phi)^2$ terms in the rest of these notes.) Assume that both $K$ and $P$ must be non-negative (we do not allow the possibility of negative P.E.) and that $P = K = 0$ only for the groundstate. Now, if there exists a solution $\phi(\mathcal{x})$, then there also exists a one parameter family of configurations $\phi(\lambda \mathcal{x})$ of which only $\lambda = 1$ is the true stable solution. The energy of this family of configurations can be related to the energy of the true solution by a simple rescaling of the integration variable $\mathcal{x}$

$$E = \lambda^{(2-d)} K + \lambda^{-d} P$$

Clearly all values of $\lambda$ have the same topological properties so we can use Bogomolny’s argument; the true solution minimizes the total energy and therefore

$$\frac{d}{d\lambda} (K + P)_{\lambda=1} = 0 = (2-d) K - d P$$

If $d > 2$ we must have $P = K = 0$. If $d = 2$ the above equation implies only the vanishing of $P$ and a small amount of further argument is needed; if $P$ vanishes $\forall \mathcal{x}$, it is constant and hence

$$\nabla P = \nabla \phi, \frac{\partial P}{\partial \phi} = 0 \ \forall \mathcal{x}$$

Either $P(\phi) = \text{constant}$ (which would be trivial) or it follows that $\nabla \phi = 0 \ \forall \mathcal{x}$ and hence $K = 0$. 

19
4.4 Summary

To be acceptable lumps must be stable. This can happen in two ways, integrability or topology;

\[ \text{KdV} \quad \text{SG} \quad \phi^4 \]

The \( \phi^4 \) model lumps are topological because the model has a \( Z_2 \) symmetry in

\[ \mathcal{P}(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2. \]

The \( Z_2 \) symmetry is invariance of the action under \( \phi \rightarrow -\phi \). (See first exercise sheet.) However they are not true solitons because they radiate during collisions.
5 Bäcklund Transformations

So far we have considered only isolated solitary wave solutions. How can we describe two interactions (after all this defines one of the essential properties of solitons)? We need another trick called the Bäcklund transformation. They have the following uses

- Generating solutions to difficult p.d.e.s from known solutions to easier ones.
- Generating new solutions to p.d.e.s from already known solutions.

The second of these (called an auto-Bäcklund transformation) is the one we will need to generate two and more soliton solutions from the solitary wave solutions we know.

5.1 Definition

Suppose that we write down an arbitrary pair of p.d.e. relations which I’ll label with a star;

\[ R_1[u,v] = 0 \]
\[ R_2[u,v] = 0. \]

These relations can be non-linear and any order, and act on two arbitrary functions which I’ll call \( u \) and \( v \). Now if we can eliminate \( u \) and \( v \) separately from these relations we will find two separate p.d.e.s which I’ll label with a heart and spade;

\[ P[u] = 0 \]
\[ Q[v] = 0. \]

(\( P[u] \) is an entire differential equation now, not to be confused with the potential energy). Let \( \heartsuit \) be an equation such as the SG equation, and let \( u \) be a solution of it. Then the relations \( \star \) generate a solution \( v \) to \( \spadesuit \) from each \( u \).

Why is this useful? Because the equations \( \star \) and \( \heartsuit \) may be easier to solve than \( \spadesuit \).

- the relations \( \star \) are collectively called a Bäcklund transformation.
- if \( P = Q \) then it’s an auto-Bäcklund transformation and we get new solutions to the same p.d.e.
5.2 Example 1: Auto-BT for the Laplace equation

Take

\[
R_1[u, v] \equiv u_x - v_y = 0 \\
R_2[u, v] \equiv u_y + v_x = 0
\]  \hspace{1cm} (60)

Differentiating \( R_1 \) by \( x \) and \( R_2 \) by \( y \) and adding we get \( P[u] \), and differentiating \( R_2 \) by \( x \) and \( R_1 \) by \( y \) and subtracting we find \( Q[v] \):

\[
P[u] \equiv u_{xx} + u_{yy} = 0 \\
Q[v] \equiv v_{xx} + v_{yy} = 0
\]  \hspace{1cm} (61)

Example: \( u(x, y) = 2xy \) is a simple solution of \( P[u] = 0 \). Then \( \star \) (eq.60) gives

\[
v_x + 2x = 0 \\
v_y - 2y = 0
\]  \hspace{1cm} (62)

Integrating gives

\[
v = y^2 - x^2,
\]  \hspace{1cm} (63)

a different solution to the Laplace equation. In this case \( R_1 \) and \( R_2 \) are the Cauchy-Riemann conditions for a function \( v + iu \) to be analytic. In this case the function is \( v + iu = (x + iy)^2 \). \( P[u] = Q[v] = 0 \) is the statement that both the real and imaginary parts satisfy Laplace’s equation.

**NB:** To go beyond just flipping between two solutions we have to introduce an extra parameter into auto-BT’s.

**NB2:** Given \( u \) there are two equations in relation \( \star \) for only one function \( v \). Normally this would be overdetermined and not have any solutions. e.g. if \( u = x^2 \) then \( \star \) gives \( v_x = 0 \), and \( v_y = 2x \) which is inconsistent. But \( u = x^2 \) is not a solution to \( P[u] = 0 \), so \( P \) is an...

**Integrability Condition:** The relations \( \star \) only consistently generate a function \( v \) from a given \( u \) provided that \( P[u] = 0 \).

I presented BT’s this way round (beginning with \( \star \) and deriving both \( P \) and \( Q \) ) so that the condition is more evident. Often as in D+J one begins with \( P \) and looks for \( \star \) to get \( Q \).
5.3 Example 2: BT’s for the Liouville equation

The Liouville equation is

$$v_{+-} = e^v. \quad (64)$$

It is similar to the SG equation which could be written as

$$v_{+-} = \frac{1}{2i}(e^{iv} - e^{-iv})$$

however unlike SG it has no static solutions because it has no minima of the potential. Here I am using light-cone coordinates described in Appendix C and in particular have used eq.464 to get the above.

Eq.64 is the equation we want to arrive at as our $Q[v]$. Consider the following BT

$$\partial_+(v - u) = \sqrt{2} e^{(v+u)/2}$$
$$\partial_-(v + u) = \sqrt{2} e^{(v-u)/2} \quad (65)$$

First cross-differentiate

$$\partial_- \partial_+(v - u) = \sqrt{2} \partial_- e^{(v+u)/2}$$
$$= \frac{1}{\sqrt{2}} e^{(v+u)/2} \partial_-(v + u)$$
$$= e^v \quad (66)$$

where I used the second BT relation in the last line. Doing the same for the second relation yields

$$\partial_+ \partial_-(v + u) = e^v \quad (67)$$

Adding and subtracting eqs.66 and 67 gives

$$u_{+-} = 0$$
$$v_{+-} = e^v \quad (68)$$

These are our $P[u]$ and $Q[v]$ respectively. The former is the linear wave equation, the latter is Liouville’s equation.

**Conclusion:** Any solution for the linear wave equation can generate via eq.65 a solution to the Liouville equation. Conversely all solutions to Liouville’s equation are related to a solution to the linear wave equation.
5.4 General solution for the Liouville equation

The above BT generates solutions to the Liouville equation from the wave equation. However for the wave equation we know a general solution due to D’Alembert, so we must be able to generate general solutions for Liouville as well. According to D’Alembert

$$u = f(x_+) + g(x_-)$$  \hspace{1cm} (69)

is a general solution for the linear wave equation (light-cone coordinates make this obvious).

Each ★ equation becomes an o.d.e. for $v$ that we have to solve to get $x_+$ and $x_-$ dependence of $v$; consider

$$\partial_+(v - u) = \sqrt{2}e^{(v+u)/2}$$  \hspace{1cm} (70)

Trick: substituting $v - u = -2 \log \psi$ this eqn. becomes

$$\frac{2}{\psi} \partial_+ \psi = -\sqrt{2}e^{-\log \psi+u} = \frac{\sqrt{2}}{\psi} e^u.$$

$$\partial_+ \psi = -\frac{e^u}{\sqrt{2}}$$  \hspace{1cm} (71)

and integrating we get

$$\psi(x_+, x_-) = -\frac{1}{\sqrt{2}} \int^{x_+} e^{u(x_+, x_-)} dz_+ + C(x_-)$$  \hspace{1cm} (72)

where $C(x_-)$ is a constant of $z_+$ integration. Now we can insert our (or rather D’Alembert’s) solution of eq.69 for $u$;

$$\psi = e^{(u-v)/2} = e^{(f(x_+)+g(x_-))/2}e^{-v/2}$$  \hspace{1cm} (73)

so that

$$e^{-v/2} = -\frac{e^{-(f(x_+)+g(x_-))/2}}{\sqrt{2}} \int^{x_+} e^{f(z_+)+g(x_-)} dz_+ + C(x_-)e^{-(f(x_+)+g(x_-))/2}$$

$$= -\frac{e^{-(f(x_+)-g(x_-))/2}}{\sqrt{2}} \int^{x_+} e^{f(z_+)} dz_+ + C(x_-)e^{-(f(x_+)+g(x_-))/2}$$

$$= -\frac{e^{-(f(x_+)-g(x_-))/2}}{\sqrt{2}} F(x_+) + C(x_-)e^{-(f(x_+)+g(x_-))/2}$$  \hspace{1cm} (74)

where I defined

$$F(x_+) = \int^{x_+} e^{f(z_+)} dz_+.$$  \hspace{1cm} (75)

Now repeat the exercise with the second BT relation: we put $u + v = -2 \log \psi$ and get

$$\partial_- \psi = -\frac{1}{\sqrt{2}} e^{-u}$$
and hence
\[
e^{-v/2} = -\frac{e^{-\left(f(x_+)-g(x_-)/2\right)}}{\sqrt{2}}G(x_-) + D(x_+)e^{\left(f(x_+)+g(x_-)/2\right)}
\] (76)

where
\[
G(x_-) = \int_{x_-}^{x_+} e^{-g(z_-)}dz_-
\] (77)

and \(D(x_+)\) is another constant of integration. All that remains is to determine \(C\) and \(D\). Comparing eqs.74 and 76 we find
\[
C(x_-) = -\frac{e^{+g(x_-)}}{\sqrt{2}}G(x_-)
\]
\[
D(x_+) = -\frac{e^{-f(x_+)}}{\sqrt{2}}F(x_+)
\] (78)

and then eq.74 gives
\[
e^{-v/2} = -\frac{1}{\sqrt{2}}e^{\left(g(x_-)-f(x_+)/2\right)}\left(F(x_+) + G(x_-)\right)
\] (79)

or
\[
v(x_+, x_-) = f(x_+) - g(x_-) - 2\log\left(-\frac{F(x_+) + G(x_-)}{\sqrt{2}}\right)
\] (80)

This is the general solution to Liouville’s equation. (first found by Liouville in 1853). Most of the time we can’t get a general solution like this. It’s fairly trivial to verify this is a solution; indeed
\[
\partial_+\partial_- v = \frac{2F'G'}{(F + G)^2}
\]
\[
e^v = e^{f-g} \frac{2}{(F + G)^2}.
\]

**Example:** Of course this solution is general, but to demonstrate, take a very simply linear solution \(ax_+ + bx_-\) of the linear wave equation with \(a\) and \(b\) constant. We have
\[
f(x_+) = ax_+
\]
\[
g(x_-) = bx_-
\]
\[
F(x_+) = \frac{e^{ax_+}}{a} + c
\]
\[
G(x_-) = \frac{e^{-bx_-}}{b} + d
\]

\((c, d\) constant) and hence a considerably less simple
\[
v = ax_+ - bx_- - 2\log\left(-\frac{be^{ax_+} + ae^{bx_-}}{\sqrt{2ab}} + \text{const}\right)
\]
One final point is that there is a one parameter family of BT’s given by

\[ \partial_+(v-u) = a\sqrt{2}e^{(v+u)/2} \]
\[ \partial_-(v+u) = \frac{1}{a}\sqrt{2}e^{(v-u)/2} \]  

Any value of \( a \) is equally good as it yields the same \( P \) and \( Q \) equations. This will be important in cases where we cannot get the full solution later.
6 The Sine-Gordon equation

\[ u_{+-} = \sin u \] (82)

Try the BT

\[ \partial_+(u-v) = \frac{2}{a} \sin \left( \frac{1}{2}(u+v) \right) \]
\[ \partial_-(u+v) = 2a \sin \left( \frac{1}{2}(u-v) \right) \] (83)

where \( a \) is a constant parameter. Again cross-differentiate

\[ \partial_+ \partial_-(u-v) = \frac{1}{a} \cos \left( \frac{1}{2}(u+v) \right) \partial_-(u+v) \]
\[ = 2 \cos \left( \frac{1}{2}(u+v) \right) \sin \left( \frac{1}{2}(u-v) \right) \]
\[ = \sin u - \sin v \] (84)

and

\[ \partial_+ \partial_-(u+v) = a \cos \left( \frac{1}{2}(u-v) \right) \partial_-(u-v) \]
\[ = 2 \cos \left( \frac{1}{2}(u-v) \right) \sin \left( \frac{1}{2}(u+v) \right) \]
\[ = \sin u + \sin v. \] (85)

Adding these two and subtracting them gives respectively

\[ u_{+-} = \sin u \]
\[ v_{+-} = \sin v. \] (86)

so this is an auto-BT for the SG equation.

6.1 Generating SG kinks from the vacuum

Let us begin with the solution \( u = 0 \) which is trivally a solution to SG (it’s the vacuum). The BT gives us

\[ u_+ = \frac{2}{a} \sin \frac{u}{2} \]
\[ u_- = 2a \sin \frac{u}{2} \] (87)

These can be easily integrated using

\[ \int \frac{du}{\sin \frac{u}{2}} = 2 \log \tan \frac{u}{4} \] (88)
giving
\[
\begin{align*}
\frac{2}{a} x_+ &= 2 \log \tan \frac{u}{4} + g(x_-) \\
2ax_- &= 2 \log \tan \frac{u}{4} + f(x_+) 
\end{align*}
\]  
(89)

with \( f \) and \( g \) being two “constant” of integration functions. Subtracting we find
\[
\frac{2}{a} x_+ + f(x_+) = 2ax_- + g(x_-). 
\]  
(90)

Since either side depends on different variables each side must be constant (like sep’n of variables in p.d.e.s) which I’ll take to be \(-2c\), so that
\[
\begin{align*}
f(x_+) &= -\frac{2}{a} x_+ - 2c \\
g(x_-) &= -2ax_- - 2c. 
\end{align*}
\]  
(91)

Hence
\[
2 \log \tan \frac{u}{4} = \frac{2}{a} x_+ + 2ax_- + 2c 
\]  
(92)

or
\[
\begin{align*}
u &= 4 \tan^{-1} \left( e^{\frac{1}{2}x_+ + ax_- + c} \right). 
\end{align*}
\]  
(93)

Finally we need to convert back to regular \( x, t \) coordinates;
\[
\begin{align*}
\frac{1}{a} x_+ + ax_- &= x \left( \frac{1}{2a} + \frac{a}{2} \right) + t \left( \frac{1}{2a} - \frac{a}{2} \right) \\
&= \frac{1 + a^2}{2a} \left( x - \frac{a^2 - 1}{a^2 + 1} t \right). 
\end{align*}
\]  
(94)

If we define
\[
v = \frac{a^2 - 1}{a^2 + 1}
\]
then it is easy to verify that
\[
\frac{1 + a^2}{2|a|} = \frac{1}{\sqrt{1 - v^2}} = \gamma
\]
as required and that
\[
u = 4 \tan^{-1} \left( e^{\pm \gamma(x-\nu t)+c} \right),
\]  
(95)

with the ± (kink or antikink) being chosen by the sign of \( a \) and direction of motion being left or right for \(|a| < 1\) and \(|a| > 1\) respectively; we have

<table>
<thead>
<tr>
<th>( a &lt; -1 )</th>
<th>(-1 &lt; a &lt; 0)</th>
<th>( 0 &lt; a &lt; 1 )</th>
<th>( 1 &lt; a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right moving anti-kink</td>
<td>Left moving anti-kink</td>
<td>Left moving kink</td>
<td>Right moving kink</td>
</tr>
</tbody>
</table>

The BT “creates” a kink or anti-kink from the vacuum solution (i.e. \( v = 0 \)). The nature of it depends on parameter \( a \).
Remarkable fact: this is true more generally! The BT has the effect of adding kinks or antikinks to any solution (even if it has dispersive waves). i.e. if $v$ has kink number $n$ and we solve $\star$ for $u$ then $u(x, t)$ will have kink number $n \pm 1$ depending on $a$.

Direct proof?: is left as a challenge (I don’t know of one)! Indirect proofs based on inverse scattering method (see next term) have been around since 1975.

So we can think of the BT as a “machine”/”black box” to add kink/antikink solutions.

We can use the BT repeatedly to add kinks and anti-kinks whereever we like (depending on the constants of integration $c$) and with whatever velocity we like; for example

Problem: The integrations get progressively harder (although still easier than SG). Try for example getting the 2 kink solution from the one kink one! Fortunately we can use a theorem.
6.2 The theorem of Permutability (Bianchi 1902) and multi-kink solutions

**General observation:** Multiple solitons have no static solution. To see this go back to the particle rolling in the inverted potential picture; If it takes an infinite “time” $\xi$ for the particle to roll into the well, then it takes an infinite time to roll out again by symmetry.

**Imagine** doing two successive BT’s with parameters $a_1$ and $a_2$;

in two different orders as shown. $u_3$ and $u_4$ both look like $u_0$ with two solitons added.

**Theorem:** For any $u_1$ and $u_2$ it is possible to choose constants of integration in the 2nd BT’s such that $u_3 = u_4$.

In terms of the diagrams we have the above. The $a_1$ and $a_2$ BTs have been arranged to commute (this depends on choosing the constants of integration correctly).

Now consider just the $\partial_+$ part of the BTs and look at the upper route
We get
\begin{align*}
\partial_+ (u_1 - u_0) &= \frac{2}{a_1} \sin \frac{1}{2} (u_1 + u_0) \\
\partial_+ (u_3 - u_1) &= \frac{2}{a_2} \sin \frac{1}{2} (u_1 + u_3). \tag{96}
\end{align*}

Adding gives
\begin{align*}
\partial_+ (u_3 - u_0) &= \frac{2}{a_1} \sin \frac{1}{2} (u_1 + u_0) + \frac{2}{a_2} \sin \frac{1}{2} (u_3 + u_1) \tag{97}
\end{align*}

Taking instead the lower route
\begin{align*}
\partial_+ (u_3 - u_0) &= \frac{2}{a_2} \sin \frac{1}{2} (u_2 + u_0) \\
\partial_+ (u_3 - u_1) &= \frac{2}{a_1} \sin \frac{1}{2} (u_2 + u_3). \tag{98}
\end{align*}

and adding gives
\begin{align*}
\partial_+ (u_3 - u_0) &= \frac{2}{a_2} \sin \frac{1}{2} (u_2 + u_0) + \frac{2}{a_1} \sin \frac{1}{2} (u_3 + u_2) \tag{99}
\end{align*}

Equating the two expressions for \( \partial_+ (u_3 - u_0) \) gives us a simple algebraic relation between the solutions;
\begin{align*}
\frac{2}{a_1} \sin \frac{1}{2} (u_1 + u_0) + \frac{2}{a_2} \sin \frac{1}{2} (u_3 + u_1) &= \frac{2}{a_2} \sin \frac{1}{2} (u_2 + u_0) + \frac{2}{a_1} \sin \frac{1}{2} (u_3 + u_2). \tag{100}
\end{align*}

Given the vacuum and the one-kink solutions we can always solve this to get the two-kink solutions. We can bootstrap our way to large kink number without doing any integrating at all! Sometimes called a non-linear superposition principle (tells us how to put solutions together).

As a check we should get the same relation if we use the \( \partial_- (u + v) \) part of the BT instead. From the top route we get
\begin{align*}
\partial_- (u_1 + u_0) &= 2a_1 \sin \frac{1}{2} (u_1 - u_0) \\
\partial_- (u_3 + u_1) &= 2a_2 \sin \frac{1}{2} (u_3 - u_1). \tag{101}
\end{align*}
Subtracting gives
\[ \partial_-(u_3 - u_0) = 2a_2 \sin \frac{1}{2}(u_3 - u_1) - 2a_1 \sin \frac{1}{2}(u_1 - u_0). \] (102)

From the bottom route we get
\[ \partial_-(u_2 + u_0) = 2a_2 \sin \frac{1}{2}(u_2 - u_0) \]
\[ \partial_-(u_3 + u_2) = 2a_1 \sin \frac{1}{2}(u_3 - u_2). \] (103)

and again subtracting gives
\[ \partial_-(u_3 - u_0) = 2a_1 \sin \frac{1}{2}(u_3 - u_2) - 2a_2 \sin \frac{1}{2}(u_2 - u_0). \] (104)

Equating the two expressions for \( \partial_-(u_3 - u_0) \) gives
\[ 2a_2 \sin \frac{1}{2}(u_3 - u_1) - 2a_1 \sin \frac{1}{2}(u_1 - u_0) = 2a_1 \sin \frac{1}{2}(u_3 - u_2) - 2a_2 \sin \frac{1}{2}(u_2 - u_0). \] (105)

What we now need to do is check that eqs.(100) and (105) are the same. Using
\[ \sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B) \]
we can rewrite eq.100 as follows;
\[ \frac{1}{a_1}(\sin \frac{1}{2}(u_1 + u_0) - \sin \frac{1}{2}(u_3 + u_2)) = \frac{1}{a_2}(\sin \frac{1}{2}(u_2 + u_0) - \sin \frac{1}{2}(u_3 + u_1)) \]
\[ \frac{2}{a_1}(\sin \frac{1}{4}(u_1 + u_0 - u_2 - u_3) = \frac{2}{a_2}(\sin \frac{1}{4}(u_2 + u_0 - u_3 - u_1)) \times \cos \frac{1}{4}(u_1 + u_0 + u_3 + u_2)) \times \cos \frac{1}{4}(u_2 + u_0 + u_3 + u_1)) \]
\[ 2a_2(\sin \frac{1}{4}(u_1 + u_0 - u_2 - u_3) = 2a_1(\sin \frac{1}{4}(u_2 + u_0 - u_3 - u_1)) \] (106)

where I cancelled the common cosine factor. Likewise we can rewrite eq.105 as
\[ a_2(\sin \frac{1}{2}(u_3 - u_1) + \sin \frac{1}{2}(u_2 - u_0)) = a_1(\sin \frac{1}{2}(u_3 - u_2) + \sin \frac{1}{2}(u_1 - u_0)) \]
\[ 2a_2(\sin \frac{1}{4}(u_3 + u_2 - u_1 - u_0) = 2a_1(\sin \frac{1}{4}(u_1 + u_3 - u_2 - u_0) \times \cos \frac{1}{4}(u_3 + u_0 - u_2 - u_1)) \times \cos \frac{1}{4}(u_3 + u_0 - u_2 - u_1)) \]
\[ 2a_2(\sin \frac{1}{4}(u_3 + u_2 - u_1 - u_0) = 2a_1(\sin \frac{1}{4}(u_1 + u_3 - u_2 - u_0)) \] (107)
i.e. the same equation so it is consistent.

