Anomalies

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1 Noether's theorem (revisited)

1.1 Overview: Why anomalies?

Although "anomalies" sounds like it might be a somewhat peripheral aspect of quantum field theory, it turns out that their study is central to the proper understanding of symmetries and conservation laws in QFT. Anomalies also find uses in a huge variety of applications, for example;

- Checking model consistency by cancellation of gauge anomalies in e.g. the Standard Model $\implies$ charge quantization for example
- 't Hooft global anomaly matching: how to tell when one theory is a weakly coupled effective description of a different strongly coupled theory
- PCAC and processes such as pion decay
- Instantons and the structure of the QCD vacuum
- The strong CP problem and the Peccei-Quinn mechanism
- B+L violation and baryogenesis
- Scaling anomalies $\implies$ renormalization group flow
- Connected with this, the a-theorem and studying RG flow in strongly coupled theories
- Supersymmetry and metastability

There are many other areas where their study has been crucial, for example anomaly cancellation established $E_8 \times E_8$ heterotic string theory as a viable and phenomenologically interesting theory. Clearly in 8 lectures I will be able to cover only a subset of these uses. However my emphasis will be on their importance in understanding gauge theory. At certain points these notes will deviate into long ramblings about the precise meaning of something or other; that is a sure sign that those points are more difficult to understand (the ramblings indicating that I spent some time reconsidering those points myself in writing these notes) and require some thought.

1.2 Symmetries and Noether's theorem

First in this course I'd like to return to Noether's theorem. Although you will have seen this before, as the subject of anomalies is really to do with conservation laws, it is vital to get this part of the story straight. In particular

- Symmetries $\implies$ conservation laws
- Anomalies $\implies$ breaking of symmetries by quantum effects which either gives real physical effects in the case of global symmetries (such as pion decay) or consistency conditions in the case of local gauge symmetries

Noether's Theorem: Any continuous global symmetry of S implies a conserved current $\partial_\mu J^\mu = 0$

Note the word "global". Actually Noether's theorem says nothing about local symmetries. Indeed local currents are often not even gauge invariant and therefore not observables. Therefore a large part of the anomalies story involves understanding the difference between local and global currents.

Corollary: The *charge*, $Q = \int d^3x J^0$, is conserved.

Proof of Corollary: In a volume V, bounding surface S, we have $\frac{dQ}{dt} = \int_V d^3\mathbf{x}\partial_t J^0 = -\int_V d^3\mathbf{x}\partial_i J^i = -\int_S d\sigma.\mathbf{J}$ $\square$

The last step follows from the divergence theorem and tells us that the rate of change of the total charge is equal to the total current flowing through S.

Proof: Let $S = \int d^4x \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$, where φ_i stands for generic fields of arbitrary spin. Consider the most general infinitessimal transformation possible under which

$$\begin{aligned} x^\mu \to (x^\mu)' &= x^\mu + X^\mu_\alpha \epsilon^\alpha \\ \varphi_i \to \varphi_i'(x') &= \varphi_i + \Phi^i_\alpha \epsilon^\alpha \end{aligned} \tag{1}$$

where ϵ^α is a set of infinitessimal parameters and $\Phi_\alpha, X^\mu_\alpha$ are functions of x. The total variation in φ can be split into that due to the shift in x, and the internal variation of the field itself $\delta_0\varphi$:

$$\begin{aligned} \varphi_i'(x') &= \varphi_i(x') + \delta_0\varphi(x') \\ &= \varphi_i(x) + \delta x^\mu \partial_\mu \varphi_i + \delta_0 \varphi_i \\ &= \varphi_i(x) + \epsilon^\alpha X^\mu_\alpha \partial_\mu \varphi_i + \delta_0 \varphi_i \end{aligned} \tag{2}$$

Thus the transformation of the field itself can also be written as

$$\delta_0 \varphi^i = \epsilon^\alpha \left(\Phi^i_\alpha - X^\mu_\alpha \partial_\mu \varphi_i \right). \tag{3}$$

The variation of the action contains contributions from both external changes (through the measure) and internal changes:

$$\delta S = \int \delta(d^4x)\mathcal{L} + \int d^4x \delta\mathcal{L}. \tag{4}$$

The change in the measure comes from the Jacobian

$$\begin{aligned} d^4x' &= \left| \det \left[\frac{\partial x'^\mu}{\partial x^\nu} \right] \right| d^4x \\ &= \left| \det \left[\delta^\mu_\mu + \partial_\nu (X^\mu_\alpha \epsilon^\alpha) \right] \right| d^4x \\ &= \left| 1 + \partial_\mu (X^\mu_\alpha \epsilon^\alpha) \right| d^4x, \end{aligned} \tag{5}$$

using $\det \exp A = \exp \mathrm{Tr} A$ infinitessimally. Hence

$$\delta(d^4x) = \partial_\mu (X^\mu_\alpha \epsilon^\alpha) d^4x. \tag{6}$$

Using (4), the total change in the action is therefore separable into a piece corresponding to the equations of motion, and a surface term,

$$\begin{aligned}\delta S &= \int \partial_\mu (X^\mu_\alpha \epsilon^\alpha)\mathcal{L} d^4x + \int d^4x \delta_0\varphi_i \frac{\partial \mathcal{L}}{\partial \varphi_i} + \partial_\mu(\delta_0\varphi_i)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \\ &= \int d^4x \partial_\mu \left(\epsilon^\alpha J^\mu_\alpha\right) - \int d^4x (X^\mu_\alpha \epsilon^\alpha)\partial_\mu \mathcal{L} + \int d^4x \delta_0 \varphi_i \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right) \end{aligned} \tag{7}$$

where we integrated by parts and where

$$\begin{aligned} J^\mu_\alpha &= X^\mu_\alpha \mathcal{L} + \left(\Phi^i_\alpha - X^\nu_\alpha \partial_\nu \varphi_i\right)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \\ &= X^\nu_\alpha \left(\delta^\mu_\nu \mathcal{L} - \partial_\nu \varphi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\right) + \Phi_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_i)}. \end{aligned} \tag{8}$$

Since by the divergence theorem the first piece of eq.(7) can be recast as a boundary term, it depends only on the value of the fields at the boundary. In particular we are free to set the variation ϵ^α to zero at the boundary (which may be at infinity). The last two pieces must therefore vanish locally and imply the Euler-Lagrange equation, and the absence of explicit dependence on x in the Lagrangian (in the case of external symmetries). Invariance of S then requires that the first piece vanishes. We can therefore conclude that

$$\partial_\mu \left(\epsilon^\alpha J^\mu_\alpha\right) = \partial_\mu K^\mu \tag{9}$$

where K^μ is some function of x that vanishes at the boundary. Therefore the combination

$$\tilde{J}^\mu = \epsilon^\alpha J^\mu_\alpha - K^\mu \tag{10}$$

obeys $\partial_\mu \tilde{J}^\mu = 0$. □

Before we look at some explicit examples, note the presence of K^μ: even if we set it to be zero classically it is generally unprotected from quantum corrections. (In practice it would have to be constructed out of other fields in the theory such as gauge fields.) *This is precisely the source of anomalies.* Thus while one may write a theory that appears to have currents J^μ_α that are conserved (i.e. that obey $\partial_\mu J^\mu_\alpha = 0$ when one uses the classical equations of motion) this is not necessarily the case at the quantum level.

Let me now help you avoid a couple of common annoyances. First note that even if there is a global anomaly (i.e. a non-zero K^μ), Noether's theorem is still satisfied (it is a theorem after all). The theorem simply arises from dividing the equation $\delta S = 0$

between local equations of motion and surface terms, so it is almost a triviality (in the technical sense), so it would be hard to see how it could not be satisfied: anomalies do not somehow "violate" Noether's theorem or anything like that. Secondly, and even more egregiously, one often comes across the statement that a symmetry is "classically obeyed". This statement is almost entirely meaningless – in a quantum world the symmetry is simply never there. What people who use this phrase probably mean to say is that there is an "accidental symmetry": that is something that at the level of the Lagrangian appears to be a symmetry but which is anomalous. This can often happen in for example string model building because certain terms happen to be absent in the Lagrangian due to so-called selection rules. The quantum violation of such symmetries is often loop suppressed when the theory is weakly coupled so it is easy to convince oneself that the symmetry is "almost there". The mistake arises because superficially any quantum theory can be decomposed into two elements: one is the Lagrangian and the other is the sum over histories. The anomaly can be understood as the non-invariance of the measure in the sum over histories, which doesn't give two hoots about accidental symmetries of the Lagrangian. However this non-invariance of the measure can usually be recast as an additional term in the Lagrangian, so the separation into Lagrangian and measure is arbitrary, and there is no sense in which the Lagrangian is somehow more fundmental than the sum over histories. Since (as we shall see) the non-invariance of the measure is a function of gauge fields it is probably most precise to say that anomalous symmetries are symmetries that can be broken by non-trivial gauge field configurations.

Indeed anomalies can come to determine the physics of the entire system. One example we shall discuss later is the violation of $B+L$ in the SM. This symmetry is anomalous. However non-trivial K^μ involve configurations of electroweak gauge fields called sphalerons. As electroweak symmetry is broken these lumps are immensely costly (in energy and entropy terms) to produce and therefore in collider experiments there is no $B + L$ violation. In the early Universe however at temperature above the electroweak phase transition, one expects rapid $B + L$ violation to have taken place. Many ideas for baryogenesis (for example leptogenesis) involve the transmission of baryon number through $B + L$ violating effects.

1.3 Examples

Scalar field with $U(1)$ symmetry

First some examples with no gauge fields in which (since there are no other field in the theory) we can safely set $K^\mu = 0$.

$$\mathcal{L} = |\partial\varphi|^2 - V(|\varphi|^2). \tag{11}$$

$\mathcal{L}$ is invariant under

$$\varphi \to e^{i\alpha}\varphi \implies \delta\varphi = i\alpha\varphi \text{ infinitessimally.} \tag{12}$$

Hence using eq(8) we have

$$\begin{aligned} \tilde{J}^\mu &= i\alpha\varphi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - i\alpha\varphi^*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)} \\ &= (\varphi\partial^\mu\phi^* - \varphi^*\partial^\mu\varphi)\,. \end{aligned} \tag{13}$$

Hence $J^\mu = i\,(\varphi\partial^\mu\phi^* - \varphi^*\partial^\mu\varphi)$ is conserved. Note that we only see this when we use the equations of motion ($\partial^2\varphi = -\frac{\partial V}{\partial\varphi^*} = -\varphi V'$) as is obviously going to be the case from the proof of Noether's theorem. i.e.

$$\begin{aligned} \partial.J &= i\left(\varphi\partial^2\phi^* - \varphi^*\partial^2\varphi\right) \\ &= -i\left(\varphi\phi^*V' - \varphi^*\varphi V'\right) = 0. \end{aligned} \tag{14}$$

Fermion field ψ_i with isospin symmetry

In this case we have two component ψ_i labelling proton and neutron say. Then

$$\mathcal{L} = i\bar{\psi}_i\gamma.\partial\psi_i. \tag{15}$$

Defining the global transformation $U_{ij} = e^{\theta^a\tau^a}$ where τ are the Pauli matrices, $\mathcal{L}$ is invariant under

$$\psi \;\to\; U^{-1}\psi \implies \delta\psi_i = -i\theta^a\tau^a_{ij}\psi_j \tag{16}$$

$$\bar{\psi} \;\to\; \bar{\psi}U \implies \delta\bar{\psi}_i = i\theta^a\bar{\psi}_j\tau^a_{ji} \text{ infinitessimally.} \tag{17}$$

Hence using eq(8) we have

$$\begin{aligned} \tilde{J}^{\mu} &= -i\theta^a \tau^a_{ij} \psi_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \\ &= \theta^a \bar{\psi}_i \gamma^\mu \tau^a_{ij} \psi_j. \end{aligned} \tag{18}$$

Hence the three currents $J^\mu_a = \bar{\psi}_i \gamma^\mu \tau^a_{ij} \psi_j$ are conserved. Again we need to use the equations of motion ($\partial\psi = \partial\bar{\psi} = 0$) to see this.