These equations are still a bit inconvenient because we want to get \( u_3 \) from \( u_0 \) (the vacuum say) via the intermediate stage single kinks \( u_1 \) and \( u_2 \). Manipulate eq.107; let \( A = u_0 - u_3 \) and \( B = u_1 - u_2 \). Then this equation reads
\[ a_2 \sin \frac{1}{4}(A + B) = a_1 \sin \frac{1}{4}(A - B) \]

32
or
\[ a_2 \left( \sin \frac{A}{4} \cos \frac{B}{4} + \sin \frac{B}{4} \cos \frac{A}{4} \right) = a_1 \left( \sin \frac{A}{4} \cos \frac{B}{4} - \sin \frac{B}{4} \cos \frac{A}{4} \right). \]

Dividing through by \( \cos \frac{B}{4} \cos \frac{A}{4} \) we get
\[ (a_1 - a_2) \tan \frac{A}{4} = (a_1 + a_2) \tan \frac{B}{4}. \]
or
\[ \tan \frac{1}{4}(u_0 - u_3) = \frac{(a_1 + a_2)}{(a_1 - a_2)} \tan \frac{1}{4}(u_1 - u_2) \] (108)

6.3 Two soliton solutions to SG

Begin with \( u_0 = 0 \) (the vacuum). Using the identity \( \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \) we get
\[ \tan \frac{u_3}{4} = \frac{(a_2 + a_1) \tan \frac{u_1}{4} - \tan \frac{u_2}{2}}{(a_2 - a_1) 1 + \tan \frac{u_1}{4} \tan \frac{u_2}{2}}. \]

Recall that the expression for the single kinks has
\[ \tan \frac{u_i=1,2}{4} = e^{\theta_i}, \]
where
\[ \theta_i = \frac{x_i}{a_i} + a_i x_- + c_i \]
\[ \epsilon(a_i) \gamma(a_i) (x - x_0 - v(a_i) t) \] (109)

where the integration constant \( c_1 \) has determined the position of the kink \( x_0 \) and \( \epsilon(a) = \text{Sign}(a) \). Then we get
\[ u_3 = 4 \tan^{-1} \left( \frac{(a_2 + a_1) e^{\theta_1} - e^{\theta_2}}{(a_2 - a_1) 1 + e^{\theta_1} + e^{\theta_2}} \right). \] (110)

Note that as expected when we try to get a static solution we have \( v = 0 \) for both kinks or \( a_1 = \pm a_2 \) and the solution breaks down. Two soliton solutions to SG

6.4 3 soliton solution

Just to show how we can now “bootstrap” to any number of kinks let’s look at the 3 kink solution. Begin with \( u_0 \) as a single kink solution parameterized by \( a_0 \) and \( u_{1,2} \) as two of the two-kink solutions parameterized by adding kinks \( a_{1,2} \) respectively. Then to get to \( u_3 \) we must add the complementary \( a_{2,1} \). Using the previous relation we have
\[ \tan \frac{(u_3 - u_0)}{4} = \frac{(a_2 + a_1) \tan \frac{u_1}{4} - \tan \frac{u_2}{2}}{(a_2 - a_1) 1 + \tan \frac{u_1}{4} \tan \frac{u_2}{2}}. \]
Substituting in the $\tan(u/4)$ expressions for the two-kink solutions we get
(using $t_i = \tan(u_i/4)$)

\[
\frac{t_3 - t_0}{1 + t_0 t_3} = \frac{a_2 + a_1}{a_2 - a_1} \left( \frac{a_0 + a_1}{a_0 - a_1} \left( \frac{t_1 - t_0}{1 + t_0 t_1} \right) - \frac{a_0 + a_2}{a_0 - a_2} \left( \frac{t_2 - t_0}{1 + t_0 t_2} \right) \right)
\]

which can be easily solved for $t_3$. It looks a bit lob-sided because one of the single kinks is on the LHS, but with a bit of effort the solution is seen not to be. To write it I’ll define

\[a_{\pm\pm} = (a_0 \pm a_1)(a_1 \pm a_2)(a_2 \pm a_0)\]

where the $\pm$ refer to the respective positions inside the brackets. The solution is

\[
\tan \frac{u_3}{4} = \frac{a_{+-} e^{\theta_0} + a_{++} e^{\theta_1} + a_{--} e^{\theta_2}}{a_{-+} e^{\theta_1+\theta_2} + a_{++} e^{\theta_0+\theta_2} + a_{--} e^{\theta_0+\theta_1} - a_{--}} \tag{111}
\]

### 6.5 Asymptotics of interactions

First note that our observation of no static solutions was correct. A static solution has $a_1 = a_2 = \pm 1$ so that $u_3 = 4\tan^{-1}(\infty)$ or $4\tan^{-1}(0)$ both of which are constant and sitting in a vacuum. The solution break down. In fact there are no solutions where two kinks have the same speed $a_1 = \pm a_2$ (this is because you could always Lorentz boost to the rest frame which would have to be a static solution). So there must always be an interaction.

We now want to look at the asymptotics of interactions to make sure solitons are preserved. The two kink solutions should contain “hidden” one kink solutions as $t \to \pm \infty$. An “after” shot of a kink/antikink solution is shown below;

The kink/antikink pair look preserved. To see this in the solution take $t \to t_1 + \delta t$ with $\delta x = x - x_0 - v_1 t_1$ where $v_1 = \frac{a_1^2 - 1}{a_1 + 1}$ and let $t_1 \to \pm \infty$ so we are looking around the location of the first kink.

34
We have
\[ \theta_1 = \epsilon_1 \gamma_1 (\delta x - v_1 \delta t), \]
\[ \theta_2 = \epsilon_2 \gamma_2 (\delta x - v_1 \delta t) + \epsilon_2 \gamma_2 (v_1 - v_2) (t_1 + \delta t) \] (112)

Near the $a_1$ kink we have (by definition) $\delta x, \delta t \approx 0$, so that $\theta_1$ is small but $\theta_2 \to \pm \infty$.

<table>
<thead>
<tr>
<th>$v_2 \lt v_1$</th>
<th>$t \to -\infty$</th>
<th>$t \to +\infty$</th>
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<tbody>
<tr>
<td>$\theta_2 \to -\epsilon_2 \infty$</td>
<td>$\theta_2 \to +\epsilon_2 \infty$</td>
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<th>$v_2 \gt v_1$</th>
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</thead>
<tbody>
<tr>
<td>$\theta_2 \to +\epsilon_2 \infty$</td>
<td>$\theta_2 \to -\epsilon_2 \infty$</td>
<td></td>
</tr>
</tbody>
</table>

Defining $\text{sign}(\theta_2) = -\eta_2$ the asymptotic form of the solution around $\delta x = 0$ can be written
\[ \lim_{t \to \pm \infty} (u_3) = 4 \tan^{-1} \left( \frac{a_2 + a_1}{a_2 - a_1} \eta_2 e^{\eta_2 \theta_1} \right). \] (113)

But defining a “phase shift”
\[ c_1 = \frac{1}{\gamma_1} \log \frac{a_2 + a_1}{a_2 - a_1} \] (114)
we can rewrite this as
\[ \lim_{t \to \pm \infty} (u_3) = 4 \tan^{-1} \left( \eta_2 e^{\eta_2 (\theta_1 + \eta_2 \gamma_1 c_1)} \right). \] (115)

Finally we can use $-1/ \tan x = \tan(x - \pi/2)$ with $x = \tan^{-1} e^{\theta_1 + \eta_2 c_1}$ to get
\[ \lim_{t \to \pm \infty} (u_3) = \begin{cases} 4 \tan^{-1} \left( e^{(\theta_1 + \gamma_1 c_1)} \right) & \eta_2 > 0 \\ -2\pi + 4 \tan^{-1} \left( e^{(\theta_1 - \gamma_1 c_1)} \right) & \eta_2 < 0 \end{cases} \] (116)

We can recognize three features

1. The form of the soliton is preserved - it remains a kink or antikink
2. There is a “phase shift” in the kink location of $x_0 \to x_0 + 2c_1$ as $\eta_2$ goes from negative to positive.
3. There is always a drop or rise of $2\pi$ in this kink between asymptotic $t \to \pm \infty$ (due to the interaction with the other kink)
The above figure shows the asymptotics in two ways. The second way is a
contour plot of $u_x^2$ which remember is a measure of the energy stored in the
field.

A subtle point:

Examine eq. 113 a bit more closely for a particular example. Say we want
to have a kink labelled “1” coming from large negative $x$, travelling to the
rightward and interacting with a kink labelled “2” travelling left. Going to
our table of kinks and antikinks we would take

$$a_1 > 1 \quad 1 > a_2 > 0$$

But now we get a bit of a surprise; at $t \to -\infty$ kink-1 is actually an antikink
because $\frac{a_2+a_1}{a_2-a_1} < 0$ and $\epsilon_2 = +1$ and so $\eta_2 = +1$ for these values! (The same
reversal happens if we begin with an antikink with $a_1 < -1$).

Not only that but look at the 2nd soliton. We now have to use the $\theta_2$ table
the other way around using the $v_2 > v_1$ row but with $1 \leftrightarrow 2$. So here we have
an $\eta_1$ which is negative as $t \to -\infty$. The two negatives cancel and this soliton
remains what you’d expect. The conclusion is that BT’s don’t necessarily
respect the “kinkness” or “antikinkness” of a solution, but just add $\pm 1$ kink
without caring much what kind it is.
Interpretation as force

There is an interaction that looks like kink/kink repulsion and kink/antikink attraction. The force gets infinite as the speeds become equal and zero - another way to think of no static solutions.

This similarity with electric charge explains the importance of this type of interacting field theory for particle physics. Promoted in the 60’s by Skyme and useful for e.g. describing nuclear interactions. This suggests that there may be bound states of kink/antikink pairs as well by comparison with what happens in nuclear models. It turns out there are, called breathers.

7 The breather solution

Look for a bound solution in the 2 soliton solution. Recall

\[
  u_3 = 4 \tan^{-1} \left( \frac{(a_2 + a_1) e^{\theta_1} - e^{\theta_2}}{(a_2 - a_1) 1 + e^{\theta_1+\theta_2}} \right). \tag{117}
\]

For any \(a_1\) and \(a_2\) the permutability theorem guarantees it’s a solution of SG, even if \(a_i\) are complex. But \(u_3\) should be real (the field theory is defined with real fields representing e.g. displacement angles). Options leading to real \(u_3\):

1. \(a_1, a_2\) (and also \(c_1, c_2\)) \(\in R\). already seen this
2. \(a_2 = a_1^\ast\) (and also \(c_2 = c_1^\ast\)) now try this

To show option 2 gives real \(u_3\) recall \(\theta_i = \frac{x_i}{a_i} + a_i x_- + c_i\), so that

\[
  \theta_2 = \theta_1^*.
\]

and

\[
  u_3^* = 4 \tan^{-1} \left( \frac{(a_2^* + a_1^*) e^{\theta_1^*} - e^{\theta_2^*}}{(a_2^* - a_1^*) 1 + e^{\theta_1^*+\theta_2^*}} \right)
  = 4 \tan^{-1} \left( \frac{(a_1 + a_2) e^{\theta_2} - e^{\theta_1}}{(a_1 - a_2) 1 + e^{\theta_1+\theta_2}} \right)
  = u_3. \tag{118}
\]

Let’s look for a solution involving arbitrary \(a_1 = a\). Define for shorthand

\[
  a = re^{i\theta}
  \theta_1 = \alpha + i\beta
\]

so that

\[
  \tan \frac{u_3}{4} = \frac{2 \cos \vartheta \ e^{\alpha+i\beta} - e^{\alpha-i\beta}}{-2i \sin \vartheta \ 1 + e^{2\alpha}}
  = \frac{\cos \vartheta \ \sin \beta}{\sin \vartheta \ \cosh \alpha}. \tag{119}
\]
So now need to determine $\alpha, \beta$. We have

$$\alpha = \Re(\theta_1)$$
$$= \left( \frac{1}{r} x_+ + rx_- \right) \cos \vartheta + \Re(c)$$
$$= \frac{1 + r^2}{2r} \cos \vartheta \left( x - x_0 - \frac{r^2 - 1}{r^2 + 1} t \right)$$
$$= \cos \vartheta \gamma(r) \left( x - x_0 - vt \right)$$

(120)

where $\gamma(r)$ and $v(r)$ are the same definitions as earlier and I absorbed $\Re(c)$ into $x_0$. We also have

$$\beta = \Im(\theta_1)$$
$$= \left( -\frac{1}{r} x_+ + rx_- \right) \sin \vartheta + \Im(c)$$
$$= \frac{1 + r^2}{2r} \sin \vartheta \left( \frac{r^2 - 1}{r^2 + 1} (x - x_0) - t \right)$$
$$= \sin \vartheta \gamma(r) \left( v(x - x'_0) - t \right)$$

(121)

So then

$$\tan \frac{u_3}{4} = \frac{\cos \vartheta \sin \vartheta \gamma(t - vx)}{\sin \vartheta \cosh(\cos \vartheta \gamma(x - vt))}$$

(122)

where I took $x_0 = x'_0 = 0$ for simplicity. So the denominator is an envelope function that moves along at speed $v = \frac{r^2 - 1}{r^2 + 1}$. Inside the envelope there is an oscillation with faster phase velocity $1/v$. (Note the speed of light =1 so that $v < 1$).

To check out the wobble choose $r = 1$ so that $v = 0$ and then

$$\tan \frac{u_3}{4} = \frac{\cos \vartheta \sin \vartheta \gamma(t - vt)}{\sin \vartheta \cosh(\cos \vartheta \gamma(x - vt))}$$

(123)

We can always “boost” this solution to a $v \neq 0$ frame. The fields values and energy density look like a bouncing kink/antikink pair as below. The figure is for $\vartheta = \pi/20$. As $\vartheta \to 0$ the kink/antikink are more loosely bound, get further apart during the bounce and the bounce takes longer. The bounce period and size are

$$\tau = \frac{2\pi}{\sin \vartheta}, \quad \lambda = \frac{1}{\cos \vartheta}$$

respectively when static, and the solutions all look like the figure if the $x$ and $t$ axes are rescaled appropriately. Notice that at speed the bounce size is

$$\frac{2\pi}{\cos \vartheta \gamma}$$

which shortens due to Lorentz contraction.
Much more “phenomenology” can be done with these solutions which are a model for many processes in particle physics.
8 BT’s for the KdV equation

We now consider another example of BT’s for the KdV equation. In the following section we will again use the theorem of permutability to get multi-soliton solitons, so we are following exactly the same procedure as for the SG equation. BT’s here are a bit (OK, a lot) more involved. The SG BT was known in the 1880’s, whereas the KdV BT had to wait until 1973 (Wahlquist and Estabrook).

Suppose
\[ u = \lambda - v^2 - v_x \] (124)
satisfies the KdV equation, where \( \lambda \) is constant (it will eventually be the free parameter we need in the BT) and \( v = v(x,t) \). After a bit of effort we find that this implies that
\[ (2v + \frac{\partial}{\partial x})(v_t + 6(\lambda - v^2)v_x + v_{xxx}) = 0 \] (125)
so that if \( v \) satisfies
\[ (v_t + 6(\lambda - v^2)v_x + v_{xxx}) = 0 \] (126)
it implies that \( u \) given by the above is a solution of the KdV equation.

Proof: KdV equation for \( u \) is \( u_t + 6uu_x + u_{xxx} = 0 \). So that
\[
0 = -2vv_t - v_{xt} + 6(\lambda - v^2 - v_x)(-2vv_x - v_{xx}) - 2vv_{xxx} - 6v_x v_{xx} - v_{xxxx} \\
= -(2v + \frac{\partial}{\partial x})v_t - 6(\lambda - v^2)(2v + \frac{\partial}{\partial x})v_x + 12vv_x^2 + 6v_x v_{xx} - 2vv_{xxx} - 6v_x v_{xx} - v_{xxxx} \\
= -(2v + \frac{\partial}{\partial x})(v_t + 6(\lambda - v^2)v_x) - 6\frac{\partial v^2}{\partial x}v_x + 12vv_x^2 - 2vv_{xxx} - v_{xxxx} \\
= -(2v + \frac{\partial}{\partial x})(v_t + 6(\lambda - v^2)v_x + v_{xxx}) \tag{127}
\]
Eq.173 with the value \( \lambda = 0 \) is called the modified KdV equation (mKdV) - c.f. question 1 on sheet 2, and \( u = -v^2 - v_x \) is called a Miura transformation (found by R.Miura in 1968). Now if \( v \) satisfies eq.173 then clearly so does \(-v\).

So if
\[
u_1 = \lambda - v^2 - v_x \\
u_2 = \lambda - v^2 + v_x \tag{128}
\]
then \( u_{1,2} \) both satisfy the KdV equation. This is the first step. The idea is now to construct an autoBT linking \( u_1 \) and \( u_2 \) going via the mKdV equation.

From the above we can rearrange to get
\[
u_1 - u_2 = -2v_x \\
u_1 + u_2 = 2(\lambda - v^2) \tag{129}
\]
It’s convenient to define $w_{1,2}$ such that

$$u_i = \frac{\partial w_i}{\partial x}$$

so then the two equations become

$$w_1 - w_2 = -2v$$
$$w_1 + w_2 = 2\lambda - \frac{1}{2}(w_1 - w_2)^2$$  \hspace{1cm} (130)

where I already substituted for $v$ in the second. This is the “$x$” part of the BT. The “$t$” part comes from the mKdV equation itself substituting $v$. We have

$$0 = -\frac{1}{2}(w_1 - w_2)_t + 6(\lambda - v^2)(-\frac{1}{2}(w_1 - w_2)_x) - \frac{1}{2}(w_1 - w_2)_{xxx}$$
$$= (w_1 - w_2)_t + 3(w_1 + w_2)_x((w_1 - w_2)_x) + (w_1 - w_2)_{xxx}$$
$$= (w_1 - w_2)_t + 3(w_1^2 - w_2^2)_x + w_1,xxx - w_2,xxx$$  \hspace{1cm} (131)

So collecting the two important equations for our BT

$$w_1 + w_2_x + \frac{1}{2}(w_1 - w_2)^2 - 2\lambda = 0$$  \hspace{1cm} (132)
$$w_1 - w_2_t + 3(w_1^2 - w_2^2)_x + w_1,xxx - w_2,xxx = 0$$  \hspace{1cm} (133)

So to repeat the point of BT’s, if we have one $w_1$ differentiating to a particular KdV solution, $u_1$, then this BT ensures that $w_2$ differentiates to a $u_2$ which is also a KdV solution, (because the relevant $v$ satisfied the mKdV equation).

### 8.1 Check;

The obvious check is to start with the Bäcklund transformation and to make sure it derives the KdV equaiton for both $u_1$ and $u_2$. In terms of $w_i$ the KdV equation is

$$w_{i,xt} + 6w_{i,x}w_{i,xx} + w_{i,xxxx} = 0 \quad i = 1, 2$$  \hspace{1cm} (134)

Differentiating eq.133 by $x$ gives immediately

$$(w_1 - w_2)_{xt} + 6(w_{1,x}w_{1,xx} - w_{2,x}w_{2,xx}) + w_{1,xxxx} - w_{2,xxxx} = 0$$  \hspace{1cm} (135)

so if we can find a similar equation but with + signs everywhere we can subtract and add and get our KdV equations. The obvious step is to differentiate 132 with respect to $t$. To simplify things I’ll define $A = w_1 + w_2$ and $B = w_1 - w_2$. The BT becomes

$$A_x + \frac{B^2}{2} - 2\lambda = 0$$
$$B_t + 3A_xB_x + B_{xxx} = 0$$  \hspace{1cm} (136)
Differentiating the first of these by \( t \) gives

\[
0 = A_{xt} + BB_t = A_{xt} - B(3A_xB_x + B_{xxx})
\]

where I substituted from the 2nd half of the BT in the last line. Now we need to get \( A_{xxxx} \). We can keep differentiating the first equation to get some identities;

\[
\begin{align*}
A_{xx} + B_xB &= 0 \quad (137) \\
A_{xxx} + B_{xx}B + B_x^2 &= 0 \quad (138) \\
A_{xxxx} + B_{xxx}B + 3B_xB_{xx} &= 0 \quad (139)
\end{align*}
\]

Adding \( A_{xxxx} \) from the last of these gives

\[
A_{xt} + A_{xxxx} - (3A_xBB_x + BB_{xxx} + A_{xxxx}) = A_{xt} + A_{xxxx} - (3A_xBB_x - 3B_xB_{xx}) = A_{xt} + A_{xxxx} + 3(A_xA_{xx} + B_xB_{xx})
\]

Finally we are ready to expand out and find (since \( A_xA_{xx} + B_xB_{xx} = 2(w_1xw_{1,xx} + w_2xw_{2,xx}) \)) that

\[
(w_1 + w_2)_{xt} + 6(w_1xw_{1,xx} + w_2xw_{2,xx}) + w_{1,xxxx} + w_{2,xxxx} = 0 \quad (140)
\]

In conjunction with eq.135 (adding and subtracting) we see both \( u_1 = w_{1,x} \) and \( u_2 = w_{2,x} \) satisfy the KdV eqn.

### 8.2 The KdV solitary wave from the vacuum

Choose \( w_2 = 0 \) and for convenience of notation write \( w_1 \equiv w \). Then eq.132 implies

\[
\begin{align*}
w_x &= 2\lambda - \frac{1}{2}w^2 \\
\int \frac{dw}{4\lambda - w^2} &= \frac{x}{2} + \frac{1}{2}A(t) \quad (141)
\end{align*}
\]

where \( A(t) \) is a constant of \( x \) integration. Now find

\[
\frac{1}{2\sqrt{\lambda}}\tanh^{-1}(w/2\sqrt{\lambda}) = \frac{x}{2} + \frac{A}{2} \quad (142)
\]

if \( \lambda > 0 \) (otherwise would find tan). Now need to fix \( A(t) \) using eq.133. Reading it off we have

\[
w_t + 3w_x^2 + w_{xxx} = 0 \quad (143)
\]
Differentiating eq.132 once we get relations $w_{xx} = -ww_x$ and $w_{xxx} = -w_x^2 - w w_{xx} = -w_x^2 + w^2 w_x$, which we can use to simplify. We have
\[
0 = w_t + 2w_x^2 + w^2 w_x \\
= w_t + 2w_x(w_x + \frac{1}{2}w^2) \\
= w_t + 4\lambda w_x 
\] (144)
So this equation gives
\[
\left( \frac{\partial}{\partial t} + 4\lambda \frac{\partial}{\partial x} \right) w = 0 
\] (145)
which means simply that $w$ is a function of $x - x_0 - 4\lambda t$ where $x_0$ is a constant, and we can write
\[
w(x, t) = 2\sqrt{\lambda} \tanh(\sqrt{\lambda}(x - x_0 - 4\lambda t)) \] (146)
To get the previous solution we identify $v = 4\lambda$ (this is velocity $v$ not to be confused with the $v$ in the mKdV equation) and we get
\[
w = \sqrt{v} \tanh(\frac{\sqrt{v}}{2}(x - x_0 - vt))
\]
Finally we need to differentiate to get $u$ which gives
\[
u = w_x = \frac{v}{2} \text{sech}^2(\frac{\sqrt{v}}{2}(x - x_0 - vt)),
\]
the solitary wave solution for KdV.

Note that there is another solution that we will need later; if $|w| > 2\sqrt{\lambda}$ then the integral would have given
\[
\frac{1}{2\sqrt{\lambda}} \coth^{-1}(w/2\sqrt{\lambda}) = \frac{x}{2} + \frac{A}{2} \\
w = 2\sqrt{\lambda} \coth(\sqrt{\lambda}(x - x_0 - 4\lambda t)) 
\] (147)
This solution is singular when the argument vanishes (i.e. where $1/\tanh = 0$).

8.3 The theorem of Permutability and non-linear superposition for KdV

Again we use that fact that for any pair of single soliton solutions $w_1$ and $w_2$ it is possible to choose constants of integration in the 2nd BT’s such that a subsequent BT yields the same $w_4$. 

43
Now we can consider just the $\partial_x$ part of the BTs (remember the two parts gave a consistency check in the case of SG, but here the $\partial_t$ part of the BT would be a bit complicated to do). The upper route gives

$$
(w_0 + w_1)_x = 2\lambda_1 - \frac{1}{2}(w_0 - w_1)^2
$$
$$
(w_1 + w_3)_x = 2\lambda_2 - \frac{1}{2}(w_1 - w_3)^2
$$

(148)

The lower route gives

$$
(w_0 + w_2)_x = 2\lambda_2 - \frac{1}{2}(w_0 - w_2)^2
$$
$$
(w_2 + w_3)_x = 2\lambda_1 - \frac{1}{2}(w_2 - w_3)^2
$$

(149)

Subtracting the second from the first equation in both cases gives

$$
(w_3 - w_0)_x = 2(\lambda_2 - \lambda_1) + \frac{1}{2}(w_0 - w_1)^2 - \frac{1}{2}(w_1 - w_3)^2
$$
$$
= 2(\lambda_1 - \lambda_2) + \frac{1}{2}(w_0 - w_2)^2 - \frac{1}{2}(w_2 - w_3)^2
$$

(150)

Equating these gives

$$
4(\lambda_2 - \lambda_1) = \frac{1}{2}(w_0 - w_2)^2 + \frac{1}{2}(w_1 - w_3)^2 - \frac{1}{2}(w_0 - w_1)^2 - \frac{1}{2}(w_2 - w_3)^2
$$
$$
= (w_3 - w_0)(w_2 - w_1)
$$

so that

$$
w_3 = w_0 + \frac{4(\lambda_2 - \lambda_1)}{w_2 - w_1}
$$

(151)

As before have a purely algebraic relation between the solutions. This is another non-linear superposition principle.
8.4 Two soliton solutions

As for the SG we can now build a two soliton solution for the KdV equation. One interesting feature is that immediately from eq.151 we can see that to avoid singularities we need to avoid \( w_1 = w_2 \). But two non-singular tanh solutions for \( w_i \) always cross whereas we can choose one singular and one non-singular solution so they do not cross, as shown below.

![Graph showing two soliton solutions](image)

The singular solution has to be the one with the larger velocity. The result is a non-singular \( w_3 \).