Example with external symmetry

Consider the shift symmetry

$$x^\mu \to x^\nu + a^\mu. \tag{19}$$

Thus in the master formula of eq(8) we have $X^\nu = a^\nu$ simply and $\Phi_\alpha = 0$, and

$$\begin{aligned} J^\nu &= a^\mu \left(\delta^\nu_\mu \mathcal{L} - \partial_\nu \varphi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \right) \\ &= a^\mu T^\nu_\mu, \end{aligned} \tag{20}$$

where $T_{\mu\nu}$ is the energy momentum tensor. Note that the conserved charges here are T^0_0 and T^0_i the energy momentum.

Axial symmetry

Consider a single fermion ψ with

$$\mathcal{L} = i\bar{\psi}\gamma.\partial\psi. \tag{21}$$

Defining the global transformation $U = e^{i\alpha\gamma_5}$, $\mathcal{L}$ is invariant under

$$\psi \quad \to \quad U\psi \implies \delta\psi = i\alpha\gamma_5\psi \tag{22}$$

$$\bar{\psi} \quad \to \quad \bar{\psi}U \implies \delta\bar{\psi}_i = i\alpha\bar{\psi}\gamma_5 \text{ infinitessimally.} \tag{23}$$

This leaves the action invariant because $[\gamma^\mu, \gamma_5]_+ = 0$ (I will use a notation where anti-commutation is denoted $[A, B]_+$) so

$$\mathcal{L} \to \mathcal{L} - \alpha\bar{\psi}\gamma\gamma_5.\partial\psi - \bar{\psi}\gamma_5\gamma.\partial(\alpha\psi) = 0. \tag{24}$$

Hence using eq(8) we have

$$\begin{aligned}\tilde{J}^{\mu} &= i\bar{\psi}\gamma(i\alpha\gamma_5\psi) \\ &= -\alpha\bar{\psi}\gamma\gamma_5\psi. \end{aligned} \tag{25}$$

(You may wonder, if $\mathcal{L}$ is invariant by itself, why do we need to integrate by parts to get the Noether current? The point is that both ψ and $\bar{\psi}$ have equations of motion. The Lagrangian in this form gives the ψ equation of motion from which we can of course get the $\bar{\psi}$ one by complex conjugation. Of course this should be consistent with the $\bar{\psi}$ equation of motion when we integrate by parts.) Hence the axial current $J_5^{\mu} = \bar{\psi}\gamma\gamma_5\psi$ is conserved, which again we see using the equations of motion ($\partial\psi = \partial\bar{\psi} = 0$).

2 Charge and Current Algebras

It is important to realize that the charges *are* the generators of global symmetry transformations. That is

$$\delta_{\epsilon}\varphi_i = i\epsilon[Q, \varphi_i]. \tag{26}$$

One familiar example is time translation: in QM for any time dependent operators $A(t)$ we have

$$\frac{dA}{dt} = i[H, A] + \frac{\partial A}{\partial t}. \tag{27}$$

As we just saw, if the operators do not explicitly depend on time, then $H = \int T_0^0 d^3x$ is the conserved charge corresponding to invariance of the action under $x^0 \to x^0 + \delta x^0$.

As such the charges generally have to satisfy the same algebras as the generators – in fact it is only because of this that the symmetry has any useful physical meaning. In particular it is the charges which are the physical observables that participate in interactions rather than gauge fields for example. Indeed (as an aside) one can carry out the following exercise. Suppose we decide to gauge the conserved U(1) symmetry above but without adding dynamical gauge fields (i.e. no Yang-Mills term). Then we would still include a gauge field inside a covariant derivative but this is understood as an auxilliary field. Since there is no Y-M term and no derivatives for it we can simply eliminate the gauge field with its equations of motion. If one does this one finds that the U(1) gauge field A^{μ} is identified with the U(1) current which is of course just a composite of scalars. Quantization proceeds as usual, but when one comes to the BRST quantization conditions one finds that the current develops a vector pole (when the U(1) symmetry is unbroken). This pole corresponds precisely to the massless photon. In other words the photon is in this sense a composite with no more or less physical meaning than the conserved current! Of course doing any calculations in this highly non-linear environment would be insanely difficult, nevertheless it illustrates the huge reduncancy of the gauge theory and the fact that the current is the crucial physical observable. For more details of this story consult Kugo and Townsend's original paper on the CP^{N-1} model.

Let us first prove eq(26) for internal symmetries:

Proof that $Q \implies \delta\varphi_i$: The charge corresponding to $\tilde{J}^\mu$ is

$$\tilde{Q}(t) = \int d^3\mathbf{x}\tilde{J}^\mu = \int d^3\mathbf{x}\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_i)}\delta_\epsilon\varphi_i. \tag{28}$$

The quantity

$$\Pi_i(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi_i)} \tag{29}$$

is by definition the momentum conjugate to φ_i; that is in order to quantize such a system of fields we begin by defining the equal time commutation relation

$$[\Pi_i(t,\mathbf{y}),\varphi_j(t,\mathbf{x})]_\pm = -i\delta_{ij}\delta^3(\mathbf{y}-\mathbf{x}), \tag{30}$$

where again $\pm$ means commutator or anticommutator. As a sanity check, for fermions $\Pi_i = -i\bar{\psi}\gamma_0$ (with a minus sign for anticommuting fermions) and we recover $[\psi_i^\dagger(t,\mathbf{y}),\psi_j(t,\mathbf{x})]_+ = \delta_{ij}\delta^3(\mathbf{y}-\mathbf{x})$ as we should. Now form

Multiply both sides by Integrating both sides over $\mathbf{y}$ gives

$$[\tilde{Q}(t),\varphi_j(t,\mathbf{x})]_\pm = \int d^3\mathbf{y}\,[\Pi_i(t,\mathbf{y}),\varphi_j(t,\mathbf{x})]\,\delta_\epsilon\varphi_i(t,\mathbf{y}) \tag{31}$$

since $[\delta\varphi_i,\varphi_j]_\pm = 0$ by assumption. Hence

$$[\tilde{Q}(t),\varphi_j(t,\mathbf{x})]_\pm = -i\delta_{ij}\int d^3\mathbf{y}\delta^3(\mathbf{y}-\mathbf{x})\delta\varphi_i(t,\mathbf{y}) = -i\delta_\epsilon\varphi_j(t,\mathbf{x}). \tag{32}$$

The unit charge operator Q is then given by $\epsilon Q = \tilde{Q}$ and the result follows.□

Now consider the charge algebra. As an example consider the isospin symmetry (or indeed any non-Abelian global symmetry with generators denoted τ^a). The generators obey a symmetry given by

$$[T_a, T_b] = iC^c_{ab}T_c \tag{33}$$

where C^c_{ab} are the structure constants. The currents are given by

$$J^\mu_a = \bar{\psi}\gamma^\mu T_a \psi \tag{34}$$

and hence

$$\begin{aligned} Q_a &= \int d^3\mathbf{x}\bar{\psi}\gamma^0 T_a \psi \\ &= \int d^3\mathbf{x}\psi^\dagger T_a \psi. \end{aligned} \tag{35}$$

The claim is that the Q_a satisfy the identical algebra to the T_a. To show this we form the commutator at equal time (so we will not repeat the t explicitly) and then perform a fairly tedious set of manipulations as follows:

$$\begin{aligned}
[Q_a, Q_b] &= \int d^3\mathbf{x} \int d^3\mathbf{y} [\psi^\dagger(\mathbf{x}) T_a \psi(\mathbf{x}), \psi^\dagger(\mathbf{y}) T_b \psi(\mathbf{y})] \\
&= \int d^3\mathbf{x} \int d^3\mathbf{y} T_{aij} T_{bkl} (\psi^\dagger_i(\mathbf{x})\psi_j(\mathbf{x})\psi^\dagger_k(\mathbf{y})\psi_l(\mathbf{y}) - \psi^\dagger_k(\mathbf{y})\psi_l(\mathbf{y})\psi^\dagger_i(\mathbf{x})\psi_j(\mathbf{x})) \\
&= \int d^3\mathbf{x} \int d^3\mathbf{y} T_{aij} T_{bkl} \left(\psi^\dagger_i(\mathbf{x}) \left\{\delta_{kj}\delta^3(\mathbf{y}-\mathbf{x}) - \psi^\dagger_k(\mathbf{y})\psi_j(\mathbf{x})\right\} \psi_l(\mathbf{y}) - \psi^\dagger_k(\mathbf{y})\psi_l(\mathbf{y})\psi^\dagger_i(\mathbf{x})\psi_j(\mathbf{x})\right) \\
&= \int d^3\mathbf{x} \int d^3\mathbf{y} T_{aij} T_{bkl} \left(\psi^\dagger_i(\mathbf{x})\delta_{kj}\delta^3(\mathbf{y}-\mathbf{x})\psi_l(\mathbf{y}) - \psi^\dagger_k(\mathbf{y})\psi^\dagger_i(\mathbf{x})\psi_l(\mathbf{y})\psi_j(\mathbf{x}) - \psi^\dagger_k(\mathbf{y})\psi_l(\mathbf{y})\psi^\dagger_i(\mathbf{x})\psi_j(\mathbf{x})\right) \\
&= \int d^3\mathbf{x} \int d^3\mathbf{y} T_{aij} T_{bkl} \left(\psi^\dagger_i(\mathbf{x})\delta_{kj}\delta^3(\mathbf{y}-\mathbf{x})\psi_l(\mathbf{y}) - \psi^\dagger_k(\mathbf{y}) \left\{\delta_{il}\delta^3(\mathbf{y}-\mathbf{x}) - \psi_l(\mathbf{y})\psi^\dagger_i(\mathbf{x})\right\} \psi_j(\mathbf{x}) - \psi^\dagger_k(\mathbf{y})\psi_l(\mathbf{y})\psi^\dagger_i(\mathbf{x})\psi_j(\mathbf{x})\right) \\
&= \int d^3\mathbf{x} \int d^3\mathbf{y} T_{aij} T_{bkl} \left(\psi^\dagger_i(\mathbf{x})\delta_{kj}\delta^3(\mathbf{y}-\mathbf{x})\psi_l(\mathbf{y}) - \psi^\dagger_k(\mathbf{y})\delta_{il}\delta^3(\mathbf{y}-\mathbf{x})\psi_j(\mathbf{x})\right) \\
&= \int d^3\mathbf{x} \left(\psi^\dagger_i T_{aij} T_{bkl}\delta_{kj}\psi_l - \psi^\dagger_k \delta_{il} T_{aij} T_{bkl}\psi_j\right) \\
&= \int d^3\mathbf{x} \left(\psi^\dagger T_a T_b \psi - \psi^\dagger T_b T_a \psi\right) \\
&= iC^c_{ab} \int d^3\mathbf{x} \left(\psi^\dagger T_c \psi\right) \\
&= iC^c_{ab} Q_c.
\end{aligned}$$

The algebra $[Q_a(t), Q_b(t)] = C^c_{ab}Q_c(t)$ is called the *charge algebra.* An important aspect of charge algebras is that they are satisfied even when the symmetry is explicitly broken.

In that case one would expect the charges $Q_a(t)$ to be functions of time, but the algebra to remain good. For example consider $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ where the second piece is a pure interaction that contains no derivatives. Then the charges are not conserved because they do not commute with the Hamiltonian, but they are built only from the $\mathcal{L}_0$ piece of the lagrangian. Therefore none of the above manipulations changes and the charge algebra is still preserved.

A good example of this phenomenon is the QCD system. Typically when discussing chiral symmetry breaking one defines quarks in $SU(3)$ flavour triplets

$$q_i = \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \tag{36}$$

Because the masses of the quarks are small (where by small we mean much smaller than the scale $\Lambda_{QCD} \sim 1GeV$ of chiral symmetry breaking in the theory). The lagrangian is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with

$$\begin{aligned} \mathcal{L}_0 &= i\bar{q}\gamma.\partial q \\ \mathcal{L}_1 &= m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s. \end{aligned} \tag{37}$$

In the absence of the second piece the global symmetry of the model is $SU(3)_A \times SU(3)_V$ with the A and V meaning axial and vector respectively. The conserved currents for these symmetries (you by now know enough to be able to guess the transformations) are

$$A^a_\mu = \bar{q}\gamma^\mu\gamma_5\frac{\lambda^a}{2}q \ ; \ V^a_\mu = \bar{q}\gamma^\mu\frac{\lambda^a}{2}q,$$

where λ^a are the 3x3 Gell-Mann matrices. One finds the following algebra always holds despite the term that violates the global symmetry:

$$\begin{aligned} [Q_a, Q_b] &= if^c_{ab}Q_c \\ [Q_a, Q_{5b}] &= if^c_{ab}Q_{5c} \\ [Q_{5a}, Q_{5b}] &= if^c_{ab}Q_c. \end{aligned} \tag{38}$$

Note that often one talks in terms of $SU(3)_R \times SU(3)_L$ with

$$Q^a_{L/R} = \frac{1}{2}(Q^a \pm Q^a_5). \tag{39}$$

In this basis it is easy to check that the symmetry is diagonalized

$$[Q^a_{L/R}, Q^b_{L/R}] = if^{ab}_c Q^c_{L/R}, \tag{40}$$

with $[Q^a_L, Q^b_R] = 0$.

3 Symmetry Breaking - summary

The previous discussion brings us briefly to the question of symmetry breaking. The QCD example above is actually a nice example because it has a number of sources of breaking of the global flavour symmetry.

- **Spontaneous symmetry breaking:** when the vacuum does not obey the same symmetry as the Lagrangian. This includes but is not restricted to the Higgs mechanism. In fact the axial part $SU(3)_A$ of the global symmetry of QCD is spontaneously broken by strong coupling effects. By the Goldstone theorem, the resultiing effective theory exhibits 8 Goldstone modes that correspond to the 8 adjoints of $SU(3)_A$. The physics of these states (The pions and K-mesons) can be described by what is known as a non-linear sigma model, a.k.a. the chiral lagrangian.
- **Explicit breaking:** the mass terms for QCD are an example of this. If they are small then symmetry breaking effects will be expressed in terms of these. Indeed the Goldstone modes (i.e. the mesons) are not quite massless precisely because of this effect. However because the masses are much smaller than Λ_{QCD} their masses are correspondingly suppressed.
- Finally there is **anomalous breaking:** i.e. non-zero K^μ. In this case the charge algebra is not preserved.

4 Abelian anomalies

I am going to introduce the anomalies - i.e. the K^μ first, and then later show you how to (in fact two ways how to) calculate them. The first kind of anomaly is abelian (usually axial) anomalies. Consider a gauge theory with massless Dirac fermions ψ and a gauge field $A^\mu_{ij} = A^\mu_a T^a_{ij}$. I will normalize the fields such that the lagrangian is

$$\mathcal{L} = i\bar{\psi}\gamma.(\partial + iA)\psi. \tag{41}$$

For future reference I'll also collect the equations of motion,

$$\begin{aligned} \gamma.(\partial + iA)\psi &= 0 \\ \bar{\psi}\gamma.(\overleftarrow{\partial} - iA) &= 0. \end{aligned} \tag{42}$$

Also for future reference I will define the gauge field strength in the usual way; defining $D^\mu_{ij} = \delta_{ij}\partial^\mu + A^\mu_{ij}$ we have

$$-i[D_\mu, D_\nu] = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu + i[A_\mu, A_\nu].$$

From this definition we see that the antisymmetrized covariant derivative of the field strength vanishes (aka the Bianchi identity)

$$D_{[\mu}F_{\rho\sigma]} = 0. \tag{43}$$

As usual this lagrangian is invariant under local gauge transformations

$$\begin{aligned} \psi &\rightarrow U^{-1}\psi \\ iA_\mu &\rightarrow U^{-1}(iA_\mu + \partial_\mu)U \end{aligned} \tag{44}$$

where $U = e^{i\alpha^a(x)T_a}$. Just to check,

$$\begin{aligned} \mathcal{L} &\rightarrow i\bar{\psi}U\gamma.(U^{-1}\partial + \partial(U^{-1}) + U^{-1}iA + U^{-1}(\partial U)U^{-1})\psi \\ &= i\bar{\psi}\gamma.(\partial + U\partial(U^{-1}) + (\partial U)U^{-1} + iA)\psi \\ &= i\bar{\psi}\gamma.(\partial + \partial(UU^{-1}) + iA)\psi = \mathcal{L} \end{aligned} \tag{45}$$

since $U^{-1}U = 1$.

There is also the axial symmetry. Defining the global transformation $U_5 = e^{i\alpha\gamma_5}$, $\mathcal{L}$ is invariant under

$$\begin{aligned} \psi &\rightarrow U_5\psi \implies \delta\psi = i\alpha\gamma_5\psi \\ \bar{\psi} &\rightarrow \bar{\psi}U_5 \implies \delta\bar{\psi}_i = i\alpha\bar{\psi}\gamma_5 \text{ infinitessimally.} \end{aligned} \quad (46)$$

Note that the relation $\{\gamma_5, \gamma_0\} = 0$ requires D to be even. As before $J_5^\mu = \bar{\psi}\gamma\gamma_5\psi$ seems to be is conserved, and there seems to be an axial charge

$$Q_5 = \int d^3\mathbf{x}\psi^\dagger\gamma_5\psi. \quad (47)$$

Again we see this using the classical equations of motion

$$\begin{aligned} \partial.J_5 &= (\partial\bar{\psi})\gamma\gamma_5\psi + \bar{\psi}\gamma\gamma_5\partial\psi \\ &= i\bar{\psi}\gamma.A\gamma_5\psi - \bar{\psi}\gamma_5(-i\gamma.A)\psi \\ &= i\bar{\psi}\gamma.A\gamma_5\psi - i\bar{\psi}\gamma.A\gamma_5\psi = 0. \end{aligned} \quad (48)$$

where we used $\{\gamma_5, \gamma_0\} = 0$ to move the gamma matrices through each other.

However this equation is wrong!! In QFT Adler Bell and Jackiw showed in 1969 that the derivative depends on background values of the field A^μ:

$$\partial.J_5 = \frac{1}{16\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\mathrm{tr}(F_{\alpha\beta}F_{\gamma\delta}) \quad (49)$$

$$= \frac{1}{4\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\left[A_\beta\partial_\gamma A_\delta + \frac{2}{3}A_\beta A_\gamma A_\delta\right]. \quad (50)$$

Note that the variation in the action is

$$\delta S = \int d^4x\partial_\mu\left[J_5^\mu - K^\mu\right] \quad (51)$$

where

$$K^\mu = \frac{1}{4\pi^2}\varepsilon^{\mu\beta\gamma\delta}\left[A_\beta\partial_\gamma A_\delta + \frac{2}{3}A_\beta A_\gamma A_\delta\right]. \quad (52)$$

This is precisely the extra piece in the derivation of Noether's theorem. It tells us that in the presence of expectation values for the gauge fields, the axial current is no longer preserved. Recall also that the K^μ is a function that is required to vanish at the boundary. On the other hand the non-conservation of current appears to be entirely determined by the configuration of the expectation values of the A_μ fields as we approach the boundary of the volume. Indeed in a static gauge configuration $\dot{Q}_5 = -\int d\sigma.(\mathbf{J}_5 - \mathbf{K})$ by the divergence theorem. Thus the non-conservation of Q_5 can be computed by integrating K^μ over the bounding surface: even though K^μ vanishes there, the integral over a bounding surface of radius r goes as r^2 which means that the sort of gauge fields VEVS that can give interesting non-conservation effects must go as $A^\mu \sim 1/r$ (otherwise we would get zero or infinity upon taking the $r \to \infty$ limit). This smacks of topological properties such as winding number, and indeed there is a close connection. Also note that more generally one would write

$$\partial.J_5 = \frac{1}{16\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\mathrm{tr}(q_5 F_{\alpha\beta}F_{\gamma\delta}), \tag{53}$$

where q_5 is the axial charge of the field. Thus if we add a second fermion there will be two axial global symmetries q_5 and q_5'. There will be a combined "axial" current $J_5^- = J_5 - J_5'$ that is anomaly free because the contributions to $\partial.J_5'$ cancel, while $J_5^+ = J_5 + J_5'$ remains anomalous.

It is tempting to say that the "action is not invariant" because of the anomaly. I think this is misleading. It is rather that, in the quantum theory, just requiring invariance of the classical action is not enough to determine $\partial.J_5$. If one instead considers the full partion function $Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}Ae^{iS}$ invariance under chiral transformations (which is a trivially true) requires $Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}Ae^{iS} = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\mathcal{D}Ae^{iS'} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}Ae^{i(S+\delta S)}$; one finds that as well as the current piece in S', δS contains an extra piece from transforming the measure, and in total $\delta S = \epsilon \int d^4x(\partial.J_5 - \partial.K)$. So we are still requiring $\delta S = 0$, it is just that part of δS comes from the path integral measure. (Saying the "*classical* action is not invariant" is just non-sensical).

5 Non-Abelian (gauge) anomalies

There are anomalies associated with the non-abelian symmetries as well. As the anomalous breaking is due to the presence of gauge fields, it turns out that there is no anomaly associated with *global* non-abelian groups such as isospin. The issue here is really to do with the consistency of gauge non-Abelian groups. Therefore consider a gauge theory with left handed massless Weyl fermions ψ and gauge field $A^\mu = A^\mu_a T^a$, with lagrangian

$$\mathcal{L} = i\bar{\psi}_L \gamma.(\partial + iA)\psi_L, \tag{54}$$

where the L subscript means projection onto the left chirality, so that a Dirac fermion would look like

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{55}$$

in the Weyl representation of γ's. The covariant derivative for the adjoints is

$$D^b_{\mu a} = \delta^b_a \partial_\mu + f^b_{ac} A^c_\mu. \tag{56}$$

and the gauge transformation is

$$\begin{aligned} A^a_\mu &\to A^a_\mu - iD^a_{\mu b}\alpha^b \\ \bar{\psi} &\to \bar{\psi} - i\alpha_a \bar{\psi} T^a. \\ \psi &\to \psi + i\alpha_a T^a \psi. \end{aligned} \tag{57}$$

The lagrangian is clearly invariant under this transformation since

$$\delta\mathcal{L} = \alpha_a \bar{\psi}_L T^a \gamma.D\psi_L - \bar{\psi}_L T^a \gamma.D(\alpha_a \psi_L) + \bar{\psi}_L \gamma.(D_\mu \alpha_a T^a)\psi_L. \tag{58}$$

The current associated with this transformation is similar to the isospin case:

$$J^\mu_a = \bar{\psi}\gamma^\mu T_a \psi. \tag{59}$$

One has to be careful about the physical meaning of such a current though because it is not gauge invariant, hence not directly observable. Although the situation looks similar to the abelian case the physics is quite different. Nevertheless, we can ask how such a current should behave in the classical theory. The easiest way to see this is from the equations of motion of the field to which the current couples, namely A_μ:

$$\begin{aligned} \partial^\nu \frac{\partial \mathcal{L}}{\partial \partial^\nu A^\mu_a} &= \frac{\partial \mathcal{L}}{\partial A^\mu_a} \Longrightarrow \\ (D^\mu F_{\mu\nu})^a &= g^2 \bar{\psi}_L \gamma^\mu T^a \psi_L = g^2 J^a_\nu. \end{aligned} \tag{60}$$

This is the usual non-abelian form of (two of) Maxwell's equations, and then

$$(D.J)^a = 0 \tag{61}$$

follows simply by hitting both sides of the equation of motion with a D^ν and from the (anti)-symmetry of the $\mu\nu$ indices on the LHS.