To get the solution define

\[
\theta_i = \sqrt{\lambda_i}(x - x_0 - 4\lambda_i t) (152)
\]

Then the two solutions are assuming \( \lambda_2 > \lambda_1 \)

\[
\begin{align*}
w_1 &= 2\sqrt{\lambda_1} \tanh \theta_1 \\
w_2 &= 2\sqrt{\lambda_2} \coth \theta_2
\end{align*}
\]

and since \( w_0 = 0 \) is the vacuum, we have

\[
w_3 = \frac{2(\lambda_2 - \lambda_1)}{\sqrt{\lambda_2 \cosh \theta_2 - \sqrt{\lambda_1 \tanh \theta_1}}} = \frac{2(\lambda_2 - \lambda_1) \sinh \theta_2 \cosh \theta_1}{\sqrt{\lambda_2 \cosh \theta_1 \cosh \theta_2 - \sqrt{\lambda_1} \sinh \theta_1 \sinh \theta_2}}
\]

Finally we need to differentiate this to get \( u_3 \). The resulting function isn’t particularly pleasant,

\[
u_3 = 2(\lambda_2 - \lambda_1) \frac{\lambda_2 \cosh^2 \theta_1 + \lambda_1 \sinh^2 \theta_2}{(\sqrt{\lambda_2 \cosh \theta_1 \cosh \theta_2 - \sqrt{\lambda_1} \sinh \theta_1 \sinh \theta_2})^2} (153)
\]

For the asymptotics we can do the same operation as for the SG equation. Look around the solution labelled \( i = 1 \) or \( 2 \). Take \( t \to t_i + \delta t \) with \( \delta x = x - x_0 - v_i t_i \) where \( v_i = 4\lambda_i \) and let \( t_i \to \pm \infty \). We have

\[
\begin{align*}
\theta_i &= \frac{\sqrt{v_i}}{2} (\delta x - v_i \delta t) \\
\theta_{j \neq i} &= \frac{\sqrt{v_j}}{2} (\delta x - v_j \delta t) + \frac{\sqrt{v_j}}{2} (v_i - v_j) t_i
\end{align*}
\]

(154)
Near the \( i \) soliton we have (by definition) \( \delta x, \delta t \approx 0 \), so that \( \theta_i \) is small but \( \theta_j \neq i \to \pm \infty \).

| \( i = 1 \) | \( \theta_j = 2 \to -\infty \) | \( \theta_j = 2 \to +\infty \) |
| \( i = 2 \) | \( \theta_j = 1 \to +\infty \) | \( \theta_j = 1 \to -\infty \) |

Defining \( \text{sign}(\theta_j) = \eta_j \) the asymptotic form of the solution around \( \delta x = 0 \) can be written by noting that \( \cosh \theta_j \to \frac{1}{2}e^{\vert \theta_j \vert} \) and \( \sinh \theta_j \to \frac{\eta_j}{2}e^{\vert \theta_j \vert} \). We get

\[
\left. u_3 \right|_i = 2(\lambda_2 - \lambda_1) \frac{\lambda_i}{(\sqrt{\lambda_2} \cosh \theta_i - \sqrt{\lambda_1} \eta_j \sinh \theta_i)^2} \quad (155)
\]

We now use

\[
\sqrt{\lambda_2} \cosh \theta_i - \sqrt{\lambda_1} \eta_j \sinh \theta_i = \frac{1}{2}(\sqrt{\lambda_2 - \eta_j \sqrt{\lambda_1}})e^{\theta_i} + \frac{1}{2}(\sqrt{\lambda_2 + \eta_j \sqrt{\lambda_1}})e^{-\theta_i} = \frac{2}{\sqrt{\lambda_2 - \eta_j \sqrt{\lambda_1}}} \left( \frac{\sqrt{\lambda_2 - \eta_j \sqrt{\lambda_1}}}{\sqrt{\lambda_2 + \eta_j \sqrt{\lambda_1}}} e^{\theta_i} + \frac{\sqrt{\lambda_2 + \eta_j \sqrt{\lambda_1}}}{\sqrt{\lambda_2 - \eta_j \sqrt{\lambda_1}}} e^{-\theta_i} \right) \quad (156)
\]

\[
= \frac{\sqrt{\lambda_2 - \lambda_1}}{2} \left( \frac{\sqrt{\lambda_2 - \eta_j \sqrt{\lambda_1}}}{\sqrt{\lambda_2 + \eta_j \sqrt{\lambda_1}}} e^{\theta_i} + \frac{\sqrt{\lambda_2 + \eta_j \sqrt{\lambda_1}}}{\sqrt{\lambda_2 - \eta_j \sqrt{\lambda_1}}} e^{-\theta_i} \right) \quad (157)
\]

Now define a phase shift

\[
c = \frac{1}{2} \log \frac{\sqrt{\lambda_2 - \lambda_1}}{\sqrt{\lambda_2 + \lambda_1}} \quad (158)
\]

We have

\[
\sqrt{\lambda_2} \cosh \theta_i - \sqrt{\lambda_1} \eta_j \sinh \theta_i = \frac{\sqrt{\lambda_2 - \lambda_1}}{2} (e^{\theta_i + \eta_j c} + e^{-(\theta_i + \eta_j c)}) = \sqrt{\lambda_2 - \lambda_1} \cosh(\theta_i + \eta_j c) \quad (159)
\]

and substituting we get

\[
\left. u_3 \right|_i = \frac{v_i}{2} \text{sech}^2 \left( \frac{\sqrt{v_i}}{2}(x - x_0 - v_it + \frac{2\eta_j}{\sqrt{v_i}}c) \right) \quad (160)
\]

Going to our table, we see that the slower \( i = 1 \) wave is shifted back (i.e. \( x_0 \) gets a negative contribution) by

\[
\frac{4}{\sqrt{v_i}} c
\]

46
and the faster $i = 1$ wave shifted *forwards* by a slightly smaller amount but otherwise they retain their form. Note we could have done the asymptotics using $w_3$ rather than $u_3$ which might have been a bit easier. The result in a 3D plot in the $(x, t)$ plane looks like

and three stages in the interaction look like

![Diagram](image-url)
9 Current/charge conservation

Current conservation is an important part of many aspects of physics. For solitons they help to explain why the motion is so restricted that it can be computed exactly. The basic idea is to construct quantities, some combination of \( u, u_x, u_{xx}, \ldots, u_t, u_{tt}, \ldots \) that are time constant during for any solution;

\[
Q = \int F(u, u_x, u_{xx}, \ldots, u_t, u_{tt}, \ldots) \, dx
\]

such that

\[
\frac{dQ}{dt} = 0.
\]

An example for the KdV equation is

\[
Q_1 = \int u \, dx
\]

which tells us that the total area under the solution is constant. Clearly this restricts the solution. We have already met the topological kink number in the SG equation, but it will turn out that the KdV equation has an infinite number of conserved charges restricting the flow. This means that not much (e.g. chaos) can happen. The fact that solitons do not break up under collision can be put down to the conservation of charges.

How do we derive this and other conserved charges? Consider the conservation of mass for gas in a pipe flowing at speed \( V(x) \) along its length. In an element \( \delta x \) long gas flow in at a rate \( \rho(x)V(x) \) where \( \rho \) is the density, and out at a rate \( \rho(x + \delta x)V(x + \delta x) = \rho(x)V(x) + \delta x \frac{\partial (\rho V)}{\partial x} \). The nett mass outflow from the element is \( \delta x \frac{\partial (\rho V)}{\partial x} \). But the rate of loss of mass is \( -\frac{\partial (\delta x \rho)}{\partial t} \). If mass is conserved then the two must be equal and

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial x} = 0 \quad (160)
\]

This is the mass conservation equation and restricts possible velocity flow. What happens to the total mass \( M = \int \rho(x)dx \) over time?

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \rho \, dx = \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial t} \, dx
\]

\[
= -\int_{-\infty}^{\infty} \frac{\partial (\rho V)}{\partial x} \, dx
\]

\[
= -[\rho V]_{-\infty}^{\infty}
\]

\[
= 0 \quad (161)
\]

if the density or velocity is zero (or even just equal) at the endpoints at \( x \to \pm \infty \). Total mass in the pipe is conserved.
9.1 Generalization of this idea

Assume two quantities $X$ and $T$ can be assembled from the $u$ $u_x$ $u_{xx}$ ... such that

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0 \quad (162)$$

For the example above the role of $X$ was played by $\rho u$ the density flow, and $T$ was the local density. Assume further that $X$ remains constant at $x \to \pm \infty$

$$X \to c \quad (163)$$

The eq.162 means that

$$\frac{d}{dt} \int_{-\infty}^{\infty} T \, dx = \int_{-\infty}^{\infty} \frac{\partial T}{\partial t} \, dx = -\int_{-\infty}^{\infty} \frac{\partial X}{\partial x} \, dx = -[X]_{-\infty}^{\infty} = 0 \quad (164)$$

so

$$Q = \int_{-\infty}^{\infty} T \, dx$$

is a conserved quantity (it was the total mass in the previous example).

9.2 Example: the wave equation

Consider

$$u_{+-} = 0 \quad (165)$$

Note that any polynomial $h$ of $u_-$ obeys $\partial_+ h(u_-) = 0$ since

$$\partial_+ h(u_-) = h'u_+ = 0.$$ 

Furthermore since $\partial_+ = \partial_t + \partial_x$ we can set

$$X = T = (u_-)^n \quad (166)$$

for any $n \in \mathbb{Z}^+$. Similarly we can also have

$$X = -T = (u_+)^n \quad (167)$$

Setting $n = 1$ gives nothing since using D’Alembert’s solution $u = f(x_+) + g(x_-)$ we get

$$Q = \int_{-\infty}^{\infty} g' \, dx = \int_{-\infty}^{\infty} g_x \, dx = 0$$
Setting \( n = 2 \) we get two independent quantities
\[
(u_x \pm u_t)^2 = u_x^2 \pm 2u_xu_t + u_t^2
\]
Equivalently we can assemble them into more recognizable combinations
\[
E = \int_{-\infty}^{\infty} \frac{1}{2}(u_t^2 + u_x^2) \, dx
\]
\[
M = \int_{-\infty}^{\infty} u_t u_x \, dx
\]
\[(168)\]
\( E \) is the total energy conservation in the absence of a potential \( \mathcal{P}(u) \). \( M \) is the momentum. Clearly we get an infinite number of conserved quantities here but then it’s somewhat trivial.

9.3 Conserved currents in the KdV equation
Looking at the KdV equation
\[
u_t + 6uu_x + u_{xxx} = 0
\]
it is obvious that the equation itself is in the form eq.162 with
\[
T = u
\]
\[
X = 3u^2 + u_{xx}
\]
\[(169)\]
so that
\[
Q_1 = \int_{-\infty}^{\infty} u \, dx
\]
must be conserved. That is the area under any solitary waves or interacting solitons is preserved as long as the interval is infinite. Now multiply KdV by \( u \) to get
\[
\frac{1}{2}(u^2)_t + (2u^3 + uu_{xx} - \frac{1}{2}u_x^2)_x = 0
\]
\[(170)\]
so that remarkably
\[
Q_2 = \int_{-\infty}^{\infty} u^2 \, dx
\]
is constant as well. As you can imagine the possibilities for \( u \) are getting increasingly restricted. The next one is more tricky. We have to do \((3u^2 - u_x \frac{\partial}{\partial x}) \times KdV \) to get
\[
(u^3 - \frac{1}{2}u_x^2)_t + (\text{some function})_x = 0
\]
\[(171)\]
\[
Q_3 = \int_{-\infty}^{\infty} u^3 - \frac{1}{2}u_x^2 \, dx
\]
Physically (in e.g. water waves) these three conservation laws correspond to conservation of mass, momentum and energy. But the series does not stop there. By trial and error conserved currents were found up to

\[ Q_{10} = \int_{-\infty}^{\infty} \left( u^{10} - 60u^7u_x^2 - \{29 \text{ terms}\} + \frac{1}{4862}(u_{xxxxx})^2 \right) dx \]

(found in the 1960’s) where \( Q_n = \int_{-\infty}^{\infty} (u^n + \ldots) dx \). This gets increasingly messy, but a general method was discovered that shows there are an infinite number of conserved currents.

### 9.4 The Gardner transform (1968)

This argument begins with the generalized mKdV seen earlier by transforming \( u \) as follows

\[ u = \lambda - v^2 - v_x \quad (172) \]

to get

\[ (2v + \frac{\partial}{\partial x})[v_t + 6(\lambda - v^2)v_x + v_{xxx}] = 0 \quad (173) \]

At the moment notice that although solutions \( v \) to the mKdV (the stuff in \([ \ ]\)'s) are also solutions to KdV, the converse is not necessarily true. Gardner modified this equation by putting

\[ v = \varepsilon w + \frac{1}{2\varepsilon} \]

\[ \lambda = \frac{1}{(2\varepsilon)^2} \quad (174) \]

Then

\[ \lambda - v^2 = -w - \varepsilon^2 w^2 \]

\[ u = -w - \varepsilon w_x - \varepsilon^2 w^2 \quad (175) \]

The extra parameter \( \varepsilon \) will help us generate the currents. We now find

\[ (2\varepsilon w + \frac{1}{\varepsilon} + \frac{\partial}{\partial x})[\varepsilon w_t - 6(w + \varepsilon w^2)\varepsilon w_x + \varepsilon w_{xxx}] = 0. \quad (176) \]

Hence \( u \) given by

\[ u = -w - \varepsilon w_x - \varepsilon^2 w^2 \]

is a solution of the KdV equation if \( w \) satisfies

\[ [w_t - 6(w + \varepsilon w^2)w_x + w_{xxx}] = 0. \quad (177) \]

We see immediately that setting \( \varepsilon = 0 \) we get the KdV equation (albeit with reversed sign in the middle term). Again this equation is in the form \( T_t + X_x = 0 \) and hence

\[ Q = \int_{-\infty}^{\infty} w \, dx \]
is conserved for any $\varepsilon$.

Any solution $w$ at non-zero $\varepsilon$ will be a function of $\varepsilon$ and so let’s expand the $w$ as
\[ w = \sum_n w_n \varepsilon^n. \tag{178} \]

$w_0$ is some solution that is valid when $\varepsilon = 0$ so it satisfies the KdV equation (with sign reversed on the $ww_x$ term. Our conserved charge can also be expanded
\[ Q = \int_{-\infty}^{\infty} w \, dx \]
\[ = \sum_n \int_{-\infty}^{\infty} w_n \varepsilon^n \, dx \]
\[ = \sum_n \tilde{Q}_n \varepsilon^n \tag{179} \]

Since $\varepsilon$ is a free parameter the $\tilde{Q}_n$ must each individually be conserved so we have an infinite number of conserved currents. (For example, $\tilde{Q}_0$ is obviously conserved since we can always take $\varepsilon = 0$. But now the combined $Q - \tilde{Q}_0$ must also be conserved. Again taking $\varepsilon \to 0$ we see that $\lim_{\varepsilon \to 0} (Q - \tilde{Q}_0)/\varepsilon = \tilde{Q}_1$ is conserved, and so on.)

All we need to find the $\tilde{Q}_n$ are the functions $w_n$. From the above we have
\[ u = -w - w_x - \varepsilon^2 w^2 \]
\[ = - \sum_n w_n \varepsilon^n - \sum_n w_{n,x} \varepsilon^{n+1} - \varepsilon^2 (\sum_n w_n \varepsilon^n)^2 \tag{180} \]

We now simply have to iteratively equate coefficients of $\varepsilon$ to find the $w_n$ in terms of $u$. In detail
\[ w_0 = -u \]
\[ w_1 + w_{0,x} = 0 \]
\[ w_2 + w_{1,x} + w_0^2 = 0 \]
\[ w_3 + w_{2,x} + 2w_0 w_1 = 0 \]
\[ w_4 + w_{3,x} + w_1^2 + 2w_0 w_2 = 0 \tag{181} \]

and so on. Then substituting $u$ and working down the ladder we finally get
\[ w_0 = -u \]
\[ w_1 = u_x \]
\[ w_2 = -(u_{xx} + u^2) \]
\[ w_3 = u_{xxx} + 2uu_x \]
\[ w_4 = -w_{3,x} - u_x^2 - 2u(u^2 + u_{xx}) \text{ etc.} \tag{182} \]
You then get $w_{n=odd}$ are total derivatives so $\int w_{n=odd} \, dx$ doesn’t tell us much.

\[
\begin{align*}
\int w_0 \, dx &= -Q_1 \\
\int w_1 \, dx &= 0 \\
\int w_2 \, dx &= -Q_2 \\
\int w_3 \, dx &= 0 \\
\int w_4 \, dx &= -2Q_3 \quad etc
\end{align*}
\]  

(183)

Note we can essentially ignore $w_{3,x}$ part in the $w_4$ integral because it is a total derivative. (To get the last one you have to use $uu_{xx} = (uu_x)_x - u_x^2$).
10 Symmetries and conservation laws

There is an important and very deep connection between symmetries and conservation laws. Symmetries can be seen most easily in the lagrangian formalism:

\[ S[u] = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} L(u, u_t, u_x) \, dx \, dt \]  
(184)

with for example

\[ L = \frac{u^2}{2} - \frac{u_x^2}{2} - (1 - \cos u) \]  
(185)

Hamilton’s principle is that if we let \( u \to u + \delta u \) then \( \delta S = 0 \) implies the Euler-Lagrange equations. One thing we neglected before though, which is important now are the boundary terms. Making the variation we actually get

\[ \delta S = 0 = \int \int L(u + \delta u, u_t + \delta u_t, u_x + \delta u_x) - L(u, u_t, u_x) \, dt \, dx \]

\[ = \int \int \left( \frac{\partial L}{\partial u} \delta u + \frac{\partial L}{\partial u_t} \delta u_t + \frac{\partial L}{\partial u_x} \delta u_x \right) \, dt \, dx \]

\[ + \text{boundary terms} \]
(186)

(Here for example \( d/dt \) means use the chain rule with \( u(x, t) \) and \( u_x(x, t) \) but do not differentiate with respect to \( x \).) Setting everything in brackets to zero gives the E-L equation as earlier.

\[ \frac{\partial L}{\partial u} - \frac{d}{dt} \frac{\partial L}{\partial u_t} - \frac{d}{dx} \frac{\partial L}{\partial u_x} = 0 \]  
(187)

However to get the last two terms I integrated by parts once and generated some “boundary terms” (i.e. complete derivatives). In other words I used

\[ \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \delta u \right) = \delta u \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \right) + \delta u_x \left( \frac{\partial L}{\partial u_x} \right) \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} \delta u \right) = \delta u \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} \right) + \delta u_t \left( \frac{\partial L}{\partial u_t} \right) \]  
(188)

and the additional boundary terms are

\[ \int \int \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} \right) \, dx \, dt. \]

These will be important for telling us about current conservation later but since they are total derivatives they make no difference to the equations of motion. For the above example we get the SG equation back from the E-L equation;

\[ u_{tt} - u_{xx} + \sin u = 0. \]
10.1 Additional Lagrangians 1: the KdV equation

Now consider
\[ \mathcal{L} = \frac{1}{2} w_x w_t + w_x^3 - \frac{1}{2} w_{xx}^2. \] (189)

This is now a function of \( w_{xx} \) as well as \( w, w_x \) and \( w_t \) so we need to modify the Euler-Lagrange equation slightly. At the risk of confusion I’ll still use \( u \) to discuss the Euler-Lagrange equation. So again let \( u \to u + \delta u \) then if \( S \to S + \delta S \) we extend fairly easily what was done before

\[ \delta S = 0 = \int \int \mathcal{L}(u + \delta u, u_t + \delta u_t, u_x + \delta u_x, u_{xx} + \delta u_{xx}) - \mathcal{L}(u, u_t, u_x, u_{xx}) \ dt \ dx \]

\[ = \int \int \frac{\partial \mathcal{L}}{\partial u} \delta u + \frac{\partial \mathcal{L}}{\partial u_t} \delta u_t + \frac{\partial \mathcal{L}}{\partial u_x} \delta u_x + \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u_{xx} \ dt \ dx \]

\[ = \int \int \left( \frac{\partial \mathcal{L}}{\partial u} - d \frac{\partial \mathcal{L}}{\partial u_t} - d \frac{\partial \mathcal{L}}{\partial u_x} + d^2 \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \delta u \ dt \ dx \]

+ boundary terms \hfill (190)

To get the middle two terms I integrated by parts once, and to get the last term I integrated by parts twice; i.e. in other words I use

\[ \frac{d^2}{dx^2} \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u \right) - 2 \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u_x \right) = \delta u \frac{d^2}{dx^2} \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) - \delta u_x \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \] (191)

since everything on the LHS of the first line is a total derivative so again giving boundary terms that are irrelevant to the E-L equations. These are as before with the extra \( u_{xx} \) term;

\[ \text{boundary terms} = \int \int \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial u_t} \delta u + 2 \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u_x - \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u \right) \right) \]

\[ + \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial u_t} \right) \ dt \ dx \]

\[ \int \int \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial u_x} \delta u + \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u_x - \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u \right) \right) \]

\[ + \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial u_t} \right) \ dt \ dx. \] (192)

The E-L equation becomes

\[ \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial \mathcal{L}}{\partial u_{xx}} = 0 \] (193)

With the choice of lagrangian we get (using \( w \) in place of \( u \)) that

\[ 0 = -\frac{d}{dt} \left( \frac{w_x}{2} \right) - \frac{d}{dx} \left( 3 w_x^2 + \frac{1}{2} w_t \right) + \frac{d^2}{dx^2} w_{xx} \]

\[ = -w_{xt} - 6 w_x w_{xx} - w_{xxxx} \] (194)

so \( w_x \) obeys the KdV equation!
10.2 Additional Lagrangians 2: The non-linear Schrodinger (NLS) equation

The BT for this equation was done in question 3 of sheet 2. Consider

\[ \mathcal{L} = \frac{i}{2}(u^* u_t - uu^*_t) - |u_x|^2 - \frac{\lambda}{2}|u|^4. \tag{195} \]

\( u \) here is complex but \( S \) is still real since \( \mathcal{L}^* = \mathcal{L} \). Treat \( u \) and \( u^* \) as independent (Like \( z \) and \( z^* \)) when differentiating. Varying by \( u^* \) we get

\[ 0 = \frac{\partial \mathcal{L}}{\partial u^*} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_t^*} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x^*} = -\frac{i}{2}u_t - u_{xx} + \lambda |u|^2 u. \tag{196} \]

Varying by \( u \) would just give the complex conjugate of this, so doesn’t add any new information. This is a version of the NLS equation. It’s an important also integrable equation with applications to fibre optics.
10.3 Noether’s theorem: conserved quantities from symmetries of the Lagrangian

Return to conservation laws. Recall we seek $X$ and $T$ such that

$$ X_x + T_t = 0 $$

$$ X|_{x=\pm\infty} = 0 $$

so that $d_t \int Tdx = \int T_t dx = -\int X_x dx = 0$.

10.3.1 Time translation symmetry

Now consider the infinitessimally small constant shift $t \rightarrow t + \varepsilon$. If $S$ is invariant under this shift it is called a time-translation symmetry of the action, and there is a conserved current associated with that given by the boundary terms. We see it as follows. Under the shift we have by Taylor expanding that

$$ t \rightarrow t + \varepsilon $$

$$ u(x, t) \rightarrow u(x, t + \varepsilon) = u(x, t) + \varepsilon u_t $$

$$ u_t(x, t) \rightarrow u_t(x, t + \varepsilon) = u_t(x, t) + \varepsilon u_{tt} $$

$$ u_x(x, t) \rightarrow u_x(x, t + \varepsilon) = u_x(x, t) + \varepsilon u_{xt} $$

$$ u_{xx}(x, t) \rightarrow u_{xx}(x, t + \varepsilon) = u_{xx}(x, t) + \varepsilon u_{xxt} $$

...etc

The extra bits on the RHS we can take to be $\delta u, \delta u_t, \delta u_x, \delta u_{xx}$ etc. Now look at the shift in the action, or rather everything inside the integral. Since the E-L equations are satisfied locally, all we have left are the boundary terms. These are (assuming for the moment that $L = L(u, u_t, u_x)$ only and does not depend on $u_{xx}$ or very importantly explicitly on $x$ or $t$)

$$ \delta L = \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \delta u_t \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} \delta u_t \right) $$

$$ = \varepsilon \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} u_t \right) + \varepsilon \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} u_t \right) $$

(199)

dividing by $\varepsilon$ and taking the $\varepsilon \rightarrow 0$ limit to get $\frac{\delta L}{\varepsilon} \rightarrow \frac{dL}{dt}$ we have the relation

$$ \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} u_t \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} u_t - L \right) = 0 $$

(200)

This relation is in precisely the form $X_x + T_t = 0$. When the action does depend on $u_{xx}$ as well, we get the slightly more complicated

$$ \frac{d}{dx} \left( \frac{\partial L}{\partial u_x} u_t + \frac{\partial L}{\partial u_{xx}} u_{xt} \right) - \frac{d}{dx} \left( \frac{\partial L}{\partial u_{xx}} u_t \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} u_t - L \right) = 0 $$

(201)
In either case the conserved current is

$$H = \frac{\partial L}{\partial u_t} u_t - L$$  \quad (202)

Some of you may recognize this as the Hamiltonian (with $p \equiv \partial L / \partial u_t$) hence the name $H$. So the Hamiltonian is a conserved current because the action is time-translation invariant. Conversely an explicit time dependence in the Lagrangian would break time translation invariance, and the current would no longer be conserved. This is a very deep relation that has many consequences.

### 10.3.2 Examples

**SG:**

$$H = \frac{\partial L}{\partial u_t} u_t - L = u_t^2 - \frac{u_t^2}{2} + \frac{u_t^2}{2} + (1 - \cos u)$$

$$= \frac{u_t^2}{2} + \frac{u_t^2}{2} + (1 - \cos u)$$

**KdV:** Here we recognize the charge $Q_3$ as the Hamiltonian! This is like the energy of the water wave;

$$H = \frac{\partial L}{\partial w_t} w_t - L = \frac{1}{2} w_x w_t - \frac{1}{2} w_x w_t - w_x^3 + \frac{1}{2} w_{xx}^2$$

$$= -w_x^3 + \frac{1}{2} w_{xx}^2$$

$$= -u^3 + \frac{1}{2} u_x^2.$$

**NLS:** Here the answer is again associated with energy

$$H = \frac{\partial L}{\partial u_t} u_t + \frac{\partial L}{\partial u_t^*} u_t^* - L = u_t \frac{i}{2} u_t^* - u_t^* \frac{i}{2} u - \frac{i}{2} (u_t^* u_t + uu_t^*) + |u_x|^2 + \frac{\lambda}{2} |u|^4.$$

$$= |u_x|^2 + \frac{\lambda}{2} |u|^4.$$

### 10.3.3 Space translation symmetry

Now consider the infinitesimally small constant shift $x \to x + \varepsilon$. If $S$ is invariant under this shift it is called a space-translation symmetry of the action, and again there is a conserved current associated with that given by the boundary terms. Under the shift we have

$$x \to x + \varepsilon$$

$$u(x, t) \to u(x, t + \varepsilon) = u(x, t) + \varepsilon u_x$$

$$u_t(x, t) \to u_t(x, t + \varepsilon) = u_t(x, t) + \varepsilon u_{tx}$$

$$u_x(x, t) \to u_x(x, t + \varepsilon) = u_x(x, t) + \varepsilon u_{xx}$$

$$u_{xx}(x, t) \to u_{xx}(x, t + \varepsilon) = u_{xx}(x, t) + \varepsilon u_{xxxx}$$

...etc

(203)
To get the conserved current we can read of from eq.201 that
\[
\frac{d}{dx} \left( \frac{\partial L}{\partial u_x} u_x + \frac{\partial L}{\partial u_{xx}} u_{xx} - \frac{d}{dx} \left( \frac{\partial L}{\partial u_{xx}} \right) u_x - L \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} u_x \right) = 0 \tag{204}
\]
The only difference from the time-translation case is that the \(-L\) went in the \(d/dx\) term. In this case the conserved current is
\[
P = \frac{\partial L}{\partial u_x} u_x \tag{205}
\]

### 10.3.4 Examples

**Sine Gordon:** \(P = \frac{\partial L}{\partial u_t} u_x = u_t u_x \)

This is the momentum density

**KdV:** \(P = \frac{\partial L}{\partial w_t} w_x = \frac{1}{2} w_x^2 = \frac{1}{2} u^2 \)

We recognize the charge \(Q_2\) as the momentum!

**NLS:** \(P = \frac{\partial L}{\partial u_t} u_x + \frac{\partial L}{\partial u_t^*} u_x^* = \frac{i}{2} (u_x u^* - u_x^* u) = \frac{i}{2} \partial_x |u|^2 \)

Again this is linked to momentum density in QM. (c.f. \(p = i\partial_x \) and \(\langle p \rangle = \int u^* \hat{p} u \))

### 10.4 Other conserved currents beyond energy and momentum

The currents derived above are the standard ones of many theories. But where does \(Q_1 = \int u \, dx\) come from? Consider instead shifts in \(w\). The KdV lagrangian has an obvious symmetry when \(w \rightarrow w + \varepsilon\) where \(\varepsilon\) is again a constant.

We will to maintain generality work with a generic lagrangian whose field is named \(u\) and assume that \(\varepsilon\) is not necessarily a constant, but can also be a function of \(u\), so \(\varepsilon = \varepsilon(u)\). Now we have simply that \(\delta u = \varepsilon\), but \(\delta u_x = \varepsilon_x\) and \(\delta u_t = \varepsilon_t\). We get
\[
\frac{d}{dx} \left( \frac{\partial L}{\partial u_x} \varepsilon + \frac{\partial L}{\partial u_{xx}} \varepsilon_x - \frac{d}{dx} \left( \frac{\partial L}{\partial u_{xx}} \right) \varepsilon \right) + \frac{d}{dt} \left( \frac{\partial L}{\partial u_t} \varepsilon \right) = 0 \tag{206}
\]
The \(\varepsilon \partial L / \partial u\) differential term in the limit goes into the E-L equations and we have left the \(\varepsilon\) inside the derivatives now, as it is not necessarily (although is for KdV) constant. The conserved current is
\[
T = \frac{\partial L}{\partial u_t} \varepsilon \tag{207}
\]

**KdV:** We see immediately (renaming \(u \rightarrow w\)) that for the symmetry \(w \rightarrow w + a\) (\(a = \text{constant}\))
\[
T = \frac{a}{2} w_x = \frac{a}{2} u. \tag{208}
\]
So the charge $Q_1$ is associated with symmetry under constant shifts in $w$.

**NLS** has a symmetry under the global rotation $u \rightarrow e^{i\theta}u$ and simultaneously $u^* \rightarrow e^{-i\theta}u^*$ where $\theta$ is constant. Here for infinitesimal $\theta$ we have $\varepsilon = i\theta u$ and also a shift $\varepsilon^* = -i\theta u^*$ for $u^*$. We find

$$T = \frac{\partial L}{\partial u_t} \varepsilon + \frac{\partial L}{\partial u^*_t} \varepsilon^*$$
$$= -\frac{\theta}{2} (u^* u - uu^*) = -\theta |u|^2$$  \hspace{1cm} (209)$$

This conservation law gives $\int |u|^2 dx = 0$, i.e. conservation of total probability.

**Generally any continuous symmetry gives a conservation law via Noether’s theorem.** In more complicated theories the set of conserved currents form a group called a Lie group. e.g. in special relativity the energy and momentum combine to form the (conserved) energy momentum tensor. The conserved group is the Poincare group. Principle underlying all modern day particle physics. Unfortunately all the other conserved currents of KdV are difficult to understand this way (current problem).
11 Hirota’s Method

It’s remarkable how many methods for solving PDEs the KdV equation has stimulated. In this section we look at another one due to Hirota in 1971. We have already seen that a linear equation of the form

$$\partial_x^n u - a \partial_t^m u = 0 \quad (210)$$

where \(a\) is a constant, can easily be solved. E.g. we can use the linear superposition principle and write

$$u(x, t) = \int_{-\infty}^{\infty} f(k)e^{i(kx-\omega t)} dk. \quad (211)$$

Applying the differential equation we get

$$(ik)^n - a(i\omega)^m = 0. \quad (212)$$

A simple single wave solution is

$$u = e^{i(kx-\omega t)} \quad (213)$$

where

$$\omega = -i ((ik)^n / a)^{\frac{1}{m}}. \quad (214)$$

Hirota noticed that there is a moderately simple way to get solutions for bilinear PDEs; that is equations of the form

$$ff_{x..x} + ft_{t..t} f_{x..x} + f_{xt} f + \ldots \quad (215)$$

i.e. all terms can have any number of \(x, t\) differentiations acting on always a product two \(f\) ’s. It involves introducing a new operator (Hirota’s operators) with their own algebra. The technique is very eneral and often gives solutions where previously none where available (e.g. recently here in Durham). Before we do so lets first look at some hints at how to get simple KdV solutions.

11.1 Hint 1: Burger’s equation

Recall from question 6 on sheet 1, that equations of the form

$$u_t + uu_x + \lambda u_{xx} = 0$$

are converted into linear equations by the transformation

$$u = 2\lambda (\log f)_x. \quad (216)$$

Comparison with the linear part of the KdV equation (that is \(u_t + u_{xxx}\)) suggests that it might be worth trying a substitution with \(\lambda = 1\);

$$u = 2(\log f)_{xx}$$
11.2 Hint 2: The one soliton solution

Now recall from last term that via the BT we found a solution \( u = w_x \) where

\[
  w = 2\sqrt{\lambda} \tanh(\sqrt{\lambda}(x - x_0 - 4\lambda t)) \tag{216}
\]

Call \( X = x - x_0 - 4\lambda t \). Then

\[
  w = 2\sqrt{\lambda} \frac{e^{2\sqrt{\lambda}X} - 1}{e^{2\sqrt{\lambda}X} + 1}
  = 2\sqrt{\lambda} \frac{2e^{2\sqrt{\lambda}X} - (1 + e^{2\sqrt{\lambda}X})}{e^{2\sqrt{\lambda}X} + 1}
  = 4\sqrt{\lambda} \frac{e^{2\sqrt{\lambda}X}}{e^{2\sqrt{\lambda}X} + 1} - 2\sqrt{\lambda}
  = 2\partial_x \log(e^{2\sqrt{\lambda}X} + 1) - 2\sqrt{\lambda} \tag{217}
\]

Since we also know \( w \to w + \text{const} \) gives the same solution we could have used

\[
  w = 2\partial_x \log(e^{2\sqrt{\lambda}X} + 1) \tag{218}
\]

This is of precisely the form guessed in hint 1, with \( f \equiv e^{2\sqrt{\lambda}X} + 1 \).

11.3 The KdV equation in Bilinear form

Motivated by this let’s try substituting \( w = 2(\log f)_x = 2f_x/f \) into the KdV equation. First write \( u = w_x \) and integrate once to get a slightly different p.d.e. for \( w \);

\[
  A(t) = \int (w_{xt} + 6w_xw_{xx} + w_{xxxx})dx
  = w_t + 3w_x^2 + w_{xxx} \tag{219}
\]

where \( A(t) \) is a \( t \)-dependent constant of integration. Now consider the boundary conditions to determine \( A(t) \). At \( x \to \pm \infty \) we look for solutions that have \( w_t = w_x = w_{xxx} = 0 \forall t \) so we can take \( A(t) = 0 \) and consider

\[
  w_t + 3w_x^2 + w_{xxx} = 0 \tag{220}
\]

Now get some identities first by differentiating:

\[
  \begin{align*}
  \frac{w}{2} &= \frac{f_x}{f} \\
  \frac{w_t}{2} &= \frac{f_{xt}}{f} - \frac{f_xf_t}{f^2} \\
  \frac{w_x}{2} &= \frac{f_{xx}}{f} - \frac{f_x^2}{f^2} \\
  \frac{w_{xx}}{2} &= \frac{f_{xxx}}{f} - \frac{3f_xf_{xx}}{f^2} + \frac{2f_x^3}{f^3} \\
  \frac{w_{xxx}}{2} &= \frac{f_{xxxx}}{f} - \frac{4f_xf_{xxx}}{f^2} - \frac{3f_x^2f_{xx}}{f^3} + \frac{12f_x^5f_{xx}}{f^4} - \frac{6f_x^4f_{xxx}}{f^4} \tag{221}
  \end{align*}
\]
(Notice that you always have the same numbers of \( f \) s on the top as bottom when you differentiate logs.) So now

\[
0 = w_t + 3w_x^2 + w_{xxx} = \frac{2fx_t}{f} - \frac{2fx_ft}{f^2} + 12 \left( \frac{fx_x}{f} - \frac{f_x^2}{f^2} \right)^2
+ \frac{2f_{xxx}}{f} - \frac{8fx_xfx_{xxx}}{f^2} - \frac{6f_{xx}^2}{f^2} + \frac{24f_x^2fx_x}{f^3} - \frac{12f_x^4}{f^4}
= \frac{2fx_t}{f} - \frac{2fx_ft}{f^2} + \frac{2f_{xxx}}{f} - \frac{8fx_xfx_{xxx}}{f^2} + \frac{6f_{xx}^2}{f^2}
\]

or

\[
ff_{xt} - fx_ft + ff_{xxx} - 4fx_xfx_{xxx} + 3f_{xx}^2 = 0 \quad (223)
\]

This is the bilinear form of the KdV equation. Despite the fact that some nontrivial cancellations took place it looks more complicated than the original problem, however its special form makes it possible to solve using bilinear operators.

### 11.4 Hirota’s Bilinear operators \( D(f.g) \).

First note that the first two terms of eq.223 are similar to

\[
\frac{1}{2} \partial_x \partial_t f^2 = ff_{xt} + fx_ft.
\]

The only difference is the sign. Hirota’s operators make it possible to write the opposite sign term in a similarly simple way. They map a pair of functions to one function as follows; For \( m, n \geq 0 \)

\[
D_t^mD_x^n(f.g) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x,t)g(x',t') \bigg|_{x'=x, \ t'=t} \quad (224)
\]

The usefulness of this operator becomes obvious when you do some examples;

**Example 1:**

\[
D_t(f.g) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right) f(x,t)g(x',t') \bigg|_{x'=x, \ t'=t} = f_t(x,t)g(x',t') - f(x,t)g_t(x',t') \bigg|_{x'=x, \ t'=t}
= ftg - fg_t \quad (225)
\]

all functions of \( x, t \). In particular this means that

\[
D_{anything}(f.f) = 0 \quad (226)
\]
Example 2:

\[ D_tD_x(fg) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) f(x,t)g(x',t') \bigg|_{x'=x, \ t'=t} \]

\[ = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right) (f_x(x,t)g(x',t') - f(x,t)g_{x'}(x',t')) \bigg|_{x'=x, \ t'=t} \]

\[ = f_{xt}g - f_{xt}g - f_{tg}g_{x} + f_{gt}g_{x} \]

(227)

all functions of \( x, t \). In particular this means that

\[ \frac{1}{2} D_tD_x(f.f) = f_{xt} - f_{tx} \]

(228)

i.e. the opposite sign combination appearing in eq.223.

Example 3:

\[ D_x^2(fg) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^2 f(x,t)g(x',t') \bigg|_{x'=x, \ t'=t} \]

\[ = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) (f_x(x,t)g(x',t') - f(x,t)g_{x'}(x',t')) \bigg|_{x'=x, \ t'=t} \]

\[ = f_{xx}g - 2f_{x}g_{x} + f_{gxx} \]

(229)

all functions of \( x, t \). Again this looks a bit like the differentiation of a product. But *be careful because*

\[ D_x^2(f.f) = 2f_{xx} - 2f_{x}^2 \]

(230)

even though \( D_x(f.f) = 0 \), so importantly the operators are not associative since \( D_x^2(f.f) \neq D_x(D_x(f.f)) \). For starters the RHS is meaningless because the two functions are already mapped by the first \( D_x \) to one function, so the second \( D_x \) doesn’t have two functions to “eat”.

Example 4:

\[ D_x^4(fg) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^4 f(x,t)g(x',t') \bigg|_{x'=x, \ t'=t} \]

\[ = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^3 (f_x(x,t)g(x',t') - f(x,t)g_{x'}(x',t')) \bigg|_{x'=x, \ t'=t} \]

\[ = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^2 (f_{xx}g - 2f_{x}g_{x} + f_{gxx'}) \bigg|_{x'=x, \ t'=t} \]

\[ = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) (f_{xxx}g - 3f_{xx}g_{x} + 3f_{x}g_{xx'} - f_{gxxx'}) \bigg|_{x'=x, \ t'=t} \]

\[ = (f_{xxxx}g - 4f_{xxx}g_{x} + 6f_{xx}g_{xx'} - 4f_{x}g_{xxx'} + f_{gxxxx'}) |_{x'=x, \ t'=t} \]

\[ = f_{xxxx}g - 4f_{xxx}g_{x} + 6f_{xx}g_{xx} - 4f_{x}g_{xxx} + f_{gxxxx} \]

(231)
all functions of $x, t$. Again it looks just like a differentiated product of $f$ and $g$ but with sign changes on the odd terms. In particular

$$D_4^4(f.f) = 2f_{xxxx}f - 8f_{xxx}fx + 6f_{xx}^2$$  \hspace{1cm} (232)$$

Comparing examples 2 and 4 we see that eq.223 can be written in a very simple way:

$$(D_tD_x + D_4^4)(f.f) = 0$$  \hspace{1cm} (233)$$

### 11.5 The 1-soliton solution

In Hirota form the multisoliton solutions are sums of exponential of linear expressions in $x$ and $t$. For the one soliton solution try

$$f = e^0 + e^\theta$$  \hspace{1cm} (234)$$

where $\theta = ax + bt + c$.

**Lemma 1:** A useful result is

$$D_t^m D_x^n(e^{\theta_1}.e^{\theta_2}) = (b_1 - b_2)^m(a_1 - a_2)^n e^{\theta_1 + \theta_2}$$  \hspace{1cm} (235)$$

Again this looks like the differentiation of products with the opposite signs.

**Proof:** by induction. Let

$$D_t^m D_x^n(e^{\theta_1}.e^{\theta_2}) = (b_1 - b_2)^m(a_1 - a_2)^n e^{\theta_1 + \theta_2}$$  \hspace{1cm} (236)$$

and similarly for $D_t$ operator. Finally the equation is obviously true for $n = 1, m = 0$ and $n = 0, m = 1$ so by induction must be true for all $n, m$.

In particular we find

$$D_t^m D_x^n(e^\theta.e^\theta) = 0$$  \hspace{1cm} (237)$$

$$D_t^m D_x^n(e^\theta.1) = b^m a^n e^\theta$$  \hspace{1cm} (238)$$

$$D_t^m D_x^n(1.e^\theta) = (-1)^{n+m}b^m a^n e^\theta$$  \hspace{1cm} (239)$$
Then KdV on \( f = 1 + e^\theta \) gives
\[
0 = (D_t D_x + D^4_x)(1 + e^{\theta}1 + e^\theta) = (D_t D_x + D^4_x)[(1.1) + (1.e^\theta) + (e^\theta.1) + (e^\theta.e^\theta)]
\]
\[
= (D_t D_x + D^4_x)[(1.1) + (1.e^\theta) + (e^\theta.1)]
\]
\[
= 2ba e^\theta + 2a^4 e^\theta
\]
(240)
so that a solution requires \( a(b + a^3) = 0 \). \( a = 0 \) would give no \( x \)-dependence and hence (by the boundary conditions) \( u = 0 \). So take \( b = -a^3 \) and then
\[
\theta = ax - a^3t + c
\]
and hence
\[
u = 2\frac{\partial^2}{\partial x^2}(\log(1 + e^{ax-a^3t+c}))
\]
\[
= \frac{a^2}{2}\text{sech}^2[\frac{1}{2}(ax - a^3t + c)]
\]
(241)
This is the one-soliton solution with \( v = a^2 \! \)!

## 11.6 Multi-soliton solutions

**General idea:** organize the calculation perturbatively in some parameter \( \varepsilon \)
(secret hope is that the series will terminate at some power of \( \varepsilon \) to give exact results rather than an infinite series in \( \varepsilon \)).

To begin with we put
\[
f(x, t) = \sum_{n=1}^{\infty} \varepsilon^n f_n(x, t)
\]
(242)
and collect powers of \( \varepsilon \). We start with 1 as the \( \varepsilon^0 \) term as it appeared in the one-soliton solution. Defining
\[
B = (D_t D_x + D^4_x)
\]
(243)
we want to find solutions such that
\[
B(f.f) = 0.
\]
(244)
Expanding as for the one soliton case we have
\[
0 = B(f.f) = B(1.1) + \varepsilon(B(1.f_1) + B(f_1.1)) + \varepsilon^2(B(1.f_2) + B(f_1.f_1) + B(f_2.1))
\]
\[
= \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=0}^{n} B(f_{n-m}.f_m)
\]
so actually have to solve an infinite set of equations
\[
\sum_{m=0}^{n} B(f_{n-m}.f_m) = 0
\]
(245)
We can think of this as a recursive set of equations to get \( f_n \) from \( f_1 \ldots f_{n-1} \). That is we begin with \( f_1 \) and solve iteratively to get \( f_2, f_3 \ldots \) hoping that it terminates at some point. To do this we need
Lemma 2: Another useful result is
\[ D_t^m D_x^n (f.1) = (-1)^{n+m} D_t^m D_x^n (1.f) \] (246)

Proof: for this we need simply note that we can swop the primed and unprimed variables since “prime” or “unprime” are simply labels; i.e.
\[ D_t^m D_x^n (f.g) = \left. \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x',t') \right|_{x'=x, t'=t} \]
\[ = (-1)^{n+m} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n g(x',t') f(x,t) \right|_{x'=x, t'=t} \]
\[ = (-1)^{n+m} D_t^m D_x^n (g.f) \] (247)

where in the 3rd line I swopped \( x \leftrightarrow x' \) and \( t \leftrightarrow t' \).

With this result eq.245 becomes
\[ \partial_x (\partial_t + \partial_x^3) f_n = -\frac{1}{2} \sum_{m=1}^{n-1} B(f_{n-m}, f_m) \] (248)
with
\[ \partial_x (\partial_t + \partial_x^3) f_1 = 0 \] (249)

Beginning with \( f_1 \) we may iterate to find all the \( f_n \) and note that at any order we are now solving linear partial differential equations. A simple particular solution to the last equation would be

\[ f_1 = e^{ax-a^3t+c} \] (250)

But now notice what happens for \( f_2 \). By eq.248 we have
\[ \partial_x (\partial_t + \partial_x^3) f_2 = -\frac{1}{2} B(f_1, f_1) \]
\[ = 0 \] (251)

where the last line follows by eq.237. Now the \( f_3...\infty \) all vanish trivially. The series has terminated with

\[ f = 1 + \varepsilon f_1 \] (252)

We can always absorb \( \varepsilon \) into the constant \( c \) and we recognize the single soliton solution.

The clever bit: Our hope is to find other solutions where the series terminates in this way, so that the solution is exact. Consider beginning with
\[ f_1 = \sum_{i=1}^{N} e^{a_i x - a_i^3 t + c_i} \] (253)
which clearly satisfies eq.250, since it is linear and we can simply add solutions. The \( f_2 \) equation now no longer vanishes by virtue of the cross terms in \( B(f_1\cdot f_1) \). However the operation of \( B \) retains the form of the exponentials which are simply multiplied together by virtue of eq.235. The \( N \) exponentials can be thought of as \( N \) separate variables in a set of simultaneous equations since they persist at every iteration. Since there are only \( N \) of them we can have at most have \( N \) equations before the system is overdetermined, and so the series must terminate after \( N \) terms. This is the \( N \) soliton solution!