Again this equation is wrong, as Bardeen (1969), and Gross and Jackiw (1972) showed. The derivative depends on background values of the field A^μ as actually

$$(D.J)^a = \frac{1}{24\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha \mathrm{tr}\left[T^a A_\beta \partial_\gamma A_\delta + \frac{1}{2}A_\beta A_\gamma A_\delta\right].$$

The situation here is much more critical than for global anomalies because there is something wrong with the equations of motion! There must be extra terms in the Lagrangian generated when we do a gauge transformation which is a sure sign that the theory is non-renormalizable.

Another sign that something is wrong can be found in the following way which is similar in spirit to the Noether procedure for global currents. Although the gauge currents are in principle not observable in the unbroken theory, there is nothing to prevent us probing the theory by turning on an expectation value for $A_\mu = \bar{A}_\mu + \delta A_\mu$. This is a supposed to be a spontaneous breaking of the gauge symmetry, and as such the partition function should be invariant under gauge transformations of the VEV. The partition function is of the form

$$\begin{aligned} Z[\bar{A}] &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[A,\psi,\bar{\psi}]} \\ &= \int \mathcal{D}\delta A e^{i\Gamma[\bar{A}]}e^{iS[\delta A,]}. \end{aligned} \tag{62}$$

The factor $e^{i\Gamma[\bar{A}]}$ is derived from the path integrals of the fermions quantixed over that background;

$$e^{i\Gamma[\bar{A}]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{i\int i\bar{\psi}\gamma.(\partial + i\bar{A} + i\delta A)\psi}. \tag{63}$$

Now consider changing the background by an infinitessimal gauge transformation, $\bar{A} \to$

$\bar{A}' = \bar{A} - iD\alpha$. We find

$$\begin{aligned}
e^{i\Gamma[\bar{A}']} - e^{i\Gamma[\bar{A}]} &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left(e^{i\int i\bar{\psi}\gamma.(\partial+i\bar{A}+i\delta A)\psi} - e^{i\int i\bar{\psi}\gamma.(\partial+i\bar{A}-iD\alpha+i\delta A)\psi}\right) \\
&= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left(e^{i\int i\bar{\psi}\gamma.(\partial+i\bar{A}+i\delta A)\psi} - e^{i\int i\bar{\psi}\gamma.(\partial+i\bar{A}-iD\alpha+i\delta A)\psi}\right) \\
&= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left[\int \bar{\psi}\gamma.(D\alpha)\psi\right] e^{i\int i\bar{\psi}\gamma.(\partial+i\bar{A}+i\delta A)\psi} \\
&= -\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left[\int \alpha_a (D.J)^a\right] e^{i\int i\bar{\psi}\gamma.(\partial+i\bar{A}+i\delta A)\psi}.
\end{aligned}$$

Thus if $D.J \neq 0$ we see that the partition function depends on the gauge that choose for our VEV, which is another way of saying that the gauge symmetry is actually already explicitly broken.

We can summarize the different meaning of the abelian versus non-abelian anomalies in the following table:

	Abelian	non-Abelian
currents	singlet	adjoint
Total divergence	Y	Y
Bad?	N	Y
Coefficients	$\frac{1}{4\pi^2}$, $\frac{2}{3}$	$\frac{1}{24\pi^2}$, $\frac{1}{2}$

6 Computing the ABJ anomaly perturbatively

There are two ways to compute the ABJ anomaly - the original analysis was based on the regularization ambiguties of the theory. For simplicity consider massive QED. Setting $g = 1$ to reduce clutter, and defining $\not{D} = \gamma.(\partial + iA)$, the lagrangian is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{64}$$

The equations of motion are,

$$\begin{aligned}
i\not{D}\psi &= m\psi \\
\partial_\mu F^{\mu\nu} &= J^\nu.
\end{aligned} \tag{65}$$

The currents are

$$
\begin{aligned}
J^\mu &= \bar{\psi}\gamma^\mu\psi \\
J_5^\mu &= \bar{\psi}\gamma^\mu\gamma_5\psi,
\end{aligned}
\tag{66}
$$

but now classically we find

$$
\begin{aligned}
\partial.J &= 0 \\
\partial.J_5 &= 2imP,
\end{aligned}
\tag{67}
$$

where $P = \bar{\psi}\gamma_5\psi$.

Proof of second equation: $\partial.J_5 = (\partial\bar{\psi})\gamma^\mu\gamma_5\psi - \bar{\psi}\gamma_5\gamma^\mu(\partial\psi) = i\bar{\psi}(\not{A} + m)\gamma_5\psi + i\bar{\psi}\gamma_5(\not{A} + m)\psi = 2im\bar{\psi}\gamma_5\psi$.

6.1 Tree-level Ward Identities

In order to calculate the anomaly we will consider the amplitudes

$$
\begin{aligned}
T_{\mu\nu\lambda}(k_1, k_2, q) &= \int d^4x_1 d^4x_2 \langle 0|\mathbf{T}\left[J_\mu(x_1)J_\nu(x_2)J_{5\lambda}(0)\right]|0\rangle e^{i(k_1x_1+k_2x_2)} \\
T_{\mu\nu}(k_1, k_2, q) &= \int d^4x_1 d^4x_2 \langle 0|\mathbf{T}\left[J_\mu(x_1)J_\nu(x_2)P(0)\right]|0\rangle e^{i(k_1x_1+k_2x_2)}
\end{aligned}
\tag{68}
$$

where $q = k_1 + k_2$.

These amplitudes should satisfy Ward identities based on the classical current conservation equations (67). In order to derive them we need to take derivatives for which we need an identitiy:

$$
\begin{aligned}
\partial_x^\mu \mathbf{T}\left[J_\mu(x)\hat{\mathcal{O}}(y)\right] &= \partial_x^\mu\left[J_\mu\hat{\mathcal{O}}\vartheta(x_0 - y_0) + \hat{\mathcal{O}}J_\mu\vartheta(y_0 - x_0)\right] \\
&= \mathbf{T}\left[(\partial_x.J)\hat{\mathcal{O}}\right] + \left[J_0,\, \hat{\mathcal{O}}\right]\delta(x_0 - y_0).
\end{aligned}
\tag{69}
$$

The last term is known as the Schwinger term. With this in mind, let us take the derivative of $T_{\mu\nu\lambda}$ with respect to x_1: we find

$$\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(k_1,k_2,q) &= \int d^4x_1 d^4x_2 k_1^\mu \langle 0|\mathbf{T}\left[J_\mu(x_1)J_\nu(x_2)J_{5\lambda}(0)\right]|0\rangle e^{i(k_1x_1+k_2x_2)} \\
&= \int d^4x_1 d^4x_2 \langle 0|\mathbf{T}\left[J_\mu J_\nu J_{5\lambda}\right]|0\rangle (-i\partial^\mu_{x_1} e^{i(k_1x_1+k_2x_2)}) \\
&= \int d^4x_1 d^4x_2 \left(i\partial^\mu_{x_1}\langle 0|\mathbf{T}\left[J_\mu J_\nu J_{5\lambda}\right]|0\rangle\right) e^{i(k_1x_1+k_2x_2)} \\
&= \int d^4x_1 d^4x_2 i\langle 0|\mathbf{T}\left[(\partial_{x_1}.J)\,J_\nu J_{5\lambda}\right] + \left[J_0(x_1),\,J_\nu J_{5\lambda}\right]\delta(x_1^0-x_2^0)|0\rangle e^{i(k_1x_1+k_2x_2)}
\end{aligned} \tag{70}$$

Because of the abelian current/charge algebra we can set the Schwinger term to zero. In addtion $\partial.J=0$ and so the RHS vanishes:

$$k_1^\mu T_{\mu\nu\lambda}(k_1,k_2,q)=0. \tag{71}$$

Likewise

$$k_2^\nu T_{\mu\nu\lambda}(k_1,k_2,q)=0. \tag{72}$$

These are the vector Ward identities.

In addition we have the axial-vector Ward identity from the x_3 derivative. We can write

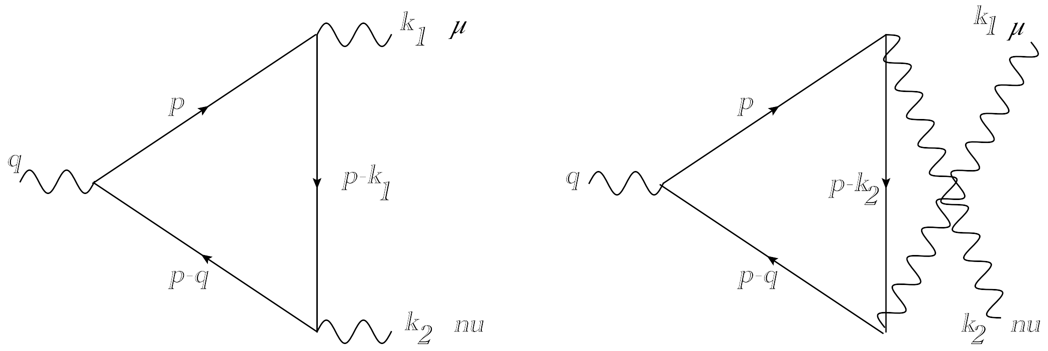
$$T_{\mu\nu\lambda}(k_1,k_2,q)=\int d^4x_1 d^4x_2 d^4x_3 \delta(x_3)\langle 0|\mathbf{T}\left[J_\mu(x_1)J_\nu(x_2)J_{5\lambda}(x_3)\right]|0\rangle e^{i(k_1x_1+k_2x_2-qx_3)}. \tag{73}$$

Then

$$\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(k_1,k_2,q) &= \int d^4x_1 d^4x_2 d^4x_3 \delta(x_3) q^\lambda \langle 0|\mathbf{T}\left[J_\mu(x_1)J_\nu(x_2)J_{5\lambda}(0)\right]|0\rangle e^{i(k_1x_1+k_2x_2-qx_3)} \\
&= \int d^4x_1 d^4x_2 d^4x_3 \delta(x_3)\langle 0|\mathbf{T}\left[J_\mu J_\nu J_{5\lambda}\right]|0\rangle (i\partial^\lambda_{x_3} e^{i(k_1x_1+k_2x_2-qx_3)}) \\
&= \int d^4x_1 d^4x_2 d^4x_3 \delta(x_3)\left(-i\partial^\lambda_{x_3}\langle 0|\mathbf{T}\left[J_\mu J_\nu J_{5\lambda}\right]|0\rangle\right) e^{i(k_1x_1+k_2x_2-qx_3)} \\
&= -i\int d^4x_1 d^4x_2 d^4x_3 \delta(x_3)\langle 0|\mathbf{T}\left[J_\mu J_\nu\,(\partial_{x_3}.J_5)\right] + \mathbf{T}\left[J_\mu\left[J_\nu,\,J_{5,0}\right]\right]\delta(x_2^0-x_3^0) \\
&\qquad +\mathbf{T}\left[J_\nu\left[J_\mu,\,J_{5,0}\right]\right]\delta(x_1^0-x_3^0)|0\rangle e^{i(k_1x_1+k_2x_2-qx_3)} \\
&= -i\int d^4x_1 d^4x_2 \langle 0|\mathbf{T}\left[J_\mu J_\nu\,(\partial.J_5)\right]|0\rangle e^{i(k_1x_1+k_2x_2)} \\
&= 2m\int d^4x_1 d^4x_2 \langle 0|\mathbf{T}\left[J_\mu J_\nu P\right]|0\rangle e^{i(k_1x_1+k_2x_2)} \\
&= 2m\,T_{\mu\nu}(k_1,k_2,q).
\end{aligned} \tag{74}$$

6.2 One loop contributions to the Ward identities

We are now ready to compute the anomaly. The one loop contribution to the anomaly is through the diagram in the figure below, where there is a J_5 current coming out of the q-vertex:



Wick contracting the fermions in the currents gives

$$T^{(1)}_{\mu\nu\lambda} = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right) + \left(\begin{array}{c} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{array}\right)\right\}. \tag{75}$$

Meanwhile the same diagram with a P on the $q-$vertex gives

$$T^{(1)}_{\mu\nu} = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{i}{\not p - m}\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right) + \left(\begin{array}{c} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{array}\right)\right\}. \tag{76}$$

6.2.1 Corrections to the axial Ward identities

In order to get the axial ward identities we can use the identity

$$\not q\gamma_5 = \gamma_5(\not p - \not q - m) + (\not p - m)\gamma_5 + 2m\gamma_5. \tag{77}$$

Thus

$$\begin{aligned} q^\lambda T^{(1)}_{\mu\nu\lambda} &= 2mT^{(1)}_{\mu\nu} + i\int\frac{d^4p}{(2\pi)^4}\left\{ \mathbf{tr}\left(\frac{1}{\not p - m}\left[\gamma_5(\not p - \not q - m) + (\not p - m)\gamma_5\right]\frac{1}{\not p - \not q - m}\gamma_\nu\frac{1}{\not p - \not k_1 - m}\gamma_\mu\right)\right. \\ &\quad \left. + \left(\begin{array}{c} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{array}\right)\right\} \\ &= 2mT^{(1)}_{\mu\nu} + D^A_{\mu\nu} + D^B_{\mu\nu} + D^C_{\mu\nu} + D^D_{\mu\nu}, \end{aligned} \tag{78}$$

where

$$
\begin{aligned}
D^A_{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p - m}\gamma_5\gamma_\nu\frac{1}{\not p - \not k_1 - m}\gamma_\mu\right)\right\} \\
D^B_{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p - \not q - m}\gamma_\nu\frac{1}{\not p - \not k_1 - m}\gamma_\mu\gamma_5\right)\right\} \\
D^C_{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p - m}\gamma_5\gamma_\mu\frac{1}{\not p - \not k_2 - m}\gamma_\nu\right)\right\} \\
D^D_{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p - \not q - m}\gamma_\mu\frac{1}{\not p - \not k_2 - m}\gamma_\nu\gamma_5\right)\right\}.
\end{aligned}
\tag{79}
$$

We can collect the four terms as follows

$$
\begin{aligned}
\Delta^A_{\mu\nu} &= D^A_{\mu\nu} + D^D_{\mu\nu} \\
&= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p - m}\gamma_5\gamma_\nu\frac{1}{\not p - \not k_1 - m}\gamma_\mu\right) - \mathbf{tr}\left(\frac{1}{\not p - \not k_2 - m}\gamma_5\gamma_\nu\frac{1}{\not p - \not q - m}\gamma_\mu\right)\right\}
\end{aligned}
\tag{80}
$$

and

$$
\begin{aligned}
\Delta^B_{\mu\nu} &= D^C_{\mu\nu} + D^B_{\mu\nu} \\
&= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p - m}\gamma_5\gamma_\mu\frac{1}{\not p - \not k_2 - m}\gamma_\nu\right) - \mathbf{tr}\left(\frac{1}{\not p - \not k_1 - m}\gamma_5\gamma_\mu\frac{1}{\not p - \not q - m}\gamma_\nu\right)\right\}
\end{aligned}
\tag{81}
$$

where we have used $\{\gamma_5, \gamma_\mu\} = 0$, $q = k_1 + k_2$ and cyclicity of the traces.

Now we note that each of these two terms $\Delta^{A,B}$ is the sum of two integrals that appear to cancel because they represent merely a shift in integration variable. That is both $\Delta' s$ are of the form

$$
\begin{aligned}
\Delta^A_{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\left\{f_{\mu\nu}(p,k_1) - f_{\mu\nu}(p-k_2,k_1)\right\} \\
\Delta^B_{\mu\nu} &= i\int \frac{d^4p}{(2\pi)^4}\left\{f_{\nu\mu}(p,k_2) - f_{\nu\mu}(p-k_1,k_2)\right\}
\end{aligned}
\tag{82}
$$

where

$$
f_{\mu\nu}(p,k) = \mathbf{tr}\left(\frac{1}{\not p - m}\gamma_5\gamma_\mu\frac{1}{\not p - \not k - m}\gamma_\nu\right). \tag{83}
$$

The crucial observation is that these terms do not vanish because we have to regulate the integrals. In general suppose we have an integral

$$\begin{aligned}\Delta(k) &= i\int \frac{d^4p}{(2\pi)^4} f(p) - f(p-k) \\ &= i\int \frac{d^4p}{(2\pi)^4}\left[k^\mu \partial_{p_\mu} f - k^\mu k^\nu \partial_{p_\mu}\partial_{p_\nu} f + \ldots\right] \end{aligned} \tag{84}$$

where the ellipsis indicates terms that are not divergent (since each $k-$derivative brings down another power of p). By the divergence theorem this can be written as an integral over the Euclidean surface $\mathbf{S}(|p|)$ at infinity,

$$\Delta(k) = -\int_{\mathbf{S}(|p|)} \frac{d\mathbf{S}.\mathbf{k}}{(2\pi)^4} f, \tag{85}$$

where higher terms can be neglected since the integrand vanishes too quickly at infinity ($d^3\sigma \sim |p|^3$ while $\partial^2 f \sim 1/|p|^4$). The surface element at constant radius $|p|$ is given by

$$dS^\mu = |p|^2 dp^\mu.$$

We may now throw away terms dependent on m and k in f so that

$$\begin{aligned}\Delta^A_{\mu\nu} &= -\int_{\mathbf{S}(|p|)} \frac{d\mathbf{S}.\mathbf{k}_2}{(2\pi)^4}\mathbf{tr}\left(\frac{1}{\not p}\gamma_5\gamma_\mu \frac{1}{\not p - \not k_1}\gamma_\nu\right) \\ &= -\int_{\mathbf{S}(|p|)} \frac{d\mathbf{p}.\mathbf{k}_2}{(2\pi)^4}\frac{\left(p^\sigma - k_1^\sigma\right)p^\rho}{p^2}\mathbf{tr}\left(\gamma_\rho\gamma_5\gamma_\mu\gamma_\sigma\gamma_\nu\right) + \ldots \\ &= -4i\int_{\mathbf{S}(|p|)} \frac{d\mathbf{p}.\mathbf{k}_2}{(2\pi)^4}\frac{p^\rho k_1^\sigma}{p^2}\varepsilon_{\mu\nu\rho\sigma} + \ldots \end{aligned} \tag{86}$$

where we used $\mathbf{tr}\left(\gamma_\rho\gamma_5\gamma_\mu\gamma_\sigma\gamma_\nu\right) = \mathbf{tr}\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) = -4i\varepsilon_{\mu\nu\rho\sigma}$, and $\varepsilon_{\mu\nu\rho\sigma}p^\rho p^\sigma = 0$. Finally using $\int dp^\sigma p^\rho = \frac{1}{4}p^2 g^{\rho\sigma}\mathrm{Vol}(S_3) = \frac{\pi^2}{2}p^2 g^{\rho\sigma}$, and permuting ρ, σ, we obtain

$$\Delta^A_{\mu\nu} = i\frac{k_1^\rho k_2^\sigma}{8\pi^2}\varepsilon_{\mu\nu\rho\sigma}. \tag{87}$$

The term for $\Delta^B_{\mu\nu}$ is identical but with $\mu \to \nu$ and $k_1 \to k_2$ hence

$$\Delta^B_{\mu\nu} = \Delta^A_{\mu\nu} = i\frac{k_1^\rho k_2^\sigma}{8\pi^2}\varepsilon_{\mu\nu\rho\sigma}. \tag{88}$$

You might fondly imagine we are now done. Unfortunately not: this ambiguity due to regularization causes us to now suspect that we may also obtain terms from regularizing the $T_{\mu\nu\lambda}$ integral. Indeed defining $\tau_{\mu\nu\lambda}(k)$ to be $T^{(1)}_{\mu\nu\lambda}$ with the momentum p shifted to $p+k$ in the integrand, we can shift $T_{\mu\nu\lambda}$ as

$$T_{\mu\nu\lambda} \to T_{\mu\nu\lambda} + \Delta_{\mu\nu\lambda}, \tag{89}$$

where $\Delta_{\mu\nu\lambda}(k) = \tau_{\mu\nu\lambda}(k) - \tau_{\mu\nu\lambda}(0)$. We have

$$\begin{aligned}\Delta_{\mu\nu\lambda} &= -\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{i}{\not p + \not k - m}\gamma_\lambda\gamma_5\frac{i}{\not p + \not k - \not q - m}\gamma_\nu\frac{i}{\not p + \not k - \not k_1 - m}\gamma_\mu\right) + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}\right\}. \\ &\quad + \int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right) + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}\right\}\end{aligned} \tag{90}$$

Applying the previous discussion this becomes

$$\begin{aligned}\Delta_{\mu\nu\lambda} &= i\int_{\mathbf{S}(|p|)} \frac{id\mathbf{S}.\mathbf{k}}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not p}\gamma_\lambda\gamma_5\frac{1}{\not p - \not q}\gamma_\nu\frac{1}{\not p - \not k_1}\gamma_\mu\right)\right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ &= i\int_{\mathbf{S}(|p|)} \frac{id\mathbf{p}.\mathbf{k}}{(2\pi)^4}\frac{p^\rho(p-q)^\sigma(p-k_1)^\alpha}{|p|^4}\left\{\mathbf{tr}\left(\gamma_\rho\gamma_\lambda\gamma_5\gamma_\sigma\gamma_\nu\gamma_\alpha\gamma_\mu\right)\right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}\end{aligned} \tag{91}$$

In order to get something non-zero as $|p| \to \infty$ we need a power of p^4 on the top, which means we need retain only the leading term

$$\begin{aligned}\Delta_{\mu\nu\lambda} &= i\int_{\mathbf{S}(|p|)} \frac{id\mathbf{p}.\mathbf{k}}{(2\pi)^4}\frac{p^\rho p^\sigma p^\alpha}{|p|^4}\left\{\mathbf{tr}\left(\gamma_\rho\gamma_\lambda\gamma_5\gamma_\sigma\gamma_\nu\gamma_\alpha\gamma_\mu\right)\right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ &= -\int_{\mathbf{S}(|p|)} \frac{d\mathbf{p}.\mathbf{k}}{(2\pi)^4}\frac{p^\rho p^\sigma p^\alpha}{|p|^4}\left\{\mathbf{tr}\left(\gamma_\rho\gamma_\lambda\gamma_5\gamma_\sigma(2g_{\nu\alpha} - \gamma_\alpha\gamma_\nu)\gamma_\mu\right)\right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ &= -\int_{\mathbf{S}(|p|)} \frac{d\mathbf{p}.\mathbf{k}}{(2\pi)^4}\frac{p^\rho p^\sigma p^\alpha}{|p|^4}\left\{\mathbf{tr}\left(\gamma_\rho\gamma_\lambda\gamma_5[2g_{\nu\alpha}\gamma_\sigma\gamma_\mu - g_{\sigma\alpha}\gamma_\nu\gamma_\mu)\right)\right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ &= -\frac{i}{4\pi^4}\int_{\mathbf{S}(|p|)} d\mathbf{p}.\mathbf{k}\frac{1}{|p|^2}\varepsilon_{\rho\lambda\nu\mu}p^\rho + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ &= -\frac{i}{8\pi^2}k^\rho\varepsilon_{\mu\nu\lambda\rho} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}.\end{aligned} \tag{92}$$

What should we take for k? The only sensible possibility is to define it in terms of the physical momenta, k_1 and k_2; let us call it $k = \alpha k_1 + \alpha' k_2$, so that

$$\begin{aligned} \Delta_{\mu\nu\lambda} &= -\frac{i}{8\pi^2}\varepsilon_{\mu\nu\rho\lambda}\left[(\alpha k_1^\rho + \alpha' k_2^\rho) - (\alpha k_2^\rho + \alpha' k_1^\rho)\right] \\ &= -\frac{i\beta}{8\pi^2}\varepsilon_{\mu\nu\rho\lambda}\left[k_1^\rho - k_2^\rho\right], \end{aligned} \tag{93}$$

where β is some other undefined parameter (as I can't calculate it I didn't even need to bother keeping track of factors of π).