**11.6.1 Two soliton example**

We have already seen the single soliton solution. Let's verify that we recover the two soliton solution. Take

\[
f_1 = e^{\theta_1} + e^{\theta_2}
\]

where \( \theta_{1,2} \) are as above chosen to satisfy the \( f_1 \) equation

\[
\theta_i = a_i x - a_i^3 t + c_i
\]

The \( f_2 \) equation becomes

\[
\partial_x(\partial_t + \partial_x^3) f_2 = -\frac{1}{2} B(f_1\cdot f_1)
\]

\[
= -\frac{1}{2} B(e^{\theta_1}\cdot e^{\theta_2}) - \frac{1}{2} B(e^{\theta_2}\cdot e^{\theta_1})
\]

\[
= -(a_1 - a_2)((b_1 - b_2) + (a_1 - a_2)^3)e^{\theta_1 + \theta_2}
\]

\[
= -(a_1 - a_2)(-(a_1^3 - a_2^3) + (a_1 - a_2)^3)e^{\theta_1 + \theta_2}
\]

\[
= (a_1 - a_2)^2((a_1^2 + a_1 a_2 + a_2^2 - a_1^2 + 2a_1 a_2 - a_2^2)e^{\theta_1 + \theta_2}
\]

\[
= 3a_1 a_2 (a_1 - a_2)^2 e^{\theta_1 + \theta_2}
\]

Since the equation is linear in \( f_2 \) it has an obvious solution of the form

\[
f_2 = Ae^{\theta_1 + \theta_2}.
\]

so we just have to determine \( A \). Substituting into the \( f_2 \) equation we get

\[
A(a_1 + a_2)(-(a_1^3 + a_2^3) + (a_1 + a_2)^3) = 3a_1 a_2 (a_1 - a_2)^2
\]

\[
A 3a_1 a_2 (a_1 + a_2)^2 = 3a_1 a_2 (a_1 - a_2)^2
\]

or

\[
A = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2}.
\]

You can now check (exercise) that the RHS of the \( f_3 \) equation vanishes and the series terminates. The solution is then

\[
f = 1 + \varepsilon e^{\theta_1} + \varepsilon e^{\theta_2} + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} \varepsilon^2 e^{\theta_1 + \theta_2}
\]

(258)
Again we can absorb the \( \varepsilon \) into the constants \( c_{1,2} \) (or equivalently set \( \varepsilon = 1 \)). Finally we just need to differentiate to get the original \( w \) (or \( u \)) using

\[
\begin{align*}
w &= 2 \frac{f_x}{f} \\
&= 2 \frac{a_1 e^{\theta_1} + a_2 e^{\theta_2} + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_1 + \theta_2}}{1 + e^{\theta_1} + e^{\theta_2} + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_1 + \theta_2}}
\end{align*}
\]

It is easier in this format to study the asymptotics of the two soliton solution. As in the BT discussion, first choose a coordinate system with origin at the centre of the around \( i = 1 \) or \( 2 \) soliton; take \( t \to t_i + \delta t \) with \( \delta x = x - x_0 - v_i t_i \) where \( v_i = a_i^2 \) and let \( t_i \to \pm \infty \). We have

\[
\begin{align*}
\theta_i &= \frac{\sqrt{v_i}}{2}(\delta x - v_i \delta t) \\
\theta_{j \neq i} &= \frac{\sqrt{v_j}}{2}(\delta x - v_j \delta t) + \frac{\sqrt{v_j}}{2}(v_i - v_j) t_i
\end{align*}
\]

Near the \( i \) soliton we have (by definition) \( \delta x, \delta t \approx 0 \), so that \( \theta_i \) is small but \( \theta_{j \neq i} \to \pm \infty \) as follows;

<table>
<thead>
<tr>
<th></th>
<th>( t \to -\infty )</th>
<th>( t \to +\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>( \theta_{j=2} \to -\infty )</td>
<td>( \theta_{j=2} \to +\infty )</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>( \theta_{j=1} \to +\infty )</td>
<td>( \theta_{j=1} \to -\infty )</td>
</tr>
</tbody>
</table>

When \( \theta_{j \neq i} \to -\infty \) the \( f \) clearly reduces to the single soliton case, \( f \to 1 + e^{\theta_i} \). When \( \theta_{j \neq i} \to +\infty \) we have to do a little more work. To leading order we have

\[
f = e^{\theta_j} \left( 1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_i} + \ldots \right) \quad (260)
\]

where the ellipsis indicates terms of order \( e^{-\theta_j} \). But then

\[
w = 2 \partial_x \log f
\]

\[
= 2 a_j + 2 \partial_x \log \left( 1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_i} \right). \quad (261)
\]

Since \( w \to w + \text{const} \) gives equivalent solutions we may ignore the constant piece. The remaining solution is equivalent to an \( f \) of the form

\[
f \equiv \left( 1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_i} \right). \quad (262)
\]

This is the “original” single \( i \)-soliton solution but with a phase shift of

\[
\delta x_0 = -\frac{2}{\sqrt{v_i}} \log \left( \frac{|\sqrt{v_1} - \sqrt{v_2}|}{\sqrt{v_1} + \sqrt{v_2}} \right), \quad (263)
\]

precisely what we found for the 2 KdV soliton solution using the much more complicated BT method.
11.6.2 N-soliton solution

Finally I’ll just present the general solution which may be proved by induction. (This is a very difficult exercise.) Define an $N \times N$ matrix $S_N$ with elements

$$(S_N)_{ij} = \delta_{ij} + \frac{2a_i e^{\theta_i}}{a_i + a_j}$$

(264)

Then the $N$-soliton solution is

$$u = 2\partial_x^2 \log f$$

(265)

where

$$f = \det(S_N).$$

You can easily check the single and two-soliton solutions.
12 Overview for the Inverse Scattering Method

12.1 Initial value problems

So far we have asked the question whether for a particular p.d.e. it is possible to construct particular solutions. We have mainly been interested in single or multi-soliton solutions. However you can also ask whether you can find a general solution, as we were able to for the Liouville equation. The question is

If we are given a wave equation and sufficient initial data at \( t = 0 \) (i.e. the functions \( u(x,0), u_t(x,0), \ldots \)), can we find \( u(x,t) \) for all \( t \)?

This is known as an initial value problem. First for there to be a unique solution what initial data do we need?

- If eqn. is 1st order in \( t \) (e.g. KdV equation) then we need \( u(x,0) \)
- If eqn. is 2nd order in \( t \) (e.g. SG equation) then we need \( u(x,0), u_t(x,0) \)

Why? Because we can use the equation itself to solve for higher \( t \) derivatives. e.g. in the KdV case, if you know \( u(x,0) \) you can solve the KdV equation to get \( u_t(x,0) \).

So far, we might be able to spot a particular solution and (assuming uniqueness) from the initial form, but in general this is not possible. e.g. consider the three initial functions

1. \( u(x,0) = 2\text{sech}^2 x \)
2. \( u(x,0) = 2.01\text{sech}^2 x \)
3. \( u(x,0) = 6\text{sech}^2 x \)

The first case can be spotted as a \( t = 0 \) snapshot of a one-soliton solution with \( v = 4; u(x,t) = 2\text{sech}^2(x-4t) \) but the other two are not so simple. Case 2 turns out to be a soliton plus oscillatory “junk” whereas case 3 is actually a snapshot of a two soliton solution. The evolution can of course be done numerically, as below for these three cases, but we want to understand it analytically.
12.2 The linear initial value problem

It helps to have an analogous situation to guide us, and the obvious one is
the use of Fourier Transforms for solving linear problems such as the Heat
(diffusion) equation and the Klein-Gordon equation. We have already seen
the former in lectures so let’s look at the latter. This is exactly the same
procedure as for the diffusion equation, but the point of going through this
again is to highlight important features

\[ u_{tt} - u_{xx} + m^2 u = 0 \]

where I have set \( v = 1 \). As it’s second order, the initial conditions we need are
\( u \) and \( u_t \). Let’s define

\[
\begin{align*}
A(x) &= u(x, 0) \\
B(x) &= u_t(x, 0)
\end{align*}
\]
As discussed previously we can solve this by using a Fourier transform in the $x-$coordinate. Let

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(k,t)e^{ikx}dk$$

$$\tilde{u}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$$

Note that $\tilde{u}(k,t)$ is a function of $t$, and (to make connection with the analysis for the KdV equation later) I’ll call this the scattering data; it tells us at any time what sinusoidal waves $u$ is composed of. Now applying the KG equation to $u$ as written above gives

$$\tilde{u}_{tt} + (k^2 + m^2)\tilde{u} = 0$$  \hfill (268)

This is an equation that can be simply solved to give the time-dependence of the scattering data;

$$\tilde{u} = \tilde{f}(k)e^{i\omega t} + \tilde{g}(k)e^{-i\omega t},$$  \hfill (269)

where $\tilde{f}$ and $\tilde{g}$ are to be determined by the initial conditions, and

$$\omega = \sqrt{k^2 + m^2}$$  \hfill (270)

is the dispersion relation.

We can now determine the “spectrum” of positive and negative modes $\tilde{f}$ and $\tilde{g}$ from the FT of the initial conditions. When $t = 0$ we have

$$\tilde{A} = \tilde{f} + \tilde{g}$$ \hfill (271)

$$\tilde{B} = i\omega(\tilde{f} - \tilde{g})$$

so that

$$\tilde{f} = \frac{i\omega \tilde{A} + \tilde{B}}{2i\omega}$$ \hfill (272)

$$\tilde{g} = \frac{i\omega \tilde{A} - \tilde{B}}{2i\omega}$$

where

$$\tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx$$

$$\tilde{B}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_t(x,0)e^{-ikx}dx.$$  \hfill (273)

Finally our solution can be written as

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(k,t)e^{ikx}dk$$

$$= \int_{-\infty}^{\infty} \left(\frac{i\omega \tilde{A} + \tilde{B}}{2i\omega}\right)e^{i(kx+\omega t)} + \left(\frac{i\omega \tilde{A} - \tilde{B}}{2i\omega}\right)e^{i(kx-\omega t)}dk$$

$$= \int_{-\infty}^{\infty} \left(\tilde{A}\cos \omega t + \frac{\tilde{B}}{\omega} \sin \omega t\right)e^{ikx}dk$$
An interesting thing just happened which has a direct analogy in the KdV equation. The reason we were able to find the solution is that the energy spectrum of positive and negative modes was time-independent (i.e. $\tilde{u}_+ = |f|^2$ and $\tilde{u}_- = |\tilde{g}|^2$ are independent of $t$). If we break the solution down into three steps,

1. Perform the FT in the $x$ coordinate and determine the initial “scattering data” $\tilde{u}(k,0)$ from the initial conditions $u(x,0), u_t(x,0)$.

2. Evolve $\tilde{u}(k,t)$ forwards in time using the evolution (determined by the KG equation) in eq.269

3. Perform the inverse FT back to get $u(x,t)$,

it is the second of these steps that was allowed by the time-independent spectrum property. Otherwise the time evolution would have mixed everything up and it would have been very difficult to get the final spectrum from the initial one (i.e. the FT would have been of no use!). It can be represented as ...

12.3 Outline of the method, and the KdV-Schroedinger connection

The method for the KdV equation was discovered by Gardner, Greene, Kruskal and Miura in the late 60’s and is analogous. At first sight the constancy-of-the-spectrum property looks like a unique property of linear equations, but linear equations are just the simplest of a much wider class of equations called the Lax KdV hierarchy (that we’ll get to much later) where this type of behaviour occurs. In particular something very similar (although much more complicated) can be done for the KdV equation. Let’s generalize the procedure above to the KdV equation;
The method now relies on a connection between KdV and the Schroedinger equation. The equation we ultimately wish to solve is the KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0 \]  

(274)

Recall the generalized Miura transformation

\[ u = \lambda - v^2 - v_x \]  

(275)

where \( \lambda \) is constant; if \( v \) satisfies

\[ v_t + 6(\lambda - v^2)v_x + v_{xxx} = 0 \]

it implies that \( u \) given by the above is a solution of the KdV equation. Now think about it the other way round; if \( u \) is a known solution of the KdV equation we can solve eq.275 to find \( v \); writing

\[ v = \frac{\psi_x}{\psi} \]  

(276)

equation 275 becomes

\[ u = \lambda - \frac{\psi_{xx}}{\psi} \]

i.e.

\[ \psi_{xx} + u\psi = \lambda\psi \]  

(277)

This is remarkable because it is the time-independent Schrödinger equation for the Sturm-Liouville problem (time appears only as a parameter in \( u(x, t) \)) with eigenvalue \( \lambda(t) \) (since \( u \) is a time dependent potential we generally expect \( \lambda \) to be time dependent as well). The QM equation for a particle in a potential \(-u\). A great deal was known at the time about such equations (otherwise the connection with QM is unimportant).
12.4 The recipe;

The general instructions for constructing the time dependent solutions go as follows

Disassembly (Scattering): Note that \( t \) is only a parameter in eq.277. Begin with initial data \( u(x,0) \) which plays the role of the potential. Now for each \( \lambda \), the \( \psi \) is a different eigenfunction. Each eigenfunction describes the scattering of a particle off the potential with certain reflection and transmission coefficients (i.e. the asymptotic values of \( \psi \) at \( x \to \pm\infty \)) - we collect these coefficients which are our initial scattering data. Here \( \lambda \) is playing the role of \( k \) in the FT, and effectively the FT has been replaced with determining the components of \( v = \psi_x/\psi \) from \( u \)

Time evolution: Now we have to evolve the eigenvalues forward in time. There is an amazing fact that helps us now - for potentials \( u \) that obey the KdV equation we will be able to show that actually any spectrum of \( \lambda \) is time independent! The spectrum is a constant and all we need to do is evolve the scattering data in a very simple manner.

Reassembly (Inverse scattering): The final step is to reconstruct the potential \( u(x,t) \) at time \( t \) by the inverse scattering. That is, given the set of scattering data can have to reconstruct the potential that the particles have scattered off. (c.f. “can you hear the shape of a drum”). That this is possible for the Sturm Liouville equation was already known.

Each of these steps is rather lengthy so I advise you to keep referring back to this overview as we go along. It will take a little time to get straight what is happening!
13 Disassembly: Scattering theory

We now begin some lengthy detours to treat each of the necessary ingredients, beginning with scattering theory. We want to analyse the possible solutions to

$$(\partial_x^2 + u)\psi = \lambda \psi$$

with $\psi$ remaining bounded $\forall x$. For shorthand I will call the operator on the LHS $L$ so that

$$L\psi = \lambda \psi.$$  

(278, 279)

Here $u$ is considered to be at some fixed time $t$. This eqn is the time independent Schroedinger eqn.

13.1 Physical interpretation

We begin with the time dependent Schroedinger equation from QM

$$(-\partial_x^2 + V(x))\Psi(x, \tau) = i\partial_\tau \Psi(x, \tau)$$

(280)

(Here I use $\tau$ for time because it has nothing to do with the time in the KdV equation or in $u(x, t)$. That time we are considering as fixed.) This describes a particle moving in a potential $V$ (this is simply a fact - but to motivate it we replace the operators $i\partial_x \rightarrow p/\sqrt{2m}$ and $i\partial_\tau \rightarrow E$ and find an equation for the non-relativistic energy, $E \equiv p^2/2m + V$). $|\Psi(x)|^2$ becomes the probability density for finding the particle at $x$.

To solve this eqn we can separate variables

$$\Psi(x, t) = \psi(x)T(\tau)$$

$$i\dot{T}/T = (V\psi - \psi'')/\psi = \text{const} = k^2 \equiv -\lambda$$

where $k^2$ is a constant. Then $T(\tau) = e^{-ik^2\tau}$ and

$$\psi'' - V\psi = k^2\psi.$$  

Hence we see the interpretation of $-\lambda$ as the energy eigenvalue and $u$ as the negative potential. If $\lim_{x \rightarrow \pm \infty}(V(x)) = 0$ and $k^2 > 0$ then the $x$ dependent factor asymptotically goes like

$$\psi(x) \rightarrow e^{\pm ikx}$$

and the full solution looks like

$$\Psi(x, t) = e^{\pm ikx - ik^2\tau}$$

which is a plane wave either going left ($-ikx$) or right ($+ikx$). Inbetween the solution $\psi(x)$ will be doing something horrible depending on $V$, but the coefficients of asymptotic left and right moving waves can be relatively easily determined and these will be the all important scattering data.
Example 1: $V(x) = 0$

This case includes the asymptotic solution above to the equation

$$\psi'' = k^2 \psi$$  \hspace{1cm} (281)

which has 2 cases

a) $k^2 > 0$; we have already seen that $\psi = Ae^{ikx} + Be^{-ikx}$ is a right or left-moving wave which is always bounded.

b) $k^2 =< 0$; This would be negative energy. Call $k^2 = -\mu^2$ and then $\psi = Ae^{i\mu x} + Be^{-i\mu x}$. The only way to keep the solution bounded at $x \to \infty$ is to have $A = B = 0$ so that there are no appropriate solutions of this kind.

Conclusion: if $u = 0$ then $L \psi = \lambda \psi$ has a solution for all $\lambda \leq 0$ (positive energy) and no bounded solutions for $\lambda > 0$ (negative energy). The energy spectrum of solutions looks like

Example 2: $V(x) = a\delta(x)$ where $a = \text{const}$ (the Dirac delta function)

This situation is particles penetrating an infinitely thin and infinitely high barrier of area under the curve =1. See the Fourier sheet for a definition of $\delta$, but for this discussion we can use the formal definition of $\delta(x)$ as a function that obeys

$$f(x) = \int f(x')\delta(x - x')dx'$$  \hspace{1cm} (282)

and is exactly zero for $x \neq 0$.

a) $k^2 > 0$; we have already seen that $\psi(x) = Ae^{ikx} + Be^{-ikx}$ is a right or left-moving wave. We can split it into two regions on either side of the spike with different constants in each region

$$\psi(x < 0) = A_-e^{ikx} + B_-e^{-ikx}$$
$$\psi(x > 0) = A_+e^{ikx} + B_+e^{-ikx}$$  \hspace{1cm} (283)
We find the constants by matching at $x = 0$. It can be shown that at $x = 0$ the function must be discontinuous (for starters a discontinuity in $\psi$ would indicate an unphysical sudden change in the probability density). So we can set

$$A_- + B_- = A_+ + B_+.$$  \hfill (284)

The equation now becomes

$$-\psi'' + a\delta(x)\psi = k^2\psi$$  \hfill (285)

Integrate over $x$ gives

$$[-\psi']_{-\epsilon}^{\epsilon} + a\psi(0) = k^2\int_{-\epsilon}^{\epsilon} \psi dx = k^2\int_{-\epsilon}^{\epsilon} \psi dx = 0$$  \hfill (286)

where $\epsilon$ is an infinitesimally small region near $x = 0$, and I used the fact that $\psi$ is bounded in the region near $x = 0$ and the integrals of sinusoidal functions over $x$ vanishes. This equation gives

$$ik(A_+ - B_+ - A_- + B_-) = a(A_- + B_-) = a(A_+ + B_+)$$  \hfill (287)

This is two equations in 4 unknowns. Solving for $A_-$ and $B_-$ we find

$$A_- = A_+(1 - \frac{a}{2ik}) + B_+\frac{a}{2ik}$$
$$B_- = B_+(1 + \frac{a}{2ik}) + A_+\frac{a}{2ik}$$  \hfill (288)

This gives the general solution. There are 2 unknowns since it is a 2nd order o.d.e. for $\psi(x)$.

**Important special case:** Consider a flow of particles incident from the left. The incoming probability density is just normalized by looking at the flux of particles so it doesn’t carry any interesting information. We can normalize it to 1 by convention, so that there is a unit flux of particles coming in from $x = -\infty$ and a zero flux from $x = +\infty$. This plus the equation above gives

$$A_- = 1$$
$$B_+ = 0$$
$$A_+ = \frac{2ik}{2ik - a}$$
$$B_- = \frac{a}{2ik - a}$$  \hfill (289)

The total solution can be written

$$\psi = \begin{cases}  
e^{ikx} + \frac{a}{2ik-a}\ne^{-ikx} & x < 0 \\ \frac{2ik}{2ik-a}\ne^{ikx} & x > 0. \end{cases}$$  \hfill (290)
The coefficient of $e^{-ikx}$ is called the **reflection coefficient**, sometimes written $R(k) = a/(2ik - a)$, and the coefficient of $e^{ikx}$ on the right of the barrier is called the **transmission coefficient**, sometimes written $T(k) = 2ik/(2ik - a)$. The physical interpretation is clear:

- $|R(k)|^2$ gives the probability that the particle is reflected by the barrier and $|T(k)|^2$ is the transmission probability
- $|R|^2 + |T|^2 = 1$ since the total probability must be one

(b) $k^2 = -\mu^2 < 0$; for the $V = 0$ case we got no solutions but here we get a surprise - an acceptable (i.e. bounded) solution exists. Begin by replacing $k = i\mu$ everywhere so that the solution is

\[
\psi = \begin{cases} 
\frac{e^{-\mu x} - \frac{a}{2\mu + a} e^{\mu x}}{2\mu} & x < 0 \\
\frac{2\mu + a}{2\mu} e^{-\mu x} - \frac{a}{2\mu} e^{\mu x} & x > 0.
\end{cases}
\] (291)

This looks unbounded, since $e^{\mu x}$ diverges for $x \to \infty$ and $e^{-\mu x}$ diverges as $x \to -\infty$. However the equation is linear so we can certainly get a new solution if we divide by the transmission coefficient $T = 2\mu/(2\mu + a)$ which is just a constant - all that changes is the normalization of $\psi$. This gives

\[
\psi = \begin{cases} 
\frac{2\mu + a}{2\mu} e^{-\mu x} - \frac{a}{2\mu} e^{\mu x} & x < 0 \\
e^{-\mu x} & x > 0.
\end{cases}
\] (292)

Generally this still diverges, but if it happens that

\[
\mu = -\frac{a}{2}
\]

and $\mu > 0$ then the diverging pieces disappear. The eigenfunction is symmetric about the origin, bounded and can be written as

\[
\psi = e^{-\frac{a}{2}|x|}.
\] (293)
Note that for $\psi(x \to \pm \infty) = 0$ so that the particle is stuck around $x = 0$. It is a QM bound state. Classically you would say that the particle is trapped because the energy ($-\lambda = k^2 = -a^2/4$) is negative. The energy spectrum looks like the figure on the left below. Generally we find a spectrum that looks like the right figure, with a discrete (possibly infinite) set of negative energy bound states, with a continuous spectrum of positive energy states.

**Example 3:** $V(x) = a \text{sech}^2(x)$

Return now to the examples with which we began the whole discussion of inverse scattering. The physics is the same but involves slightly more technical (but standard) functions. The equation we wish to solve is

$$-\psi'' + a \text{sech}^2(x)\psi = k^2\psi \quad (294)$$

First we make the substitution $\eta = \tanh x$, so that

$$\frac{d}{dx} = \text{sech}^2(x) \frac{d}{d\eta}$$

$$= (1 - \eta^2) \frac{d}{d\eta}$$

We now need to collect a bit of technology; the equation we have to solve is now

$$\frac{d}{d\eta} (1 - \eta^2) \frac{d\psi}{d\eta} + \left( \frac{k^2}{1 - \eta^2} - a \right)\psi = 0 \quad (295)$$

This is a standard equation that gives associated Legendre polynomials of the first kind (which I’ll define in a minute) commonly called $P_n$ and $Q_n$. $Q_n$ are unbounded so we neglect them and keep $P_n$. It is defined as

$$P_n(\eta) = \frac{1}{n!2^n} \frac{d^n}{d\eta^n}(\eta^2 - 1)^n. \quad (296)$$
To remove the mystery a bit the first few functions are
\[
\begin{align*}
P_1 &= \eta \\
P_2 &= -\frac{1}{2} + \frac{3\eta^2}{2} \\
P_3 &= -\frac{3\eta}{2} + \frac{5\eta^3}{2}.
\end{align*}
\]

The associated Legendre polynomials are defined as
\[
P^m_n = (-1)^m (1 - \eta^2)^{\frac{m}{2}} \frac{d^m P_n}{d\eta^m}
\]

When \( m \) is non-integer or even complex, the latter can be defined in terms of the hypergeometric function \( {}_2F_1 \) (you have to be a bit careful about regions of convergence here as there are various definitions);
\[
\Gamma(1 - m) P^m_n = \frac{(1 + \eta)^m}{(1 - \eta)^m} {}_2F_1(-n, n + 1, 1 - m; \frac{1 - \eta}{2})
\]

where \( {}_2F_1 \) has the series expansion
\[
{}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + a)\Gamma(k + b)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(k + c)} \frac{z^k}{k!}
\]
\[
\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy.
\]

That is all the technology we need apart from the fact that for integer \( N \), \( \Gamma(N) = (N - 1)! \). The solution is found to be (e.g. plug the differential equation into Maple)
\[
\psi = AP^{ik}_\nu
\]

where
\[
\nu = \sqrt{1 - 4a^2} - \frac{1}{2}
\]

a) \( k^2 > 0 \); In this case we expect a continuous spectrum of solutions. To get the reflection and transmission coefficients consider first the limit \( x \to +\infty \). Then \( \eta \to \pm 1 \) and the hypergeometric function gives 1. On the other hand we have to use \( 1 - \eta \sim 2e^{-2x} \) and then find just the transmitted wave
\[
\psi \sim \frac{A}{\Gamma(1 - ik)} e^{ikx}
\]

In the limit \( x \to -\infty \) we have \( \eta \to -1 \) and so \( 1 + \eta \sim 2e^{2x} \) and in this limit we find
\[
\psi \sim A \frac{(1 + \eta)^m}{(1 - \eta)^m} \left( \frac{2^m \Gamma(m)(1 + \eta)^{-m}}{\Gamma(-n)\Gamma(1 + n)} + \frac{\Gamma(-m)}{\Gamma(-m - n)\Gamma(1 - m + n)} \right)
\]
\[
\sim A \left( \frac{\Gamma(ik)}{\Gamma(-\nu)\Gamma(1 + \nu)} e^{-ikx} + \frac{\Gamma(-ik)}{\Gamma(-ik - \nu)\Gamma(1 - ik + \nu)} e^{ikx} \right)
\]
Again the solution represents a wave coming in from infinity, except it must be travelling to the right. Part of the wave is reflected and part the incident wave. We can choose $A$ so that instead the incoming wave has unit flux; then

\[
R(k) = \frac{\Gamma(-ik - \nu)\Gamma(1 - ik + \nu)\Gamma(ik)}{\Gamma(-\nu)\Gamma(1 + \nu)\Gamma(-ik)} \\
T(k) = \frac{(-ik - \nu)\Gamma(1 - ik + \nu)}{\Gamma(1 - ik)\Gamma(-ik)}.
\] (304)

In order to simplify this expression we can use the identity $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$. This gives

\[
R(k) = \frac{\sin(\pi \nu) \Gamma(-ik - \nu)\Gamma(1 - ik + \nu)\Gamma(ik)}{\pi \Gamma(-\nu)\Gamma(1 + \nu)}.
\] (305)

\[
T(k) = \frac{\sqrt{\sin(\pi(\nu + ik))\sin(\pi(\nu - ik))}}{\sinh(\pi k)}.
\]

After a bit of work you can verify that $|R|^2 + |T|^2 = 1$. Interestingly $R = 0$ when

\[
\nu = N
\] (306)

where $N$ is any integer, or

\[
a = -N(N + 1).
\] (307)

In this case the potential is called reflectionless.

b) $k^2 < 0$; Again we can put $\mu = -ik$ and we will assume that $\mu > 0$. In this case we have asymptotically unbounded solutions going like $e^{\pm \mu x}$ unless we can kill one of them as we did for the $\delta$ function bound state. Repeating that procedure, we multiply the solutions by $1/T(k)$ so that we have

\[
\psi = \\left\{ \begin{array}{ll}
\frac{\sin(\pi \nu)}{\sin(\pi \mu)} e^{\mu x} + \frac{\Gamma(1+\mu)\Gamma(\mu)}{\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} e^{-\mu x} & x \to -\infty \\
e^{-\mu x} & x \to +\infty
\end{array} \right.
\] (308)

Now only the second piece of the $x \to -\infty$ region of the solution is unbounded. This coefficient vanishes where the $\Gamma$-functions have poles, at $\mu - \nu = -M$ (where $M$ is a positive integer). In the case

\[
a = -N(N + 1)
\]

we have $\mu = N - M$ giving $N$ discrete eigenvalues corresponding to the $N$ solitons in the solution, and there are no non-trivial positive energy waves. (Note asymptotically these look like the delta function.) In more general cases (e.g. $u(x,0) = 4\text{sech}^2(x)$) there are both oscillatory reflections and negative energy bound state eigenfunctions. Thus there is a pure bound state solution only for the special values of $a$, with the solutions being of the form $\psi = AP_{\mu}^N$ where $\mu = 1...N$. 83
Note that we can interpret the wave as naturally splitting into \textit{solitons plus oscillatory parts}. The discrete eigenvalues correspond to the soliton component, and the continuous ones are the oscillatory (dispersive) component. E.g. $a = -6$ corresponds to the pure 2 soliton case. (You might guess at this point that $u(x,0) = N(N+1) \text{sech}^2(x)$ is a snapshot of a pure $N$ soliton solution which will proceed to fly off if we let it evolve in time with no oscillatory component.) Naturally you expect no oscillations at infinity for such values and the potential is then reflectionless. (i.e. any oscillations coming in from infinity are just “on top of” the soliton component). On the other hand when $a$ does not take one of these values we have a reflected wave going off to infinity in one direction.
14 Evolving the scattering data

14.1 What data do we need?

We have found that any particular \( u(x,0) \) leads to a set of possible eigenfunctions with a continuous positive spectrum and a discrete negative energy one. However it would be tedious if we always had to determine all the eigenfunctions for \( \psi \) in a general \( u \). Fortunately we only need a limited set of information about the asymptotic behaviour of the eigenfunctions.