Together, eqs.(93), (88) and (78) imply

$$\begin{aligned} q^\lambda T^{(1)}_{\mu\nu\lambda} &= 2mT^{(1)}_{\mu\nu} + i\frac{k_1^\rho k_2^\lambda}{4\pi^2}\varepsilon_{\mu\nu\rho\lambda} - \frac{i\beta}{8\pi^2}\varepsilon_{\mu\nu\rho\lambda}(k_1^\rho - k_2^\rho)(k_1^\lambda + k_2^\lambda) \\ &= 2mT^{(1)}_{\mu\nu} + i(1-\beta)\frac{k_1^\rho k_2^\lambda}{4\pi^2}\varepsilon_{\mu\nu\rho\lambda}. \end{aligned} \tag{94}$$

6.2.2 Corrections to the vector Ward identities

Now we turn to the contributions to the vector ward identities. The simplest way to get these is to write $\not{k}_1 = \not{p} - m - (\not{p} - \not{k}_1 - m)$. Going back to eq.(75) we find

$$k_1^\mu T^{(1)}_{\mu\nu\lambda} = ik_1^\mu \int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m}\gamma_\mu\right) + \left(\begin{array}{c} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{array}\right)\right\}. \tag{95}$$

The first piece gives

$$\begin{aligned} k_1^\mu T^A_{\mu\nu\lambda} &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m} - \frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\right)\right\} \\ &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m} - \gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-m}\right)\right\} \\ &= i\int \frac{d^4p}{(2\pi)^4}\left\{-\mathbf{tr}\left(\frac{1}{\not{p}-\not{k}_1-m}\gamma_5\gamma_\lambda\frac{1}{\not{p}-\not{q}-m}\gamma_\nu + \frac{1}{\not{p}-m}\gamma_5\gamma_\lambda\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\right)\right\} \end{aligned} \tag{96}$$

For the second piece use $\not{k}_1 = \not{p} - \not{k}_2 - m - (\not{p} - \not{q} - m)$

$$\begin{aligned} k_1^\mu T^B_{\mu\nu\lambda} &= i\int \frac{d^4p}{(2\pi)^4}\left\{\mathbf{tr}\left(\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\not{k}_2\frac{1}{\not{p}-\not{k}_2-m}\gamma_\nu\right)\right\} \\ &= i\int \frac{d^4p}{(2\pi)^4}\left\{-\mathbf{tr}\left(\frac{1}{\not{p}-m}\gamma_5\gamma_\lambda\frac{1}{\not{p}-\not{q}-m}\gamma_\nu + \frac{1}{\not{p}-m}\gamma_5\gamma_\lambda\frac{1}{\not{p}-\not{k}_2-m}\gamma_\nu\right)\right\} \end{aligned} \tag{97}$$

Mercifully these expressions are similar to previous ones. Summing them (and noting that two terms cancel) we can write

$$\begin{aligned} k_1^\mu T^{(1)}_{\mu\nu\lambda} &= i\int \frac{d^4p}{(2\pi)^4}\left\{f_{\lambda\nu}(p,k_2) - f_{\lambda\nu}(p-k_1,k_2)\right\} = \Delta^B_{\nu\lambda} \\ &= \frac{i}{8\pi^2}\varepsilon_{\nu\lambda\rho\sigma}k_1^\rho k_2^\sigma + \Delta - \text{shift} \end{aligned} \tag{98}$$

Finally we need to add the same Δ shift to $T_{\mu\nu\lambda}$ so that

$$\begin{aligned} k_1^\mu T^{(1)}_{\mu\nu\lambda} &= \frac{i}{8\pi^2}\varepsilon_{\nu\lambda\rho\sigma}k_1^\rho k_2^\sigma + \frac{i\beta}{8\pi^2}\varepsilon_{\rho\nu\sigma\lambda}k_1^\rho\left[k_1^\sigma - k_2^\sigma\right] \\ &= \frac{i}{8\pi^2}\varepsilon_{\nu\lambda\rho\sigma}k_1^\rho k_2^\sigma + \frac{i\beta}{8\pi^2}\varepsilon_{\nu\lambda\rho\sigma}k_1^\rho k_2^\sigma \\ &= \frac{i(1+\beta)}{8\pi^2}\varepsilon_{\nu\lambda\rho\sigma}k_1^\rho k_2^\sigma. \end{aligned} \tag{99}$$

6.2.3 The end result ...

Our main results are eqs.(94) and (99). There are two Ward identities and only one free parameter β. If we wish to keep our original abelian gauge group preserved the only consistent choice is to satisfy the vector Ward identify and take $\beta = -1$. This then leaves us with an anomalous axial Ward identity

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + i\frac{k_1^\rho k_2^\lambda}{2\pi^2}\varepsilon_{\mu\nu\rho\lambda}. \tag{100}$$

Going back to the original derivation of the Ward identity we can see what this means for the divergence of the axial current. Setting $m = 0$ we can check that the proposed anomaly correctly reproduces the anomalous axial Ward identity:

$$\begin{aligned}
q^{\lambda}T_{\mu\nu\lambda} &= -i\int d^4x_1d^4x_2\langle 0|\mathbf{T}\left[J_\mu J_\nu\left(\partial.J_5\right)\right]|0\rangle e^{i(k_1x_1+k_2x_2)} \\
&= -i\int d^4x_1d^4x_2\langle 0|\mathbf{T}\left[J_\mu J_\nu\frac{1}{4\pi^2}\varepsilon_{\mu'\nu'\rho\sigma}\partial^{\mu'}A^{\nu'}\partial^\rho A^\sigma\right]|0\rangle e^{i(k_1x_1+k_2x_2)} \\
&= \frac{-i}{4\pi^2}\varepsilon_{\mu'\nu'\rho\sigma}\int d^4x_1d^4x_2\langle 0|J_\mu\partial^{\mu'}A^{\nu'}|0\rangle\langle 0|J_\nu\partial^\rho A^\sigma|0\rangle e^{i(k_1x_1+k_2x_2)}+\text{perm} \\
&= \frac{i}{4\pi^2}\varepsilon_{\mu'\nu'\rho\sigma}\int d^4x_1d^4x_2k_1^{\mu'}k_2^{\rho}\langle 0|J_\mu A^{\nu'}|0\rangle\langle 0|J_\nu A^\sigma|0\rangle e^{i(k_1x_1+k_2x_2)}+\text{perm} \\
&= -\frac{i}{4\pi^2}\varepsilon_{\mu'\mu\rho\nu}\int d^4x_1d^4x_2k_1^{\mu'}k_2^{\rho}\delta^4(x_1)\delta^4(x_2)e^{i(k_1x_1+k_2x_2)}+\text{perm} \\
&= \frac{i}{2\pi^2}\varepsilon_{\mu\nu\mu'\rho}k_1^{\mu'}k_2^{\sigma}, \qquad (101)
\end{aligned}$$

where we used that $\langle 0|J_\mu A^\nu|0\rangle = \langle 0|\bar{\psi}\gamma_\mu\psi A^\nu|0\rangle = ig\delta^\nu_\mu\delta^4(x)$ is just the gauge vertex.

6.3 One loop exactness

The anomaly we have just derived receives no further corrections at any order in perturbation theory and is finite. We can see this by simple power counting. Let us introduce a loop counting parameter $S \to S/\lambda$. In the Feynman rules, the propagators will receive a factor λ while vertices receive a factor λ. A diagram has a total λ^{I-V} where I is the number of internal lines and V the number of vertices. These are related to the number of loops as $I = L + V - 1$. (An informal induction proof goes as follows: each internal propagator either connects $2V$'s or forms a loop, thus always adding 1 to the number of loops L. On the other hand inserting a new vertex $V \to V+1$, increases I by 1 but leaves L unchanged. Thus $I = V + L + const$. The simplest diagram, the loop with $I = L = V = 1$, yields $I = L + V - 1$). The one loop diagram has $L = 1$ and $I = 3$ so is finite.

Now the amplitude $T_{\mu\nu\lambda}$ has scaling dimension 1, so the ward identity for $q^\lambda T_{\mu\nu\lambda}$ must have scaling dimension 2. Furthermore the anomaly must be proportional to ε (if it were not we could have performed dimensional regularization on our integrals which would give no momentum ambiguity). Because of the symmetry in k_1, k_2, the anomaly must be of the form $Ak_1k_2\varepsilon$. Thus the anomaly coefficient A is dimensionless and therefore cannot contain $|p| \to \infty$ divergences. Because of the power counting argument above, a higher than one-loop diagram has $I > V$, and the diagrams must have $V \geq 3$ (since three vertices are required for the currents to emerge). $V = 3$ corresponds to the one-loop case so higher order diagrams yielding operators of the form $k_1 k_2 \varepsilon$ would have to have $V \geq 4$ and hence $I \geq 5$, or to be of the form $\sim \int d^4 p k_1 k_2 \varepsilon / |p|^{n \geq 5}$.

7 Fujikawa's Method

The determination of the anomaly by perturbative methods looks a little obscure, since it is bound up with ambiguities in infinities. The real source of the anomaly becomes much clearer using a different method due to Fujikawa[1]. Consider electrodynamics again with the axil symmetry

$$\begin{aligned} \psi &\to U_5 \psi \implies \delta\psi = i\alpha\gamma_5\psi \\ \bar{\psi} &\to \bar{\psi} U_5 \implies \delta\bar{\psi}_i = i\alpha\bar{\psi}\gamma_5 \text{ infinitessimally.} \end{aligned} \tag{102}$$

and the gauge symmetry

$$\begin{aligned} A_\mu &\to A_\mu - i\partial_\mu \lambda \\ \bar{\psi} &\to \bar{\psi} - i\lambda\bar{\psi} \\ \psi &\to \psi + i\lambda\psi. \end{aligned} \tag{103}$$

Thus the total (finite) transformation of the fermions under a chiral transformation is

$$\begin{aligned} \bar{\psi} &\to \bar{\psi}' = e^{i\alpha\gamma_5}\bar{\psi} \\ \psi &\to \psi' = e^{i\alpha\gamma_5}\psi. \end{aligned} \tag{104}$$

Consider the transformation of the partition function

$$\begin{aligned} Z &= \int \mathcal{D}A\mathcal{D}\psi'\mathcal{D}\bar{\psi}' e^{iS[A,\psi',\bar{\psi}']} \\ &= \int \mathcal{D}A\mathcal{D}\psi'\mathcal{D}\bar{\psi}' e^{iS[A,\psi,\bar{\psi}]+i\int d^4x\alpha\partial J_5} \\ &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi} \det\left(\exp 2i\alpha\gamma_5\right) e^{iS[A,\psi,\bar{\psi}]+i\int d^4x\alpha\partial J_5} \\ &= \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{\mathrm{Tr}(2i\alpha\gamma_5)} e^{iS[A,\psi,\bar{\psi}]+i\int d^4x\alpha\partial J_5}, \end{aligned} \tag{105}$$

[1] You should always be wary when some-one explains something to you in two different ways: inevitably it means that neither is really satisfactory.

where the determinant is from the Jacobian of the measure, and the trace with a capital T means over both Dirac and space-time indices. Our claim is that the anomaly derives from precisely this term.

Our task is to evaluate

$$\mathrm{Tr}(2i\alpha\gamma_5).$$

In order to take the trace we need to represent the Hilbert space of fermionic wavefunctions such that (bearing in mind that this is a gauge theory) it can be regulated in a gauge invariant way. The most efficient way to do this is to decompose the fermions into an orthonormal basis of eigenfunctions of the $\not{D} = \not{\partial} + i\not{A}$ operator:

$$\begin{aligned} i\not{D}\varphi_n &= \lambda_n \varphi_n, \\ \sum_n \varphi_n(x)\varphi_n^\dagger(y) &= \delta^4(x-y)\mathbf{1} \end{aligned} \tag{106}$$

where the identity in the completeness relation refers to spinor indices. (In Dirac notation $\sum_n \langle x,\alpha|n\rangle\langle n|y,\beta\rangle = \delta_{\alpha\beta}\delta^4(x-y)$). Using this basis

$$\mathrm{Tr}(2i\alpha\gamma_5) = 2i\int d^4x \sum_n \alpha\varphi_n^\dagger(x)\gamma_5\varphi_n(x), \tag{107}$$

where we *sum over spinor indices* in order to get that trace. This expression is divergent and needs to be regulated. We can do this in a gauge invariant way by introducing a regulating function $f(k^2)$ with the properties that $f(0) = 1$, and

$$\lim_{k\to\infty} f(k^2) = \lim_{k\to\infty} f'(k^2) = \lim_{k\to\infty} f'(k^2) = \ldots = 0. \tag{108}$$

For example $f(k^2) = e^{-k^2}$ would do, but the choice is irrelevant. We can regulate the trace using this function as

$$\mathrm{Tr}(2i\alpha\gamma_5) = \lim_{\Lambda\to\infty}\left[2i\int d^4x \sum_n \alpha\varphi_n^\dagger\gamma_5 f\left(\frac{\lambda_n^2}{\Lambda^2}\right)\varphi_n\right], \tag{109}$$

so that f cuts off the sum at some scale Λ. Note that the regularization is gauge independent as required. In order to reduce clutter I will henceforth omit the $\lim_{\Lambda\to\infty}$ and take it as read.

Using eq.(106) the trace becomes

$$\mathrm{Tr}(2i\alpha\gamma_5) = 2i \int d^4x \sum_n \alpha\varphi_n^\dagger \gamma_5 f\left(\frac{-\not{D}^2}{\Lambda^2}\right)\varphi_n. \tag{110}$$

Next we Fourier transform the basis;

$$\mathrm{Tr}(2i\alpha\gamma_5) = 2i \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\sum_n \alpha e^{-ik'x}\tilde{\varphi}_n^\dagger(k')\gamma_5 f\left(\frac{-\not{D}^2}{\Lambda^2}\right)\tilde{\varphi}_n(k)e^{ikx}. \tag{111}$$

We now use the identity

$$\begin{aligned} \not{D}^2 &= \gamma^\mu\gamma^\nu D_\mu D_\nu = \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]D_\mu D_\nu + \frac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\}D_\mu D_\nu \\ &= \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + D^2 \end{aligned} \tag{112}$$

and note that for any function $g(x)$, we have $D_\mu\left[g(x)e^{ikx}\right] = e^{ikx}\left(ik_\mu + D_\mu\right)g$. Hence using completeness of the momentum eigenstates (derived from Fourier transforming eq.(106) giving $(\sum_n \tilde{\varphi}_n(k)\tilde{\varphi}_n^\dagger(k') = (2\pi)^4\delta(k-k')\mathbf{1})$) we have

$$\begin{aligned} \mathrm{Tr}(2i\alpha\gamma_5) &= 2i\int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\sum_n \alpha\tilde{\varphi}_n^\dagger e^{ikx}e^{-ik'x}\gamma_5 f\left(\frac{-\frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - (-ik+D)^2}{\Lambda^2}\right)\tilde{\varphi}_n \\ &= 2i\int d^4x \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\alpha\gamma_5 f\left(\frac{-\frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - (-ik+D)^2}{\Lambda^2}\right) \\ &= 2i\int d^4x \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\alpha\gamma_5 f\left(\frac{-\frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - (-ik+D)^2}{\Lambda^2}\right). \end{aligned} \tag{113}$$

[In lectures there was a question about where the spinor indices are in this - writing the spinor indices explicitly we just used $\sum_n \left(\tilde{\varphi}_n^\dagger(k')\right)_\alpha \hat{\mathcal{O}}_{\alpha\beta}\left(\tilde{\varphi}_n(k)\right)_\beta = (2\pi)^4\delta(k-k')\hat{\mathcal{O}}_{\alpha\beta}\delta_{\beta\alpha} = (2\pi)^4\delta(k-k')\mathrm{tr}\hat{\mathcal{O}}$].

Next we notice that the trace over fermion indices always contains a γ_5, and that $\mathrm{tr}(\gamma_5) = \mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu) = 0$. In fact the first non-zero contribution comes from the second term in the

Taylor series of f. Thus we have

$$\begin{aligned}\mathrm{Tr}(2i\alpha\gamma_5) &= 2i\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\alpha\gamma_5\frac{1}{2}\left[\frac{-\frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}}{\Lambda^2}\right]^2 f''\left(\frac{k^2}{\Lambda^2}\right)+\mathcal{O}(1/\Lambda^6)\\ &= -i\int d^4x\int\frac{d^4k}{\Lambda^4(2\pi)^4}\alpha\frac{1}{4}\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)F^{\mu\nu}F^{\rho\sigma}f''\left(\frac{k^2}{\Lambda^2}\right)+\mathcal{O}(1/\Lambda^6)\\ &= -\int d^4x\frac{\alpha}{16\pi^4}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\int\frac{d^4k}{\Lambda^4}f''\left(\frac{k^2}{\Lambda^2}\right)+\mathcal{O}(1/\Lambda^6)\end{aligned}$$

We may now take the $\Lambda\to\infty$ limit whereupon the subleading terms vanish.

$$\mathrm{Tr}(2i\alpha\gamma_5) = -\int d^4x\frac{\alpha}{16\pi^4}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}A, \tag{114}$$

where

$$\begin{aligned}A &= \int d^4q\, f''\left(q^2\right)\\ &= i\int_0^\infty \pi^2 ydy\, f''(y)\,. \end{aligned}\tag{115}$$

where $q=k/\Lambda$ is a dimensionless four vector and the factor i comes from the Wick rotation to Euclidean space. Integrating by parts $\int_0^\infty ydy\,f''(y) = [yf'(y)]_0^\infty - \int_0^\infty dy\,f'(y) = -[f'(y)]_0^\infty = 1$. Hence $A=i\pi^2$, and

$$\mathrm{Tr}(2i\alpha\gamma_5) = -i\int d^4x\frac{\alpha}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}. \tag{116}$$

This is our main result: combining it with eq.(105) the total effective change in the action can be written

$$i\delta S = i\int d^4x\alpha\left(\partial.J_5 - \frac{\alpha}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right), \tag{117}$$

giving the anomaly!!

8 Anomaly coefficients and constructing anomaly free models

We now return to the issue of how to construct anomaly free models, taking the SM as our example. First recall the general form of the gauge anomalies: a gauge theory with left handed massless Weyl fermions ψ and gauge field $A^\mu = A^\mu_a T^a$ has lagrangian

$$\mathcal{L} = i\bar{\psi_L}\gamma.(\partial + iA)\psi_L, \tag{118}$$

where the L subscript means projection onto the left chirality, so that a Dirac fermion would look like

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{119}$$

in the Weyl representation of γ's. The covariant derivative for the adjoints is

$$D^b_{\mu a} = \delta^b_a \partial_\mu + f^b_{ac} A^c_\mu. \tag{120}$$

and the gauge transformation is

$$\begin{aligned} A^a_\mu &\rightarrow A^a_\mu - iD^a_{\mu b}\alpha^b \\ \bar{\psi} &\rightarrow \bar{\psi} - i\alpha_a \bar{\psi} T^a. \\ \psi &\rightarrow \psi + i\alpha_a T^a \psi. \end{aligned} \tag{121}$$

Using the Fujikawa technique, a gauge transformation results in the contribution to the action of

$$\int d^4x \ \alpha_a \left(D.J_{L/R}\right)^a \mp \frac{1}{24\pi^2} \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \mathrm{tr}\left[T^a A_\beta \partial_\gamma A_\delta + \frac{1}{2} A_\beta A_\gamma A_\delta\right], \tag{122}$$

where the $\pm$ corresponds to left or right chiralities. Note that the trilinear term comes from a box diagram.

8.1 The importance of d^{abc}

Because the terms are always of the same form, in order to discuss the cancellation of anomalies we can restrict our attention to just the first term:

$$\begin{aligned} \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha(T^a A_\beta \partial_\gamma A_\delta) &= \varepsilon^{\alpha\beta\gamma\delta}\mathbf{tr}\left(T^a T^b T^c\right)(\partial_\alpha A^b_\beta)(\partial_\gamma A^c_\delta) \\ &= \frac{\varepsilon^{\alpha\beta\gamma\delta}}{2}\mathbf{tr}\left(T^a\{T^b, T^c\}\right)(\partial_\alpha A^b_\beta)(\partial_\gamma A^c_\delta) \end{aligned} \tag{123}$$

where we used the symmetries of the indices. (In terms of diagrams it is of this form because the diagram with crossed-over legs effectively just swaps the indices on the generators). This the anomalies will be zero if

$$d^{abc} = \mathbf{tr}\left(T^a\{T^b, T^c\}\right) \tag{124}$$

is zero.

Therefore, for simple groups, we can get a long way by noting the general properties of d^{abc}:

1. d^{abc} is gauge invariant (in the sense that if we had gauge transformed the theory we would have still found the same term, or equivalently under the transformation $T \to g^{-1}Tg$).
2. For a representation R,

$$d^{abc}(R) = \mathbf{tr}\left(T^a_R\{T^b_R, T^c_R\}\right). \tag{125}$$

3. Define $d(R) = A(R)d(\Box)$ (where $\Box$ means the fundamental). Then using the properties of the generators, we find **a)** $A(\bar{R}) = -A(R)$, **b)** $A(R_1 \oplus R_2) = A(R_1) + A(R_2)$, **c)** $A(R_1 \otimes R_2) = A(R_1)\dim(R_2) + \dim(R_1)A(R_2)$
4. From 3) it follows that chiral fermions in real or pseudoreal cannot contribute. Explicitly, a pseudo-real representation has $\epsilon\psi^* = \psi$ for some $\epsilon \in G$. Trivially, $\psi \to U\psi \implies \psi^* \to U^*\psi^*$, but also $\psi^* \to \epsilon^{-1}U\psi = \epsilon^{-1}U\epsilon\psi^* \implies U^* = \epsilon^{-1}U\epsilon$. Then since d is gauge invariant $d(\psi^*) = d(\psi)$ by **3a)** but $d(\psi^*) = A(\psi^*)d(\psi) = -d(\psi)$, by **1)**. Hence $d(\psi) = 0$.
5. Clearly if there are no reprentations that contribute then there are no anomalies; the following groups have only real or pseudo-real representations so are automatically anomaly free;

$$SU(2),\ SO(2N+1),\ SO(4N),\ Sp(2N),\ G_2,\ F_4,\ E_7,\ E_8.$$

6. $SO(4N+2)$ and E_6 do have complex representations but they still have $d = 0$. e.g. $SO(10)$ and E_6 have chiral representations but they are always anomaly free.

7. $SU(N \geq 3)$ do in general allow anomalies and we have to make sure they cancel.