If the asymptotic behaviour of the discrete eigenfunctions (normalized so that \( \int |\psi|^2 dx = 1 \)) is

\[
\psi_n \sim c_n e^{-\mu_n x} \quad x \to \infty
\]

with \( \lambda = -\mu_n^2 \) for each individual eigenvalue, and the asymptotic behaviour of the discrete eigenfunctions (normalized so that the incoming flux = unity) is

\[
\psi \sim \begin{cases} 
 e^{ikx} + R(k)e^{-ikx} & x \to -\infty \\
 T(k)e^{ikx} & x \to +\infty 
\end{cases}
\]

then the collection of scattering data

\[
\{R(k), c_n, \mu_n\}
\]

is completely determined by \( u \). Conversely (and this is the important thing) \( u \) can be reconstructed entirely from these scattering data. Consequently just knowing how these data evolve in time, enables us to reconstruct \( u \) at any later time (which is of course our eventual goal). The reason this procedure works at all is our claim is that the eigenvalues \( \lambda \) remains constant (if \( u \) is a solution of the KdV equation) which we shall verify shortly.

The subsequent reconstruction of \( u \) is then equivalent to an inverse scattering problem that we have to tackle later at the “reassembly” stage. (For that stage, you should have in mind throwing particles at a potential and seeing what data you need to measure to reconstruct the entire potential - “can you hear the shape of a drum?”)

14.2 Examples

To give an idea of how to collect this date consider the examples we have seen

The delta function: Here we have \( R = \frac{a}{2ik-a} \). There is a single discrete eigenvalue with \( \psi = A e^{-\frac{k}{2} |x|} \) where \( A \) is a constant. Conventionally we normalize the discrete eigenvalues such that \( \int |\psi|^2 dx = 1 \) so we need \( A = \sqrt{a/2} \). Consequently \( \psi \sim \sqrt{\frac{a}{2}} e^{-\frac{k}{2} x} \) and so

\[
\mu_1(0) = c_1(0) = \sqrt{\frac{a}{2}}
\]

These are the values taken at \( t = 0 \).
Reflectionless potentials: Here we have $R = 0$ simply. We still have to normalize the eigenfunctions though. They are

$$\psi = AP^\mu_N$$

(312)

where $\mu = 1 \ldots N$. When $\mu$ is integer we find that

$$\int |P^\mu_N|^2 dx = \frac{(N + \mu)!}{\mu (N - \mu)!}.$$ 

(313)

In addition the asymptotics of $P^\mu_N$ (which after some tedious effort we can get from the series expansion of the hypergeometric function) is

$$P^\mu_N \sim \frac{(N + \mu)!}{\mu! (N - \mu)!} e^{-\mu x}$$

(314)

so that the normalized eigenfunction goes as

$$\psi^\mu_N \sim \frac{1}{\mu!} \sqrt{\frac{\mu (N + \mu)!}{(N - \mu)!}} e^{-\mu x}$$

(315)

and therefore

$$c^\mu_N = \frac{1}{\mu!} \sqrt{\frac{\mu (N + \mu)!}{(N - \mu)!}}$$

(316)

e.g. when $N = 2$ we get $c_1 = \sqrt{6}$ and $c_2 = \sqrt{12}$.

Almost reflectionless potentials: This corresponds to the $u = 2.001 \text{sech}^2(x)$ example, where we expect to get a very small oscillatory component (the dispersive wave that is initially shed by the soliton). We can adapt the previous case using a small perturbation. Let

$$a = -N'(N' + 1)$$

where $N' = N + \varepsilon$. The eigenvalues are $\mu' = N' - M = \mu + \varepsilon$ where $M$ is integer and since $\varepsilon$ is small we can set $M = 1 \ldots N - 1$ again. The normalization of eigenfunction proceeds exactly as before, except that we must replace factorials with $\Gamma$-functions. The expansion of a $\Gamma$–function goes like

$$\Gamma(N') = \Gamma(N)(1 + \varepsilon \psi(N) + \ldots)$$

(317)

where $\psi = \Gamma'/\Gamma$ is called the digamma function (you don’t need to know anything about it). Expanding the terms and retaining the first $\varepsilon$ piece I get

$$c^\mu_{N'} = c^\mu_N(1 + \varepsilon \left[ \frac{1}{\mu} + \frac{\psi(N + \mu + 1)}{2} - \psi(\mu + 1) \right]).$$

(318)
\[ R(k) = \frac{\Gamma(-ik - N')\Gamma(1 - ik + N')\Gamma(ik)}{\Gamma(-N')\Gamma(1 + N')\Gamma(-ik)} \]

\[ \approx \varepsilon \frac{\sin(\pi N')\Gamma(-ik - N')\Gamma(1 - ik + N')\Gamma(ik)}{\Gamma(-ik)} \] (319)

It will be our dispersive wave component eventually.

### 14.3 Evolving the data.

Now we have collected our data we need to evolve it forwards in time. For the moment I will outline the straightforward brute force method in Drazin and Johnson and concentrate mainly on the discrete data. In the next section we will see the smarter and much more profound Lax pair method, which also teaches us something about why this method works at all.

Begin with the time independent SE. I’ll reinstate the \( u = -V \) since we’ll need at some point to use the KdV equation for this. The equation is

\[ -\psi_{xx} + (\lambda - u)\psi = 0 \] (320)

and we want to show that \( \lambda_t = 0 \) if \( u \) evolves through time according to the KdV equation.

**Proof:** First differentiate the SE by \( x \) and \( t \),

\[ -\psi_{xxx} - u_x\psi_x + (\lambda - u)\psi_x = 0 \]

\[ -\psi_{xxt} + (\lambda_t - u_t)\psi + (\lambda - u)\psi_t = 0. \]

I stress that \( \psi \) is here differentiated w.r.t. time even though it is the time-independent SE that it solves - it can at the moment be a function of time purely because of time appearing as a parameter in \( u, \lambda \).

Now define

\[ K(x,t) = \psi_t - u_x\psi + 2(u + 2\lambda)\psi_x \] (321)

and construct the identity

\[ \partial_x(\psi_xK - \psi x) = \psi_{xx}(\psi_t - u_x\psi + 2(u + 2\lambda)\psi_x) - \psi(\psi_{xxt} - u_{xxx}\psi + 3u_x\psi_{xx} + 2(u + 2\lambda)\psi_{xxx}) \] (322)

(Note that the sign differences with D+J are due to their opposite sign \( uu_x \) term in the KdV equation. This expression and the next are tedious.
to get - use mathematica or maple if you want to check them.) Next substitute for \( \psi_{xxx} \) and \( \psi_{xxt} \) and \( \psi_{xx} \) gives

\[
\partial_x(\psi_x K - \psi K_x) = \psi^2(-\lambda_t + u_x + 6uu_x + u_{xxx}) = -\lambda_t \psi^2
\]

(323)

where in the last line I used the fact that \( u \) satisfies the KdV equation. Finally we note that the LHS is a total derivative. Thus we can integrate by \( x \) to get

\[
[\psi_x K - \psi K_x]_{-\infty}^{\infty} = -\lambda_t \int \psi^2 dx.
\]

(324)

Consider the bound-state eigenfunctions that have \( \lambda = \mu^2 > 0 \) (i.e. negative energy) then the eigenfunctions are real and are normalized such that \( \int \psi^2 dx = 1 \). In addition they exponentially decay at infinity, so that \( \lim_{|x| \to \infty} (K, \psi, \psi_x, K_x) = 0 \) and hence \( \lambda_t = 0 \).

### 14.3.1 Evolution of the Discrete spectrum

Thus the eigenvalues are constants of the motion, and asymptotically the discrete eigenfunctions have to evolve as

\[
\psi \sim c_n(t) e^{-\mu_n|x|}; \quad |x| \to \infty
\]

(325)

where now \( \mu_n \) is constant. Having established this fact we can work with \( K \) to get the time evolution of the scattering data. Integrating by \( x \) we have

\[
\psi_x K - K_x \psi = 0
\]

(326)

where the RHS is zero (not a time dependent \( x \)-integration constant) because of the boundary condition that the LHS vanishes at spatial infinity. A further integration gives

\[
K = h_\lambda(t) \psi
\]

(327)

where \( h_\lambda \) is the \( t \)-dependent integration constant that can in principle be different for every \( \lambda \). However it turns out (see D+J for details - insert \( K \) and use the SE and integrate again) that \( h_\lambda = 0 \) is the only choice consistent with the boundary conditions of \( \psi, K \) vanishing at spatial infinity. Consequently

\[
K \sim 0 = \psi_t - u_x \psi + 2(u + 2\lambda)\psi_x
\]

(328)

This is our evolution equation for \( \psi \) and hence our scattering data.

We can now insert the asymptotic form of \( \psi \) of eq.325 (along with the asymptotic boundary condition \( u, u_x \to 0 \)) into the above to find

\[
(c_n)_t - 4\mu_n^3 c_n = 0
\]

\[
c_n(0) = c_n(0) e^{4\mu_n^3 t}.
\]
14.3.2 The continuous spectrum

For $\lambda = -k^2 < 0$, we have a continuous spectrum. We may follow the time evolution of the reflection and transmission coefficients of the continuous eigenfunctions by keeping $\lambda = -k^2$ fixed. First we find the asymptotic form of $K$ from its definition in eq.321

$$K \sim (T_t - 4ik^3T)e^{ikx} \quad ; \quad x \to +\infty$$

$$K \sim (R_t + 4ik^3R)e^{-ikx} - 4ik^3e^{ikx} \quad ; \quad x \to -\infty$$

(329)

Now inserting this into eq.324 and insisting that $\lambda_t = 0$ gives us

$$R_t + 8ik^3R = 0.$$ (330)

To get $T$ we need to integrate eq.324.

$$\psi_x K - \psi K_x = g_k(t)$$ (331)

but since this is true for all $x$ and we have just chosen $R$ so the RHS vanishes at the $x \to -\infty$ limit we must have $g_k = 0$. Integrating again

$$K = h_k(t) \psi$$ (332)

In the $x \to -\infty$ limit

$$h_k(Re^{-ikx} + e^{ikx}) = (R_t + 4ik^3R)e^{-ikx} - 4ik^3e^{ikx}$$

$$= -4ik^3(Re^{-ikx} + e^{ikx})$$

determining $h_k = -4ik^3$. In the $x \to +\infty$ limit we find

$$T_t - 4ik^3T = h_k T$$

$$\to T_t = 0.$$ (333)

Taken together then we can summarize the evolution of the scattering data as

$$\mu_n = \text{constant}$$

$$c_n(t) = c_n(0)e^{4\mu_n t}$$

$$R(k, t) = R(k, 0)e^{-8ik^3t}$$

$$T(k, t) = T(k, 0)$$ (334)

Note that the change in $R$ is a phase so that we still have $|T|^2 + |R|^2 = 1$. 

89
15 (Long) Digression: Lax pairs and the KdV hierarchy

The analysis in the previous section can be improved because

- It was rather clumsy and the importance of the KdV equation was a bit opaque
- We would like to be able to see if there are other equations that have the same isospectral properties and can also be solved using inverse scattering

These questions can be addressed using the Lax pair formalism. The result is the discovery of an infinite hierarchy of equations that can be solved by inverse scattering like the KdV equation, and a deep connection with the conserved currents discussed earlier.

15.1 The idea of a Lax pair

Consider the eigenvalue problem of the time independent SE

\[
(\partial_x^2 + u)\psi = \lambda \psi
\]  

with \(\psi\) remaining bounded \(\forall x\). The Lax method concentrates on the properties of the operator \(L = \partial_x^2 + u\) so that

\[
L\psi = \lambda \psi.
\]

I’m going to generalize the discussion so that later we can derive equations similar to the KdV equation, so I’ll say that the equation we want to solve is

\[
u_t = N(u)
\]

where for the KdV eqn we have \(N(u) = -(6uu_x + uu_{xxx})\) but in principle \(N\) can be any function of \(u\). What we would eventually like to know is what form of \(N\) gives isospectral evolution of eigenfunctions. The key properties are

1. The set of eigenvalues \(\lambda\) of \(L(u)\) is independent of \(t\)
2. There is a set of eigenfunctions \(\psi\) of \(L\) that evolves in \(t\) as \(\psi(t) = M\psi\), where \(M\) is another time-independent differential operator

We have seen property (1) for the KdV equation already, but we still need to confirm property (2) as we have so far been concentrating on the asymptotic behaviour.

First assume that \(\exists\) an operator \(M\) such that the equation \(u_t = N(u)\) can be written

\[
L_t + [L, M] = 0
\]
Here all of these objects are operators, but the $[L, M]$ term should be “multiplicative” if the equation is to make sense – this will become clear shortly. Also the $t$ differentiation is on the $u$ dependence in $L$. One has to imagine all of the operators acting on functions of $x$ so really it should be

$$L_t \psi(x) + [L, M] \psi(x) = 0. \quad (339)$$

Now let’s use the eigenvalue equation to determine $\lambda_t$ by differentiating $(L - \lambda) \psi$

$$\lambda_t \psi = L_t \psi + L \psi_t - \lambda \psi_t = (ML - LM) \psi + (L - \lambda) \psi_t = (ML - LM) \psi + (L - \lambda) \psi_t = (L - \lambda)(\psi_t - M \psi) \quad (340)$$

Now multiply by $\psi^*$ and integrate over $x$ gives,

$$\lambda_t \int_{-\infty}^{\infty} |\psi|^2 dx = \lambda_t = \int_{-\infty}^{\infty} \psi^*(L - \lambda)(\psi_t - M \psi) dx \quad (341)$$

where I assumed a conventionally normalized eigenfunction $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$. At this point I’ll introduce an important property of $L$, self-adjointness.

**Definition:** An operator $L$ is called self-adjoint if for any pair of functions $\eta(x), \chi(x)$ it obeys

$$\int_{-\infty}^{\infty} \eta^*L \chi dx = \int_{-\infty}^{\infty} (L \eta)^* \chi dx \quad (342)$$

**Proof:** to prove $L$ is self adjoint integrate by parts twice

$$\int \eta^*(\partial_x^2 + u) \chi dx = \text{surface terms} + \int (\partial_x^2 \eta^* + u \eta^*) \chi dx$$

$$= \int_{-\infty}^{\infty} (L \eta)^* \chi dx$$

where all the surface terms vanished in the $x \rightarrow \pm \infty$ limit.

---

1For example if $L = \partial_x$ and $M = u + \partial_x$ (which they don’t but never mind) then

$$[L, M] \psi = (LM - ML) \psi = \partial_x(u \psi + \partial_x \psi) - (u \partial_x \psi + \partial_x^2 \psi) = u_x \psi.$$  

This is multiplicative in the sense that there is no $\partial_x \psi$ term. This is all longhand. In shorthand it is easier to drop the $\psi$ and write

$$[L, M] = (LM - ML) = \partial_x u + \partial_x^2 - u \partial_x - \partial_x^2 = u_x + u \partial_x - u \partial_x = u_x$$

where the last line follows once you remember to let the differential operators act on all functions to the right, i.e. $u$ and the unwritten $\psi$. 

91
Using this result, eq.341 becomes

\[ \lambda_t = \int_{-\infty}^{\infty} ((L - \lambda)\psi^* (\psi_t - M\psi) \, dx = 0 \quad (343) \]

where we used \((L - \lambda)\psi = 0\) in the last line. This completes the proof that \(\lambda_t = 0\) if the operator \(L\) obeys \(L_t + [L, M] = 0\).\(^2\) We have yet to determine the time evolution of \(\psi\). Having established that \(\lambda_t = 0\) we can return to eq.340

\[ (L - \lambda)(\psi_t - M\psi) = 0 \quad (346) \]

This is in the form of an eigenvalue equation, and (because for each \(\lambda\) the eigenfunctions are unique) we must have

\[ (\psi_t - M\psi) = a\psi \quad (347) \]

where \(a\) is some constant. However we can define a new \(\hat{M}\) given by

\[ \hat{M} = M + a \]

for which the above gives

\[ \psi_t = \hat{M}\psi \quad (348) \]

But in terms of \(\hat{M}\) we have

\[ L_t + [L, \hat{M}] = 0 \quad (349) \]

so we may as well have used \(\hat{M}\) instead of \(M\) in the first place.

\(^2\text{Generalization: The above discussion can be generalized. For example the operators } L \text{ and } M \text{ could be matrix valued. The generalization is then that the operators } L, M \text{ act on some Hilbert space, and we work with the inner product } (\eta, \chi). \text{ For the example above we would define } (\eta, \chi) = \int \eta^* \chi \, dx \text{ and the Hilbert space is the space spanned by normalized eigenfunctions, but in general the inner product may involve a trace of matrices, or whatever. (You may have seen the equivalent in QM as } \langle \eta | \chi \rangle). \text{ In this notation self adjointess is written } (\eta, L\chi) = (L\eta, \chi) \quad (344) \text{ and the argument would have been written } \]

\[ \lambda_t(\psi, \psi) = (\psi, (L - \lambda)(\psi_t - M\psi)) \]

\[ = ((L - \lambda)\psi, (\psi_t - M\psi)) \]

\[ = 0. \quad (345) \]
15.2 The Lax pair for KdV

All that needs to be done now is to find correct $M$ for the differential equation, in this case the KdV equation. It is possible to be systematic, but at this stage I’ll just present it and confirm it works; the Lax pair is

$$
L = \partial_x^2 + u \\
M = -(4\partial_x^2 + 6u\partial_x + 3u_x).
$$  \hspace{1cm} (350)

In order to avoid injury, it is conventional to define $D = \partial_x$. Remember that everything has the sense of operators acting on functions to the right so that for example

$$
[u, D]f = (u\partial_x - \partial_x u)f \\
= u\partial_x f - \partial_x (uf) \\
= u\partial_x f - (\partial_x u)f - u(\partial_x f) \\
= -(\partial_x u)f \\
$$  \hspace{1cm} (351)

so we would just write $[u, D] = -u_x$. Now for the calculation. First, obviously

$$
L_t = u_t.
$$

Then we have $[L, M] = [D^2 + u, -(4D^3 + 6uD + 3u_x)]$. Note that $[D^n, D^m] = 0$. Also we can expand the commutators to make things a bit easier before we start, and then note that the commutations involving $[D^n, f(x)]$ acts essentially like a straight differentiation where you drop the $fD^n$ term and keep trailing $D^{n-m}$ terms; so

$$
= 4(u_{xxx} + 3u_{xx}D + 3u_x D^2) \\
+ 6uu_x - 6(u_{xx} + 2u_x D)D \\
- 3(u_{xx} + 2u_x D) \\
= u_{xxx} + 6uu_x.
$$

Note that a crucial property is that the end result is “multiplicative”, all the terms with trailing $D$ and $D^2$ cancelled. That is $[L, M]$ when acting on some function $f(x)$ just multiplies it by $(6uu_x + u_{xxx})$. This will be very important when it comes to making a more systematic search for similar KdV-like systems later. So finally

$$
L_t + [L, M] = u_t + 6uu_x + u_{xxx}
$$

as required.
15.3 Time evolution of scattering data revisited

Let’s check that the Lax pair gives the correct time evolution of the scattering data. The equation is

\[ \psi_t = M \psi. \] (352)

For the scattering data we need only get \( M \) in the asymptotic \((|x| \to \infty)\) region;

\[ M = -(4 \partial_x^3 + 6u \partial_x + 3u_x) \sim -4 \partial_x^3 \] (353)

where we used that \( u, u_x \to 0 \) at spatial infinity. For the reflection and transmission coefficients, we have

\[ \psi \sim \begin{cases} A_-(t)e^{ikx} + B_-(t)e^{-ikx} & x \to -\infty \\ A_+(t)e^{ikx} & x \to +\infty \end{cases} \] (354)

I have set \( B_+ = 0 \) (i.e. there is never any left moving wave on the right side of the scattering region) but the normalization \( A_- \) I leave as time dependent. Inserting this into eq.352 we get

\[ \partial_t A_- e^{ikx} + \partial_t B_- e^{-ikx} = 4ik^3 A_- e^{ikx} - 4ik^3 B_- e^{-ikx} \quad x \to -\infty \]
\[ \partial_t A_+ e^{ikx} = 4ik^3 A_+ e^{ikx} \quad x \to +\infty \] (355)

Equating coefficients of \( e^{\pm ikx} \) we get

\[ A_- = A_-(0)e^{4ik^3 t} \]
\[ A_+ = A_+(0)e^{4ik^3 t} \]
\[ B_- = B_-(0)e^{-4ik^3 t}. \] (356)

Finally to get \( R(t, k) \) and \( T(t, k) \) we need to divide the solution by \( A_- (t) \) to normalize the incoming wave to one. This gives \( R(t, k) = R(0, k)e^{-8ik^3 t} \) and \( T(t, k) = T(0, k) \) as we found earlier.

For the time development of \( c_n \), we have asymptotically

\[ \psi \sim c_n(t)e^{-\mu_n |x|}. \] (357)

Inserting this into eq.352 we get

\[ \partial_t c_n = 4 \mu_n^3 c_n \] (358)

so as found earlier,

\[ c_n(t) = c_n(0)e^{4 \mu_n^3 t} \] (359)

Note that for the scattering data we only need the asymptotic form of \( \psi \) and the \( u, u_x \) dependence dropped out of \( M \). If this hadn’t happened the method would not have worked (i.e. we would have had to have known the form of \( u(t) \) already in order to make the inverse transform).
15.4 The KdV hierarchy

It is natural to ask now if the KdV equation was the only equation with isospectral properties. The answer is no, and in fact the Lax pair method shows us how to construct and infinite series of such equations.

**Key point:** the proof of an invariant spectrum only used the property that the evolution of \( L(u) \) can be written as

\[
L_t + [L, M] = 0
\]  

(360)

but didn’t actually rely on the form of \( M \). In fact we required \( M = -(4D^3 + 6uD + 3ux) \) to get the KdV equation, but this wasn’t used in the proof.

The idea is to pick different \( M(u) \) such that the \( L \) evolution can be written in the same way but the equation generated doesn’t have to be the KdV equation. This will be enough to guarantee the constancy of the spectrum (i.e. time independence of the eigenvalues, \( \lambda_t = 0 \)).

However \( M(u) \) is not arbitrary. Since \( L_t = u_t \) is purely multiplicative then so must \([L, M]\) be. That is all the \( D \)'s must cancel in \([L, M]\). If they do then \( L_t + [L, M] \) is the required equation for \( u \) as a p.d.e. Let’s try some examples. In what follows I’ll always be assuming \( L = D^2 + u \);

**Example 1:** \( M = \alpha(x) \) for some function \( \alpha \). Then

\[
[L, M] = [D^2 + u, \alpha]
\]

\[
= [D^2, \alpha]
\]

\[
= \alpha_{xx} + 2\alpha_x D
\]  

(361)

So in this case we require \( \alpha_x = 0 \) or \( \alpha = constant \). But then \( \alpha_{xx} = 0 \) as well and \( L_t = u_t = 0 \) is the final equation. So this certainly has \( \lambda_t = 0 \) but it is a bit trivial.

**Example 2:** \( M = \alpha(x)D + \beta(x) \) for some functions \( \alpha \) and \( \beta \). Then

\[
[L, M] = [D^2 + u, \alpha D + B]
\]

\[
= [D^2, B] + [D^2, \alpha]D - \alpha[D, u]
\]

\[
= \beta_{xx} + 2\beta_x D + \alpha_{xx}D + 2\alpha_x D^2 - \alpha u_x.
\]

So in this case we need \( 2\beta_x + \alpha_{xx} = 0 \) and \( \alpha_x = 0 \). Again we have \( \alpha \) and \( \beta \) both constant so \( \alpha_{xx} = \beta_{xx} = 0 \), but

\[
L_t + [L, M] = u_t - \alpha u_x = 0.
\]  

(362)

This is a linear equation that can again be trivially solved by D’Alembert’s solution. i.e. any function \( f(x + \alpha t) \) solves it. Again it is obvious that the eigenvalues remain constant because the initial “wave” profile of \( u \) is just translated. This is maybe the first hint that the simplicity of the KdV equation and the linear equation is in some sense related.
As an exercise we can check that the eigenfunctions evolve according to

\[ \psi_t = M(u)\psi = \alpha \psi_x + \beta \psi \]

Let \( z_\pm = t \pm \alpha x \). Then this can be rewritten as

\[ \frac{\psi_{z_+}}{\psi} = \beta \]

or

\[ \psi = \exp(\beta z_+ + \gamma(x_+)) \]

where \( \gamma \) is any function of \( t + \alpha x \) that has to be set by the initial scattering data.

**Example 3:** The final example to try “by hand” is \( M = \alpha(x)D^3 + \beta(x)D + \gamma(x) \) with \( \alpha, \beta, \gamma \) again being arbitrary functions of \( x \). Then

\[
[L, M] = [D^2 + u, \alpha D^3 + \beta D + \gamma] \\
= [D^2, \alpha]D^2 + [D^2, \beta]D + [D^2, \gamma] \\
- \alpha[D^3, u] - \beta[D, u] \\
= \alpha_{xx}D^2 + 2\alpha_xD^3 + \beta_{xx}D + 2\beta_xD^2 + \gamma_{xx} + 2\gamma_xD \\
- \alpha(u_{xxx} + 3u_{xx}D + 3u_xD^2) - \beta u_x
\]

(363)

Setting the coefficients of \( D^3, D^2 \) and \( D \) equal to zero we get

\[
2\alpha_x = 0 \\
\alpha_{xx} + 2\beta_x - 3\alpha u_x = 0 \\
\beta_{xx} + 2\gamma_x - 3\alpha u_{xx} = 0
\]

(364)

The first relation gives \( \alpha = \text{constant} \), so \( \alpha_x = \alpha_{xx} = 0 \). The second then gives

\[ \beta = \frac{3}{2} \alpha u + c \]

where \( c \) is a constant. The third gives

\[ \gamma = \frac{3}{4} \alpha u_x + d \]

where \( d \) is another constant. Substituting into eq.363 we find

\[ L_t + [L, M] = u_t - \frac{\alpha}{4} u_{xxx} - \frac{3\alpha}{2} u u_x - cu_x. \]

Choosing \( \alpha = -4 \) and \( c = 0 \) gives the KdV equation! For more general values we can just rescale the coordinates; you can check that

\[ \tilde{u}(x, t) = u(x + \frac{\alpha c}{4} t, -\frac{\alpha}{4} t) \]

satisfies the KdV equation. This shows for the first time that the KdV equation is in some sense “natural”, and is one of the lower members of a *hierarchy* of such equations.
15.4.1 Hints for the general case

In principle we could keep going, but the calculations are getting pretty grim. Let’s look at some hints about how to simplify things a little. We need some new technology.

1. **Hermitian inner product**: \((\eta, \chi)\). For the example above we would define \((\eta, \chi) = \int \eta^* \chi \, dx\) and the Hilbert space is the space spanned by normalized eigenfunctions, but in general the inner product may involve a trace of matrices, or whatever. (You may have seen the equivalent in QM as \(\langle \eta | \chi \rangle\)). The argument for \(\lambda_t = 0\) would have been written

\[
\lambda_t (\psi, \psi) = (\psi, (L - \lambda)(\psi_t - M\psi)) \\
= ((L - \lambda)\psi, (\psi_t - M\psi)) \\
= 0. \tag{365}
\]

Now we take into account that the \(\psi\) and \(\eta\) may be matrix valued or vectors.

2. **The adjoint of an operator**: Before we used the property of self-adjointness of \(L\). We can define the “adjoint” of \(L\) with a dagger such that as

\[
(\eta, L\chi) = (L^\dagger \eta, \chi) \tag{366}
\]

Then self-adjointness is the condition that

\[
L^\dagger = L \tag{367}
\]

so that

\[
(\eta, L\chi) = (L\eta, \chi) \tag{368}
\]

as required. The dagger is the like the hermitian conjugate of a matrix. The analogy is with a matrix \(M\) and vectors \(U, V\) where you can check things work in the way you’d expect. The hermitian inner product would be \((V, U) = V^\dagger U\) where now \((V^\dagger = (V^*)^T)\). You can check that

\[
(M^\dagger V, U) = (M^\dagger V)^\dagger U \\
= V^\dagger MU \\
= (V, MU) \tag{369}
\]

and self-adjointness means that the matrix is hermitian \(M = M^\dagger\). With rather sloppy nomenclature self-adjoint operators are often called symmetric, \(L^\dagger = L\). The opposite is called anti-symmetric or skew-symmetric \(M^\dagger = -M\).

Now note the condition that \([L, M]\) is a real, multiplicative function and contains no operators. By definition of self-adjointness we must have

\[
[L, M] = [L, M]^\dagger. \tag{370}
\]
What can we learn about $M$ from this? Using the rules in appendix D we can do the following

\[
[L, M] = (LM)^\dagger - (ML)^\dagger \\
LM - ML = M^\dagger L^\dagger - L^\dagger M^\dagger \\
LM - ML = M^\dagger L - LM^\dagger
\]

\[\rightarrow [L, M + M^\dagger] = 0 \quad (371)\]

Just as for matrices we can decompose $M$ into symmetric and antisymmetric parts as follows

\[
M_S = \frac{(M + M^\dagger)}{2} \\
M_A = \frac{(M - M^\dagger)}{2}.
\]

The above equation tells us that the symmetric part of $M$ commutes with $L$. Since the equation of interest is $L_t + [L, M] = 0$ we may as well take $M$ to be antisymmetric.

### 15.4.2 How to write $M$?

Instead of writing the general $M$ of order $m$ as $\sum_j^m \alpha_j(x)D_j^j$ where $\alpha_j$ are some functions of $u$ $u_x$ etc, we can choose a different basis by choosing

\[
\sum_j^m \beta_j D_j^j + D_j \beta_j
\]

where $\beta_j$ are some other functions. Now by integration by parts we have $D^\dagger = -D$ and of course the functions obey $\beta_j^\dagger = \beta_j$. So then

\[
(D^{2k})^\dagger = D^{2k} \\
(D^{2k-1})^\dagger = -D^{2k-1}
\]

so we may take the odd powers of $D$ only

\[
\sum_{k=1}^m \beta_{2k-1} D^{2k-1} + D^{2k-1} \beta_{2k-1}
\]

Consider the KdV operator

\[
M(u) = -4D^3 - 6uD - 3u_x \\
= ( -2D^3 - 2D^3 ) + ( -3uD - 3Du )
\]

so the above is a consistent way to write is. There is one additional simplifying feature; if $[L, M]$ is multiplicative, then the leading term has $\beta_{2m-1} = \text{constant}$. (as an exercise show this.)
Now there is no alternative but hard work. The constraint of multiplicative
operators means that the vanishing of each term in $D^n$ in $M$ imposes exactly
one constraint, so that there are exactly as many constraints as functions $\beta_j$
and therefore the latter are completely constrained. Let's look at the first few
examples;

1. $m = 1$; the highest power is 0 so trivially
   \[ u_t = 0 \] \hspace{1cm} (376)

2. $m = 2$; the highest power is 1, we have $M = -D$, (i.e. recall that the
   leading $\beta$ is a constant) and
   \[
   [L, M] = -[D^2 + u, D] \\
   = u_x
   \]

   So $u_t + u_x = 0$

3. $m = 2$; the highest power is 3 and we have $M = -D^3 + \beta_1 D + D \beta_1$.
   \[
   [L, M] = [D^2 + u, -D^3 + \beta_1 D + D \beta_1] \\
   = [D^2, \beta_1]D + D[D^2, \beta_1] + [D^3, u] - \beta_1[D, u] - [D, u] \beta_1 \\
   = \beta_{1xx} D + 2\beta_{1x} D^2 + D(\beta_{1xx} + 2\beta_{1x} D) + u_{xxx} + 3 u_{xx} D + 3 u_x D^2 - 2 \beta_1 u_x \\
   = \beta_{1xx} D + 2\beta_{1x} D^2 + (\beta_{1xx} + 2\beta_{1x} D) D + \\
   (\beta_{1xxx} + 2\beta_{1xx} D) - u_{xxx} - 3 u_{xx} D - 3 u_x D^2 - 2 \beta_1 u_x \\
   = (4 \beta_{1xx} - 3 u_{xx}) D + (4 \beta_{1x} - 3 u_x) D^2 + \beta_{1xxx} - u_{xxx} - 2 \beta_1 u_x \] \hspace{1cm} (377)

   So that $D$ and $D^2$ vanishing implies $\beta_1 = \frac{3}{4} u$. Inserting this we get
   \[ 0 = 4u_t - u_{xxx} - 6 u u_x. \]

   A simple rescaling (e.g. $t \rightarrow -4 t$ or $M \rightarrow -4 M$) yields the usual KdV
equation.

4. $m = 3$ : The highest power is 5. This gets messy (left as character
   building exercise)

   \[ u_t + 30 u^2 u_x + 20 u_x u_{xx} + 10 u u_{xxx} + u_{xxxxx} = 0 \] \hspace{1cm} (378)

This is a new 5th order equation with the same properties as KdV. The series
continues to infinity.
16 Connection with conservation laws

There is an amazing connection with the conserved charges and conservation laws. Recall the the KdV equation has an infinite sequence of charges, given by

\[ Q_n = \int_{-\infty}^{\infty} T_n dx \]

where \((T_n)_t + (X_n)_x = 0\) for some \(X_n\) with \([X]_{-\infty}^{\infty} = 0\). The series is

\[
\begin{align*}
T_1 &= u \\
T_2 &= u^2 \\
T_3 &= u^3 - \frac{1}{2} u_x^2 \\
T_4 &= u^4 - 2uu_x^2 + \frac{1}{3} u_{xx}^2 \\
T_5 &= u^5 - \frac{105}{21} u^2 u_x^2 + uu_{xx}^2 - \frac{1}{21} u_{xxx}^2 \\
T_n &= u^n + \ldots
\end{align*}
\]

(379)

where \(T_{4,5}\) are new to you. We now have two infinite sequences

- For the KdV eqn itself an infinite sequence of \(T_n\)’s
- Going beyond the KdV eqn, an infinite sequence of \(N_n(u)\) such that \(u_t = N_n(u)\) leaves the eigenvalues of \(L = \partial_x^2 + u\) invariant

How do these two sequences tie together if at all? The most boring possibility is that each \(N_n(u)\) has its own completely separate sequence of \(T_n\)’s however the answer is much more mysterious.

16.1 Functional or Frechet derivatives

To connect the two sequences we need an extra tool, the functional derivative. Recall from the variational idea from Lagrangian Mechanics (see Appendix B). We first defined the Lagrange density, \(\mathcal{L}\) such that

\[
S[u] = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \mathcal{L}(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) \, dx \, dt
\]

(380)

I have here generalized things to allow dependence on higher derivatives of \(u\) than you would normally consider. Let \(u \rightarrow u + \delta u\) then if \(S \rightarrow S + \delta S\) we have

\[
\delta S = 0 = \int \int \mathcal{L}(u + \delta u, u_t + \delta u_t, u_x + \delta u_x, \ldots) - \mathcal{L}(u, u_t, u_x) \, dt \, dx
\]

\[
= \int \int \left( \frac{\partial \mathcal{L}}{\partial u} \delta u + \frac{\partial \mathcal{L}}{\partial u_t} \delta u_t + \frac{\partial \mathcal{L}}{\partial u_x} \delta u_x + \frac{\partial \mathcal{L}}{\partial u_{xx}} \delta u_{xx} + \frac{\partial \mathcal{L}}{\partial u_{xxx}} \delta u_{xxx} + \ldots \right) \, dt \, dx
\]

\[
= \int \int \left( \frac{\partial \mathcal{L}}{\partial u} \frac{d}{dt} u_t + \frac{\partial \mathcal{L}}{dx} \frac{d}{dx} u_x + \frac{\partial^2 \mathcal{L}}{dx^2} \frac{d}{dx} u_{xx} + \frac{\partial^3 \mathcal{L}}{dx^3} \frac{d}{dx} u_{xxx} + \ldots \right) \delta u \, dt \, dx
\]

(381)
Here we can get the last line by repeatedly integrating by parts in the relevant integrand, $x$ or $t$, and there is a plus or minus sign depending on whether we need an even or odd number of integration by parts. The main point was to get everything multiplying $\delta u$ which leads to the Euler-Lagrange equations upon setting the function inside the brackets to zero. The object inside the brackets is defined to be the functional derivative of $S$ with respect to $u$.

$$\frac{\delta S}{\delta u} = \frac{\partial L}{\partial u} - \frac{d}{dt} \frac{\partial L}{\partial u_t} - \frac{d}{dx} \frac{\partial L}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial L}{\partial u_x} - \frac{d^3}{dx^3} \frac{\partial L}{\partial u_x} + \ldots$$

(382)

so that the concept of derivatives (in particular $f(x + \delta x) = f(x) + \frac{df}{dx} \delta x$) is generalized to integrals of functions;

$$S[u + \delta u] = S[u] + \int \frac{\delta S}{\delta u} \delta u \, dx + O(\delta u^2).$$

(383)

We can also find functional derivatives for integrals just over $x$. E.g. if

$$F[u] = \int f(u, u_x, u_{xx} \ldots) \, dx$$

then

$$F[u + \delta u] = F[u] + \int \left( \frac{\partial f}{\partial u} - \frac{d}{dx} \frac{\partial f}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial f}{\partial u_x} - \frac{d^3}{dx^3} \frac{\partial f}{\partial u_x} + \ldots \right) \delta u \, dx.$$ 

(385)

The only difference to the action principle is the disappearance of the “time”-pieces, so actually this is simpler than the action principle with which you may be more familiar. Some examples:

- $f = u \Rightarrow \frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} = 1$
- $f = u^3 \Rightarrow \frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} = 3u^2$
- $f = u_x^2 \Rightarrow \frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx} \frac{\partial f}{\partial u_x} = -2u_{xx}$

One additional feature that will become important in a minute is that if $f$ is a total derivative (i.e. $f = \partial_x g$) then

$$\frac{\delta F}{\delta u} = 0.$$ 

(386)

The proof of this I’ll leave as an exercise, but it is clearly true for example for $g$ being a function of $u$ only; by the chain rule

$$\frac{\delta F}{\delta u} = u_x g_u - \partial_x g_u = u_x g_{uu} - u_x g_{uu} = 0$$

Now consider the conserved quantities, $Q_n = \int T_n \, dx$
\[ \delta Q_1 = \frac{\delta}{\delta u} (\delta u) = 1 \]
\[ \delta Q_2 = \frac{\delta}{\delta u} (\delta u) = 2u \]
\[ \delta Q_3 = \frac{\delta}{\delta u} \left( u^3 - \frac{1}{2} u^2_+ \frac{1}{2} u^2_{xx} \right) = 3u^2 + u_{xx} \]
\[ \delta Q_4 = \frac{\delta}{\delta u} \left( u^4 - 4uu_{uxx} - 2u_{xx}^2 + \frac{1}{2} u_{xxxx} \right) = 4u^3 + 8u_x^2 + 4uu_{xx} - 2u_{xx}^2 + \frac{2}{5} u_{xxxx} \]

Taking the derivatives with respect to \( x \) gives

\[ \partial_x \left( \frac{\delta Q_n}{\delta u} \right) = 0, \ 2u_x, \ 6uu_x + u_{xxx}, \ \frac{1}{5} (u_{xxxxx} + 10uu_{xxxx} + 20u_xu_{xx} + 30u^2u_x), \ldots \]

This should be compared with the \( \mathbb{N}_n(u) \) operators (rescaled)

\[ \mathbb{N}_n(u) = 0, \ u_x, \ 6uu_x + u_{xxx}, \ u_{xxxxx} + 10uu_{xxxx} + 20u_xu_{xx} + 30u^2u_x \]

So the two sequences are exactly the same in this sense. There are some other remarkable facts

- If \( u \) evolves by the \( m \)th KdV equation exactly the same set of \( Q'_n \)'s are conserved quantities

- Imagine we have one time for each equation \( t_1, t_2 \ldots \) etc, so that we have \( u(x, t_1, t_2 \ldots) \), and that evolution through \( t_m \) is given by \( \mathbb{N}_m \), so we have a set of evolution equations;

\[ u_{tm} = \mathbb{N}_m(u). \]

Then if we evolve for a while in \( t_n \) and then in \( t_{m \neq n} \), it doesn’t matter which way round we do it. This is the idea of commuting flows, and is a very important aspect of “modern” soliton theory.
Finally we have to reconstruct the potential at time $t$ by performing the inverse scattering transform. We have collected the data

\begin{align*}
\mu_n(t) &= \mu_n(0) \\
c_n(t) &= c_n(0)e^{4\mu_n^3 t} \\
R(k,t) &= R(k,0)e^{-8ik^3 t}
\end{align*}

(388)

confident that we can determine the form of $u(t)$ from it, but is this possible? In other words, if you are allowed to sit at spatial infinity and bounce particles of the potential, can you tell the shape of the potential by just measuring the data. This is a question of some practical importance, for example in sonar.

For the 1d SE the answer was already well known going back to Gel’fand and Levitan in 1951 and Marchenko in 1955. The recipe for reconstructing $u(t)$ is as follows (I’ll use it without proof - for the proof see D+J).

### 17.1 The Marchenko equation

To reconstruct $u(x,t)$ from the scattering data

1. construct the function

\begin{equation}
F(\xi) = \sum_{n=1}^{N} e^{\mu_n^3 \xi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k)e^{ik\xi}dk
\end{equation}

(389)

where $N$ is the number of solitons in the spectrum, and both $c_n, R(k)$ are to be evaluated at time $t$.

2. Solve the following linear integral equation for $K(x, y)$

\begin{equation}
K(x, z) + F(x + z) + \int_{x}^{\infty} K(x, y)F(y + z)dy = 0
\end{equation}

(390)

3. $u(x,t)$ (which is equal to $-V(x,t)$) is given by

\begin{equation}
u(x,t) = 2\partial_x K(x,x).
\end{equation}

(391)

The equation may or may not be solvable, but the main point is that the inverse problem is essentially the same at all times, with the time dependence in $F$ being given by

\begin{equation}
F(\xi; t) = \sum_{n=1}^{N} c_n^2(0)e^{4\mu_n^3 (t-t_0)\xi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k,0)e^{ik\xi-8ik^3 t}dk
\end{equation}

(392)

Taken together the technique is a remarkable discovery, and is a kind of “non-linear Fourier analysis”. To make the final step clearer we’ll look at some examples.
17.2 The single KdV soliton

Returning to the single soliton case; do we recover the travelling wave solution? Here as discussed our starting function is \( u(0) = 2\text{sech}^2 x \). Inserting into the \( c_N^\mu \) equation 316 with \( N = \mu = 1 \) we have simply

\[
c_1^1 = \sqrt{2} \quad (393)
\]

and \( R(t, k) = 0 \), so that

\[
F(\xi, t) = 2e^{4t-\xi} \quad (394)
\]

The Marchenko equation is

\[
K(x, z) + 2e^{8t-(x+z)} + 2 \int_x^\infty K(x, y) e^{8t-(y+z)} dy = 0 \quad (395)
\]

Since \( y \) is just a dummy variable, we can extract the \( e^{-z} \) factors and the equation is in the form

\[
K(x, z) + e^{-z} (2e^{8t-x} + 2 \int_x^\infty K(x, y) e^{8t-y} dy) = 0 \quad (396)
\]

The solution must clearly be in the form \( K(x, z) = J(x)e^{-z} \) where

\[
J(x) = -2e^{8t-x} - 2J(x) \int_x^\infty e^{8t-2y} dy = -2e^{8t-x} - J(x)e^{8t-2x}
\]

so that

\[
K(x, z) = -\frac{2e^{8t-(x+z)}}{1 + e^{8t-2x}}. \quad (397)
\]

So that

\[
u(x, t) = 2\partial_x K(x, x) = -4\partial_x (e^{2x-8t} - 1)^{-1} = 8e^{2x-8t} (e^{2x-8t} - 1)^{-2} = 2\text{sech}^2(x - 4t) \quad (398)
\]

which is the travelling wave solution!

17.3 The almost reflectionless case; \( u(0) = 2.001\text{sech}^2 x \)

In this case it is possible to develop a perturbation series to find solutions. Previously we found that the scattering data for the almost reflectionless potential (that is \( u = N'(N' + 1)\text{sech}^2 x \) where \( N' = N + \varepsilon \)) can be written

\[
c_N^\mu(0) = c_N^\mu(1 + \varepsilon \alpha(\mu, N))
\]

\[
R(k, 0) = \varepsilon \beta(k, N)
\]
where $\alpha, \beta$ are some not very nice functions of $N, \mu, k$. We can do a perturbation series in $\varepsilon$ by defining $K = K_0 + \varepsilon K_1 + \ldots$ and $F = F_0 + \varepsilon F_1 + \ldots$. The Marchenko equation can then be solved iteratively; i.e. the full equation plus the zeroth order solution we have just found

$$K_0(x, z) + F_0(x + z) + \int_x^\infty K_0(x, y) F_0(y + z) \, dy = 0$$

$$\Rightarrow K_1(x, z) + F_1(x + z) + \int_x^\infty (K_1(x, y) F_0(y + z) + K_0(x, y) F_1(y + z)) \, dy = 0 \quad (399)$$

Since this equation is linear in exponentials things become more tractable although not very pleasant (especially the second term in the integral). Have a go at the $N=\mu=1$ single soliton case using

$$\alpha(1, 1) = \frac{1}{4} (3 + 2\gamma_E)$$

$$\beta(k, 1) = -\frac{\Gamma(2 - ik)}{1 + ik}$$

where $\gamma_E = 0.577216$. If you can solve it you should find a dispersive component travelling to the left. It probably helps to swap the order of $k$ and $y$ integrations, but I suspect you'll still be left with a single difficult integral to do.

### 17.4 The $N$-soliton solution

The single soliton solution indicates how to proceed with the multi-soliton one, and we actually get new results. Begin by constructing $F$ again; it can be written as

$$F(\xi; t) = \sum_{\mu=1}^N (c_\mu^N(0))^2 e^{8\mu^3 t - \mu \xi}$$

since as we have seen $R = 0 \forall t$. The Marchenko equation is in the form

$$K(x, z) + \sum_{\mu} (c_\mu^N(t))^2 e^{-\mu(x+z)} + \sum_{\mu} (c_\mu^N(t))^2 \int_x^\infty K(x, y) e^{-\mu(y+z)} \, dy = 0 \quad (401)$$

We now define a $N$ component vector consisting of entries $e^{-\mu z}$ with $\mu = 1 \ldots N$, which I'll call $\eta$. Now suppose the solution is in the form

$$K(x, z) = \sum_{\mu} J(x)_{\mu} \eta_{\mu}(z)$$

where $J$ is now another $N$ vector. The Marchenko equation is now

$$\sum_{\mu} J_{\mu} \eta_{\mu}(z) = \sum_{\mu} (c_\mu^N(t))^2 \eta_{\mu}(z) (e^{-\mu x} + \sum_{\nu} J_{\nu} \int_x^\infty e^{-(\mu+\nu)y} \, dy)$$

$$= \sum_{\mu} (c_\mu^N(t))^2 \eta_{\mu}(z) (e^{-\mu x} + \sum_{\nu} J_{\nu} \frac{e^{-(\mu+\nu)x}}{\mu + \nu}) \quad (403)$$
Following our previous success, look for a solution where all the coefficients vanish for every $\eta_\mu$. The above equation becomes a matrix equation

$$J_\mu = (c^\mu_N(t))^2 (e^{-\mu x} + \sum_\nu J_\nu \frac{e^{-(\mu+\nu)x}}{\mu + \nu})$$

which can be written

$$J = S^{-1}_N L$$

where

$$S_{N\mu\nu} = \delta_{\mu\nu} - (c^\mu_N(t))^2 \frac{e^{-(\mu+\nu)x}}{\mu + \nu}$$

$$L_\mu = (c^\mu_N(t))^2 e^{-\mu x}$$

To find the solution we now note that

$$\partial_x S_{N\mu\nu} = (c^\mu_N(t))^2 e^{-(\mu+\nu)x}$$

$$= L_\mu e^{-\nu x}$$

so

$$J_\mu = \sum_\nu (S^{-1}_N)_{\mu\nu} (\partial_x S_N)_{\nu\mu} e^{\mu x}$$

and

$$K(x, x) = \sum_\mu J_\mu e^{-\mu x}$$

$$= \sum_{\mu\nu} (S^{-1}_N)_{\mu\nu} (\partial_x S_N)_{\nu\mu}$$

$$= \text{Tr}(S^{-1}_N \partial_x S_N)$$

This can (may) be recognized formally as

$$\partial_x \text{Tr}(\log(S_N)) = \partial_x \log(\det S_N).$$

Finally the $u(x, t)$ can be written as

$$u(x, t) = 2\partial_x K(x, x)$$

$$= 2\partial_x^2 \log(\det S_N)$$

As an exercise you can show that this is equivalent to the previous N-soliton solution we wrote derived using Hirota’s method.
18 Integrable systems in classical mechanics

The systems we have been looking at so far can be thought of as an integrable infinite Hamiltonian system - it’s a one dimensional field theory which can be thought of as an infinite number of coupled oscillators (c.f. the Sine-Gordon discussion). Not surprisingly then, Lax pairs can be applied to finite systems as well, and there are various famous examples of completely integrable systems in classical mechanics. A Hamiltonian system is defined by a set of coordinates $q_k=1...n$ momenta $p_k=1...n$ and a Hamiltonian, $H(q,p)$ defined such that

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$ (410)

Definition: A Hamiltonian system $H(q_k,p_k)$ where $k = 1...n$ is called completely integrable if it has $n$ integrals of motion $Q_k(q,p)$ where $\{H, Q_k\} = 0$ which are time independent, and in mutual involution $\{Q_k, Q_j \neq k\} = 0$.

The interest for our discussion is that the integrability of classical systems can be shown by the existence of a Lax pair, which allow it to be solved. And the integrals of motion are precisely the eigenvalues of the Lax equation;

$$\dot{L} = [L, M]$$ (411)

(I’ll use a dot notation for time derivatives, $\dot{L} \equiv L_t$) To clarify what the form of $L$ or $M$ should be consider some examples.

18.1 The Lax pair for the simple harmonic oscillator

In the field theory, we saw that there Lax pair equation involved differential operators acting on function. In finite systems this is replaced by matrix equations instead (e.g. think of the differential operator before taking the limit, it would be $(q_{k+1} - q_{k})/\varepsilon$, and think of $q_k$ as being an $n$-vector). The simplest example is the single simple harmonic oscillator with Lax pair

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix}$$ (412)

The Lax equation becomes

$$\begin{pmatrix} \dot{p} & \omega \dot{q} \\ \omega \dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix}$$ (413)
From this we can read off \( \dot{q} \) and \( \dot{p} \) and find
\[
\dot{q} = p \\
\dot{p} = -\omega^2 q
\] (414)
which gives \( H(q, p) = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 \). There is a formal solution to the Lax equation given by
\[
L(t) = U(t)L(0)U(t)^{-1}
\] (415)
where \( U \) is a solution to the initial value problem
\[
\dot{U} = -MU \\
U(0) = 1
\] (416)
I’ll assume without proof that the inverse of \( U \) exists. It is easy to differentiate the \( L(t) \) solution to see that \( L(t) \) indeed satisfies the Lax equation;
\[
\begin{align*}
\dot{L} &= \dot{UL}(0)U^{-1} - ULU^{-1}\dot{U}U^{-1} \\
&= -MUL(0)U^{-1} + UL(0)U^{-1}M \\
&= -ML(t) + L(t)M
\end{align*}
\] (417)
The formal solution to the IVP for \( U \) is given by
\[
U(t) = e^{-Mt}.
\] (418)
All we need to know is what exponential of a matrix means! The easiest way to the answer is to Taylor expand the exponential
\[
e^{Mt} = 1 + Mt + \frac{M^2t^2}{2!} + \ldots
\] (419)
It is easy to check that
\[
M^n = \left(\frac{\omega}{2}\right)^n \begin{cases} 
(-1)^{n-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & n \text{ odd} \\
(-1)^{n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & n \text{ even}
\end{cases}
\] (420)
so resumming the series in each element we have
\[
\begin{align*}
e^{Mt} &= \begin{pmatrix} \cos \frac{\omega t}{2} & -\sin \frac{\omega t}{2} \\ \sin \frac{\omega t}{2} & \cos \frac{\omega t}{2} \end{pmatrix} \\
\end{align*}
\] (421)
giving
\[
L = \begin{pmatrix} \cos \frac{\omega t}{2} & -\sin \frac{\omega t}{2} \\ \sin \frac{\omega t}{2} & \cos \frac{\omega t}{2} \end{pmatrix} \begin{pmatrix} p_0 & \omega q_0 \\ \omega q_0 & -p_0 \end{pmatrix} \begin{pmatrix} \cos \frac{\omega t}{2} & \sin \frac{\omega t}{2} \\ -\sin \frac{\omega t}{2} & \cos \frac{\omega t}{2} \end{pmatrix}
\begin{pmatrix} p(t) & \omega q(t) \\ \omega q(t) & -p(t) \end{pmatrix} = \begin{pmatrix} p_0 \cos \omega t - \omega q_0 \sin \omega t & p_0 \sin \omega t + \omega q_0 \cos \omega t \\ \omega q_0 \sin \omega t - \omega q_0 \cos \omega t & -p_0 \cos \omega t + \omega q_0 \sin \omega t \end{pmatrix}
\]
from which we can read off
\[ q(t) = \frac{p_0}{\omega} \sin \omega t + q_0 \cos \omega t \quad (422) \]
the standard solution for the SHO. The fact that there is a Lax equation tells us there must be a single constant of the motion which in this case is the Hamiltonian itself; it can be written
\[ H = \frac{1}{4} \text{Tr}(L^2). \quad (423) \]
It is easy to see that this should be a constant because by this definition
\[ \dot{H} = \frac{1}{2} \text{Tr}(L \dot{L}) = \frac{1}{2} \text{Tr}(L(LM - ML)) = 0 \quad (424) \]
by the cyclic properties of the trace.

### 18.2 The Toda Lattice

The SHO is a rather trivial example; in particular we can solve it extremely easily using usual techniques anyway. A less trivial example of a matrix system with a Lax pair solution is the Toda lattice, which describes the dynamics of \( n \) "particles" (labelled \( k = 1 \ldots n \)) on a line, interacting with the closest set of neighbours. The Hamiltonian is given by
\[ H(p, q) = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n-1} g_k^2 e^{2(q_k - q_{k+1})} \quad (425) \]
The system can be thought of as similar to the suspended oscillators on a line we used for the Sine-Gordon example, but with an exponential potential whose coupling \( g_k \) depends on the position, \( k \), so in this respect it is actually quite general.

Hamilton's equations (the equations of motion) are quite simple
\[
\dot{q}_k = \frac{\partial H}{\partial p_k} = p_k \\
\dot{p}_k = -\frac{\partial H}{\partial q_k} = 2\left(g_{k-1}^2 e^{2(q_{k-1} - q_k)} - g_k^2 e^{2(q_k - q_{k+1})}\right) \quad (426)
\]
A bit of matrix manipulation shows that these equations are realized by Lax equation with the following Lax pair;
\[
L = \begin{pmatrix}
p_1 & g_1 e^{2(q_1 - q_2)} & g_2 e^{2(q_2 - q_3)} & \cdots \\
g_1 & p_2 & g_2 & \cdots \\
g_2 & \ddots & \ddots & \ddots \\
g_{n-1} & \ddots & \ddots & g_{n-1} e^{2(q_{n-1} - q_n)} \\
g_{n-1} & \ddots & \ddots & g_{n-1} & p_n
\end{pmatrix}
\]
\[
M = -2 \begin{pmatrix}
0 & g_1 e^{2(q_1-\eta_2)} \\
0 & 0 & g_2 e^{2(q_2-\eta_3)} \\
& & \ddots & \ddots \\
& & & g_{n-1} e^{2(q_{n-1}-\eta_n)} \\
0 & 0
\end{pmatrix}
\] (427)

The conserved quantities follow as for the SHO by noting that
\[
Q_k = \frac{1}{k} \text{Tr}(L^k)
\] (428)
is independent of time since by the Lax equation
\[
\frac{d}{dt} Q_k = \text{Tr}(L^{-1} \dot{L}) = \text{Tr}(L^{-1}(LM - ML)) = 0
\] (429)
The solution is by no means as obvious as the SHO one; it was provided by Olshanetsky and Perelomov in 1979.

We wish to find the solution (i.e. the time dependent functions of \(q_k(t)\)) to \(L_t = [L, M]\) with \(L, M\) defined as above. Begin by noting that every Hermitian matrix \((Y = Y^\dagger)\) has a unique decomposition
\[
Y = ZXZ^\dagger
\] (430)
where \(X\) is the diagonal matrix of real eigenvalues. We will eventually have
\[
X = \begin{pmatrix}
e^{2q_1} & \\
e^{2q_2} & \\
& \ddots \\
e^{2q_n}
\end{pmatrix}
\] (431)
and will take \(Y(0) = X(0)\) is the matrix of initial values of \(X\), and \(Z\) encodes how they mix up on time evolution with \(Z(0) = 1\). Differentiating,
\[
\dot{Y} = \dot{Z}XZ^\dagger + Z\dot{X}Z^\dagger + ZX\dot{Z}^\dagger
\]
and
\[
\dot{Y}Y^{-1} = (\dot{Z}XZ^\dagger + Z\dot{X}Z^\dagger + ZX\dot{Z}^\dagger)(ZXZ^\dagger)^{-1}
\]
\[
= (\dot{Z}XZ^\dagger + Z\dot{X}Z^\dagger + ZX\dot{Z}^\dagger)(Z^\dagger)^{-1}X^{-1}Z^{-1}
\]
\[
= \dot{Z}Z^{-1} + Z\dot{X}X^{-1}Z^{-1} + ZX\dot{Z}^\dagger(Z^\dagger)^{-1}X^{-1}Z^{-1}
\]
\[
= Z \{ Z^{-1}\dot{Z} + \dot{X}X^{-1} + X\dot{Z}^\dagger(Z^\dagger)^{-1}X^{-1} \} Z^{-1}
\] (432)
Now suppose that \(Z\) is the solution to \(\dot{Z} = -ZM\) (and also by taking the hermitian conjugate \(\dot{Z}^\dagger = -M^\dagger Z^\dagger\)) and substitute this into the above equation
\[
\dot{Y}Y^{-1} = Z \{ -M + \dot{X}X^{-1} - XM^\dagger Z^{-1} \} Z^{-1}.
\] (433)
It is straightforward to verify that
\[ \dot{YY}^{-1} = 2ZLZ^{-1}. \]  \hspace{1cm} (434)

Now if we can determine \( Y \) as well then we should be able to get \( X \) which will give us \( q_k(t) \). Differentiating the above by \( t \) we find
\[
\frac{d}{dt}(\dot{YY}^{-1}) = 2(\dot{Z}LZ^{-1} + Z\dot{L}Z^{-1} + ZL\dot{Z}^{-1})
= 2Z\{Z^{-1}\dot{Z}L + \dot{L} + L\dot{Z}^{-1}Z\}Z^{-1}
= 2Z\{-ML + \dot{L} + LM\}Z^{-1} = 0 \hspace{1cm} (435)
\]

So then we must have
\[ \dot{YY}^{-1} = D \hspace{1cm} (436) \]

where \( D \) is a constant \( n \times n \) matrix, and
\[ \dot{Y} = DY \]

for some \( D \). This equation can be integrated as usual (normally we would have \( \log(\frac{y}{b}) = d \) where \( b \) is some constant, but now we need simply note that \( Y \) is hermitian by definition) to give
\[ Y = Be^{Dt}B^\dagger \hspace{1cm} (437) \]

where \( Y(0) = BB^\dagger \). The definition \( Y(0) = X(0) \) means that the \( B \) constant of integration is
\[ B = X(0)^{1/2} \hspace{1cm} (438) \]

i.e. the matrix of \( e^{q_k(0)} \) on the diagonals. Also \( \dot{Y}(0)Y(0)^{-1} = 2L(0) \) gives the \( D \) constant of integration as
\[ D = B^{-1}L(0)B \hspace{1cm} (439) \]

This determines \( y(t) \). So finally
\[ Y(t) = ZXZ^\dagger = Be^{Dt}B^\dagger \]
\[ \implies X(t) = Z^{-1}Be^{Dt}B^\dagger(Z^\dagger)^{-1}. \hspace{1cm} (440) \]
Appendix A: Numerically solving p.d.e.s

The mathematica program that produced all the solution plots is as shown below. This can easily be modified to solve Sine Gordon, the dispersive equation etc. There are a number of example starting waves, only one of which corresponds to the actual KdV soliton.

Clear["\"\""]
kdv[(a_, t_, x_) := D[p[t, x], t] + a p[t, x] D[p[t, x], x] + D[p[t, x], (x, 3)] = 0;
cell = 40;
s[0] = 3 c0 / a (Sech[Sqrt[c0]/2 (x - x0 - c0 t)]^2; s[0] = 0.53 c0 / a (Exp[- Sqrt[c0]/2 (x - x0 - c0 t)]^2);
phi = s[1, 3, 25, 0, x] + 0 s[1, 3/2, 35, 0, x] + s[1, 3, 25, cell, 0, x] + 0 s[1, 3/2, 35, cell, 0, x] + s[1, 3, 25, cell, 0, x] + 0 s[1, 3/2, 35, cell, 0, x];
phi = phi / x = 0. = (phi / x = cell)
ss = NDSolve[kdv[(1, t, x] && p[0, x] = phi && p[t, 0] = p[t, cell], p, (t, 0, 40), (x, 0, cell), Method -> StiffnessSwitching]; // Timing
<< Graphics 'Animation'
Table[Plot[Evaluate[p[1/4, x] /. First[ss]], (x, 0, cell), 
PlotRange = [-2, 15], ImageSize = 600], {i, 0, 20}];
Appendix B: Sketch of Lagrangian Mechanics

B.1 Variational principle

The principle of variational mechanics is that $u(t)$ minimises an action $S$ (here if it were a particle in motion it would be more conventional to call the position $x(t)$ but this would be confusing here, since later we will define a $u$ as a field theory over $x,t$):

\[ S[u] = \int_{t_1}^{t_2} L(u, u_t) \, dt \]  \hfill (441)

In general $L$ can also depend explicitly on $t$ but we won’t consider this here. $L$ is called the \textbf{Lagrangian}, and is given by

\[ L = T - V \]  \hfill (442)

$S$ is called the \textbf{action}. The boundary conditions are that $(u(t_1)$ and $u(t_2)$ are fixed.) For a particle in falling under gravity

\[ L = \frac{m}{2} u_t^2 - mg u \]  \hfill (443)

To find the solution, if the functional $S[u]$ minimizes the action, it should be a variational minimum: let $u \rightarrow u + \delta u$ then if $S \rightarrow S + \delta S$ we have

\[
\delta S = 0 = \int L(u + \delta u, u_t + \delta u_t) - L(u, u_t) \, dt \\
= \int L(u, u_t) + \frac{\partial L}{\partial u} \delta u + \frac{\partial L}{\partial u_t} \delta u_t - L(u, u_t) \, dt \\
= \int \left( \frac{\partial L}{\partial u} - \frac{d}{dt} \frac{\partial L}{\partial u_t} \right) \delta u \, dt \]  \hfill (444)

where to get the last line I integrated by parts. This should be true for any $\delta u$ so that we get the Euler-Lagrange eqn’s

\[ \frac{\partial L}{\partial u} - \frac{d}{dt} \frac{\partial L}{\partial u_t} = 0 \quad \forall t \]  \hfill (445)

For the example $\frac{\partial L}{\partial u} = mu_t$ and $\frac{\partial L}{\partial u_t} = -mg$ so $u_{tt} = -g$.

More generally for a non-relativistic particle moving in a potential $V(u)$ we get

\[ L = \frac{m}{2} u_t^2 - V(u) \]  \hfill (446)

and hence

\[ mu_{tt} = -V'(u) \]  \hfill (447)

(The rate of gain of K.E. = rate of loss of P.E.) The variational principle replaces equation of motion (Newton’s law in this case).

\textbf{Note} it’s $T - V$. The best “path” $u(t)$ tries to share the energy equally between $T$ and $V$ as it goes
B.2 Variational principle in field theory

For our equations we have e.g. \( \theta(x,t) \) and \( u(x,t) \) so that we have a quantity (field) defined over \( x,t \). Assume \( x \in \mathbb{R} \) and \( u(\pm \infty) = \text{constant} \). Certainly sufficient to get solitons plus much more. Now define the Lagrange density, \( \mathcal{L} \) such that

\[
S[u] = \int_{t_1}^{t_2} \int_{\infty}^{\infty} \mathcal{L}(u, u_t, u_x) \, dx \, dt \tag{448}
\]

let \( u \to u + \delta u \) then if \( S \to S + \delta S \) we have

\[
\delta S = 0 = \int \int \mathcal{L}(u + \delta u, u_t + \delta u_t, u_x + \delta u_x) - \mathcal{L}(u, u_t, u_x) \, dt \, dx
\]

\[
= \int \int \left( \frac{\partial \mathcal{L}}{\partial u} \delta u + \frac{\partial \mathcal{L}}{\partial u_t} \delta u_t + \frac{\partial \mathcal{L}}{\partial u_x} \delta u_x \right) \, dt \, dx
\]

\[
= \int \int \left( \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x} \right) \delta u \, dt \, dx. \tag{449}
\]

Here we can get the last line by integrating by parts in the relevant integrand, \( x \) or \( t \). (Note we are neglecting surface terms here which are important for conserved currents. In more general cases we would use Gauss’s theorem in 2 dimensions to do the final step.) The E-L equations become

\[
\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x} = 0 \tag{450}
\]

B.3 Example - the Sine-Gordon equation

To get the SG equation from the variational principle define

\[
\mathcal{T} = \frac{ml^2}{2} \theta^2_t
\]

\[
\mathcal{V} = mg(1 - \cos \theta) + \frac{ml^2}{2} \theta^2_x \tag{451}
\]

Recall that \( \theta_{xx} \) came from the linear force law of stretched elastic. Here \( \theta^2_x \) is therefore the contribution to the potential energy per unit length stored in the elastic, above the “vacuum” energy. (Go back to the swinging pendulum picture and convince yourself this is the case.)

Now take

\[
\mathcal{L} = \mathcal{T} - \mathcal{V}
\]

\[
= \frac{ml^2}{2} \theta^2_t - \frac{ml^2}{2} \theta^2_x - mg(1 - \cos \theta) \tag{452}
\]

The E-L equations \( \delta S = 0 \) then require substituting

\[
\frac{\partial \mathcal{L}}{\partial \theta_t} = ml^2 \theta_t
\]
\[
\frac{\partial L}{\partial \theta_x} = -ml^2 \theta_x \\
\frac{\partial L}{\partial \theta} = -mgl \sin \theta
\]
giving
\[
ml^2 \theta_{tt} - ml^2 \theta_{xx} + mgl \sin \theta = 0 \tag{453}
\]

**B.4 One dimensional relativistic field theory**

We can define a relativistic field theory with a general potential \( P(u) \) by analogy with the above. Define
\[
\mathcal{T} = \frac{1}{2}u^2_t \\
\mathcal{V} = \frac{1}{2}u^2_x + P(u) \tag{454}
\]
and hence
\[
\mathcal{L} = u^2_t - u^2_x - P(u) \tag{455}
\]
The Euler-Lagrange equation 450 now give
\[
u_{tt} - u_{xx} + P' = 0 \tag{456}
\]
Note that a travelling solution of the form \( u = f(x - vt) \) yields the equation
\[
f'' = \gamma^2 P' \tag{457}
\]
Integrating once gives
\[
\frac{1}{2}(f')^2 - \gamma^2 P = A \tag{458}
\]
where \( A \) is a constant of integration. Again we see that any travelling wave solution will be equivalent to a particle rolling in the inverse potential. From here we could integrate again to find an implicit solution for \( f \).

Note that we can also define the total energy density as we did for Bogomolny;
\[
\mathcal{E} = \mathcal{T} + \mathcal{V} \tag{459}
\]

- In some cases (e.g. QM) the lagrangian is more fundamental (there is no equation of “motion” of quantum particles)
- It’s often quicker and more intuitive - e.g. Bogomolny
- Important later for conservation laws
Appendix C: Lightcone coordinates

Also called characteristic or Null coordinates. In many relativistic cases it is useful to define coordinates where light rays travel along the axes. Recall that the worldlines of light rays are on the lightcone given by $x = \pm t$ (where I’ll set the speed of light = 1), as in the figure below;

Coordinates of any point $(x, t)$ relative to the $x_\pm$ axes are given by a rotation through $45^\circ$;

$$x_\pm = \frac{1}{2} (x \pm t)$$

(By convention there is a factor $\sqrt{2}$ with respect to a pure rotation which is done by the matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$).

Then differentiation can be done by

\[
\begin{align*}
\frac{\partial}{\partial x_+} &= \frac{\partial t}{\partial x_+} \frac{\partial}{\partial t} + \frac{\partial x}{\partial x_+} \frac{\partial}{\partial x} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x_-} &= \frac{\partial t}{\partial x_-} \frac{\partial}{\partial t} + \frac{\partial x}{\partial x_-} \frac{\partial}{\partial x} = -\frac{\partial}{\partial t} + \frac{\partial}{\partial x}
\end{align*}
\]

where I used

\[
\begin{align*}
t &= x_+ - x_- \\
x &= x_+ + x_-
\end{align*}
\]

You can check that

\[
\frac{\partial x_\pm}{\partial x_\pm} = 1; \quad \frac{\partial x_\pm}{\partial x_\mp} = 0,
\]

and also that

\[
\frac{\partial}{\partial x_+ \partial x_-} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}
\]

so that

\[
u_{xx} - u_{tt} = u_{+-}
\]

where I’ll use subscripts $++$ for the differentiation by $x_+$ and $x_-$ respectively.
Appendix D: Commutators

Commutators are the Quantum Mechanical equivalent of Poisson brackets. They act on operators and are defined as follows

\[ [A, B] = AB - BA \]  \hspace{1cm} (465)

where both \( A \) and \( B \) are operators, e.g. matrices, or differential operators such as \( \partial_x \). If \( [A, B] = 0 \) then the two operators are said to commute, and their ordering is irrelevant. If two operators do not commute then the ordering is important. In other words, if \( [A, B] = C \), then by definition

\[ AB = BA + C \]  \hspace{1cm} (466)

Using the above definition the following rules are useful in expanding and evaluating complicated expressions;

\[
\begin{align*}
[A, A] &= 0 \quad \forall A \\
[A, B + C] &= [A, B] + [A, C] \\
[A + B, C + D] &= [A, C] + [A, D] + [B, C] + [B, D] \\
\end{align*}
\]  \hspace{1cm} (467)

These rules are not hard to derive. For example the last one follows by writing out explicitly

\[
\begin{align*}
[A, BC] &= ABC - BCA \\
&= ABC - BAC + BAC - BCA \\
&= (AB - BA)C + B(AC - CA) \\
\end{align*}
\]

Be careful to maintain the ordering. Calling \( [A, B] = \phi \) then the above rules clearly imply

\[ [A^n, B] = \phi A^{n-1} + A\phi A^{n-1} + \ldots + A^{n-1}\phi \]

If also \( \phi, A \) = 0 then it follows that

\[ [f(A), B] = \phi f'(A) \]

where \( f(A) \) is any differentiable function of \( A \).

From these rules more specific ones follow. For example if \( [A, B] = 0 \) then

\[ [A, f(B)C] = f(B)[A, C] \]

where \( f(B) \) is any function of \( B \). A final identity (which is useful in QM although probably not for solitons) is the famous Baker-Campbell-Hausdorff theorem;

\[ e^{A}e^{B} = e^{B}e^{A}e^{[A,B]} \].

117
This is easy to show to second order in $A, B$;

$$
(1 + A + \frac{A^2}{2} + \ldots)(1 + B + \frac{B^2}{2} + \ldots) = (1 + B + \frac{B^2}{2} + \ldots)(1 + A + \frac{A^2}{2} + \ldots)(1 + [A, B] + \ldots)
$$

$$
1 + A + B + AB + \frac{A^2}{2} + \frac{B^2}{2} = 1 + B + A + \frac{A^2}{2} + \frac{B^2}{2} + BA + [A, B]
$$