8.2 The SM

For more general product groups the situation is much more complicated as we have to ensure that not only the cubic anomalies cancel but also the mixed anomalies. As our first example we shall show that the Standard Model is anomaly free. The gauge group is $SU(3) \times SU(2)_L \times U(1)_Y$ so we will encounter of course the same cubic anomalies as above, but also mixed anomalies when the three generators in the triangle come from different factors. Before going to the precise particle content we can make some general comments. In all of the below, we use tr as short for $\text{tr}_{Left} - \text{tr}_{Right}$:

1. The theory is vector-like with respect to $SU(3)$ (i.e. there are as many $\mathbf{3}$ as $\bar{\mathbf{3}}$ quarks) so that $SU(3)^3$ anomalies cancel.
2. The $SU(2)^3$ anomalies cancel for the reason cited above. Indeed just to check, we have $d^{abc} = \text{tr}(\tau^a\{\tau^b, \tau^c\})$. But $\{\tau^b, \tau^c\} = 2\delta^{bc}\mathbf{1}$ and hence $d^{abc} = 2\text{tr}(\tau^a\delta^{bc}) = 0$ as advertised.
3. The $SU(2)^2 \times SU(3)$ and $SU(2) \times SU(3)^2$ anomalies are zero because we are always taking the trace over an $SU(N)$ generator.
4. Using for example trace Y to imply summing over the hypercharges of all particles that can run in the anomaly loop, we see that $SU(2) \times Y^2$ and $SU(3) \times Y^2$ anomalies are zero again because of the tracelessness of the generators.
5. Cancellation of $SU(2)^2 \times Y$ is equivalent to $\text{tr}(Y\{\tau^b, \tau^c\}) = 2\delta^{bc}\text{tr}_{Left}Y$. Hence we require
$$\text{tr}_{Left}Y = 0. \tag{126}$$
6. Cancellation of $SU(3)^2 \times Y$ is equivalent to $\text{tr}(Y\{\lambda^b, \lambda^c\}) = 2\delta^{bc}\text{tr}_{quarks}Y$. Hence
$$\text{tr}_{L-quarks}Y - \text{tr}_{R-quarks}Y = 0. \tag{127}$$
7. Noting that left and right chiralities contribute with opposite sign, cancellation of

Y^3 anomalies requires

$$\mathrm{tr} Y^3 \equiv \mathrm{tr}_{Left} Y^3 - \mathrm{tr}_{Right} Y^3 = 0. \tag{128}$$

Hence we are left with three highly non-trivial conditions on the particle content of the SM. The content is

$$\begin{aligned} \ell_L &= (\mathbf{1}, \mathbf{2}, -1) \\ e_R &= (\mathbf{1}, \mathbf{1}, -2) \\ q_L &= (\mathbf{3}, \mathbf{2}, \frac{1}{3}) \\ u_R &= (\bar{\mathbf{3}}, \mathbf{1}, \frac{4}{3}) \\ d_R &= (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3}). \end{aligned} \tag{129}$$

One can check that there are as many **3** as $\bar{\mathbf{3}}$'s as above. We now need to simply correctly count the multiplicity of states when taking the trace. As generations simply repeat we may as well consider only one generation, but we must ensure to sum over colours and flavours. Thus (working down the list for each anomaly),

$$\begin{aligned} \mathrm{tr}_{L-quark} Y - \mathrm{tr}_{R-quark} Y &= \frac{1}{3} \times 3_c \times 2_f - \frac{4}{3} \times 3_c + \frac{2}{3} \times 3_c = 0 \\ \mathrm{tr}_{Left} Y &= -1 \times 2_f + \frac{1}{3} \times 3_c \times 2_f = 0 \\ \mathrm{tr} Y^3 &= (-1)^3 \times 2_f - (-2)^3 + \left(\frac{1}{3}\right)^3 \times 3_c \times 2_f - \left(\frac{4}{3}\right)^3 \times 3_c - \left(-\frac{2}{3}\right)^3 \times 3_c \\ &= -2 + 8 + \frac{2}{9} - \frac{64}{9} + \frac{8}{9} = 0. \end{aligned} \tag{130}$$

The SM is indeed anomaly free.

Note how constraining these conditions are. Often it is said that the fact that the SM can be fit into multiplets of $SO(10)$ is a strong hint at Grand Unification. Actually it is more that anomalies have to cancel for consistency, and the conditions are so constraining that any consistent theory, even if it does not really unify, is likely to resemble one that does since as we saw above large classes of simple gauge theories automatically have no anomalies. Alas anomaly cancellation makes the case for unification weaker rather than stronger.

8.3 SQCD

Another important example of anomalies is to be found in supersymmetric QCD (SQCD). Consider an $SU(N_c)$ supersymmetric theory with N_f quark superfields, Q and N_f antiquark superfields, $\tilde{Q}$. The theory has a global $SU(N_f)_L \times SU(N_f)_R$ symmetry under which the quarks and antiquarks are rotated amongst themselves. There are also some $U(1)$symmetries which I will tell you about in a moment, but let me first remark that such symmetries are immensely important in for example 't Hooft anomaly matching. If any global symmetries do not "see" the gauged $SU(N_c)$ through anomalies (or indeed anything else), we can be fairly sure that even if the theory becomes strongly coupled they will remain intact and can be used as a probe of the theory through the strong coupling regime. [Think about flavour symmetries in the standard model when $SU(3)_c$ confines yielding the chiral lagrangian description of the low energy theory.] Therefore we need to find which global symmetries are anomaly free.

Regarding $U(1)'s$ there is one obvious anomaly free one which is $Q \to e^{i\alpha}Q$ and $\tilde{Q} \to e^{-i\alpha}\tilde{Q}$. This charge (call it B) is traceless and hence $\mathrm{tr}(U(1)^3_B) = \mathrm{tr}(U(1)_B \times SU(N_c)^2) = \mathrm{tr}(U(1)^2_B \times SU(N_c)) = 0$. The orthogonal axial symmetry under which quark and antiquark rotate the same way (call it $U(1)_A$) is traceful and hence obviously anomalous. It also has mixed $\mathrm{tr}(U(1)_A \times SU(N_c)^2)$ anomalies and hence is likely to be affected by $SU(N_c)$.

There is one other symmetry known as R-symmetry that is peculiar to SUSY. Take the Grassman variables θ^α and allow then to rotate by a phase λ, $\theta \to e^{i\lambda}\theta$. This means that $\bar{\theta} \to e^{-i\lambda}\bar{\theta}$, and the fact that $\int d^2\theta\theta^2 = 1$ implies $d^2\theta \to e^{-2i\lambda}d^2\theta$. If we at the same time rotate the chiral multiplets appearing in the superpotential by their own phases such that the sum of the charges in any term in the superpotential is always 2, so that $W \to e^{2i\lambda}W$, it follows that the interaction terms$\int d^2\theta W$ are invariant. Likewise the kinetic terms are invariant (since $K = K^\dagger$ and $d^2\theta d^2\bar{\theta} \to d^2\theta d^2\bar{\theta}$). Finally the Yang-Mills terms are if the gauge field-strength superfield $\mathcal{W}^\alpha$ has $R-$charge 1.

Note that we have to be careful with charge assignements because the R-symmetry does not commute ith supersymmetry. In particular since $\Phi \sim \phi + \sqrt{2}\theta\psi$ we see that if $\Phi \to e^{ir\lambda}\Phi$ then the fermions transform as $\psi \to e^{i(r-1)\lambda}\psi$: the R-charge of the fermions in a chiral supermultiplet is one less than the R-charge of the multiplet itself. On the other hand the field strength superfield is expanded as $\mathcal{W}^\alpha \sim \lambda^\alpha + \ldots$ and the R-charge of any gaugino is always 1. (This also tells us that any gaugino Majorana mass term breaks all R-symmetries by the way).

After this short introduction to R-symmetry let us return to the issue of which one in the SQCD system is anomaly free. We can write the quantum numbers of the quark and antiquark multiplets under their $[SU(N_c); SU(N_f)_L, SU(N_f)_R, U(1)_B, U(1)_R]$ symmetries as

$$
\begin{aligned}
Q &= (\mathbf{N_c}; \bar{\mathbf{N}}_f, \mathbf{1}, \frac{1}{N_c}, R_Q) \\
\tilde{Q} &= (\bar{\mathbf{N}}_\mathbf{c}; \mathbf{1}, \mathbf{N}_f, -\frac{1}{N_c}, R_Q).
\end{aligned} \tag{131}
$$

The $1/N_c$ is by convention since then a baryon Q^{N_c} has charge unity (actually the more popular convention seems to be to take the charge of the quarks to to be 1 which I find reprehensible) . Also note that we have chosen Q and $\tilde{Q}$ to have the same R-charge R_Q: if I had chosen different charges I would have simply been able to add a linear combination of that R-symmetry and $U(1)_B$ to get this R-symmetry.

Now consider the mixed anomalies in order to determine R_Q. The only relevant anomaly is $SU(N_c)^2 \times U(1)_R$. We need to know the Dynkin indices of both the fundamentals and the adjoint (since the gauginos have non-zero R-charge as well. For future reference the cubic $A(R)$ and quadratic Dynkin indices $T(R)$ for $SU(N)$ are given in the Table below

Irrep	dim(R)	$T(R)$	$A(R)$
$\Box$	N	1	1
Adj	N^2-1	$2N$	0
$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\frac{N((N-1)}{2}$	$N-2$	$N-4$
$\Box\Box$	$\frac{N((N+1)}{2}$	$N+2$	$N+4$
$\begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}$	$\frac{N((N-1)(N-2)}{6}$	$\frac{(N-3)((N-2)}{2}$	$\frac{(N-3)((N-6)}{2}$
$\Box\Box\Box$	$\frac{N((N+1)(N+2)}{6}$	$\frac{(N+3)((N+2)}{2}$	$\frac{(N+3)((N+4)}{2}$

From this we see that the total $SU(N_c)^2 \times U(1)_R$ anomaly is proportional to

$$
2N_f (R_Q - 1) + 2N_c \implies R_Q = 1 - \frac{N_c}{N_f}. \tag{132}
$$

We conclude that the anomaly free (with respect to the mixed gauge×global) charges are

$$\begin{aligned} Q &= (\mathbf{N_c}; \bar{\mathbf{N}}_f, \mathbf{1}, \frac{1}{N_c}, 1 - \frac{N_c}{N_f}) \\ \tilde{Q} &= (\bar{\mathbf{N}}_\mathbf{c}; \mathbf{1}, \mathbf{N}_f, -\frac{1}{N_c}, 1 - \frac{N_c}{N_f}). \end{aligned} \quad (133)$$

Note that the global symmetries themselves are not anomaly free which we shall see is an important property.

9 Anomaly matching and Seiberg duality

One of the most interesting uses of anomalies is in 't Hooft anomaly matching. The general idea is as follows. Suppose I have a theory like SQCD above and that it becomes strongly coupled. In such a case we might want to propose a weakly coupled description of the physics, for example involving mesons and hadrons. How can we tell if our proposal is the right one?

't Hooft came up with the following idea. Suppose we wanted to gauge the large global symmetry of the theory. Those symmetries will themselves have anomalies: for example SQCD has a cubic $SU(N_f)^3$ anomaly of $2N_c$. If order to make the theory consistent one would have to add a weakly coupled sector uncharged under the original $SU(N_c)$ to cancel these anomalies. I will call it a *spectator* sector. [Think of the leptons contributing to the cancellation of $SU(2)^2 - Y$ anomalies in the SM.] This sector would be unaffected by the fact that $SU(N_c)$ becomes strongly coupled. [Continue to think of the leptons in the SM.] If the proposed weakly coupled description of the strongly coupled theory is a good one, the spectator sector should cancel the anomalies in that theory too. In other words both descriptions have to have the same set of global anomalies.

In most cases the matching of global anomalies is relatively weak and one can find a number of candidate descriptions. Occasionally they are very strong though. One example is supersymmetric $SU(N_c)$ QCD with N_f flavours, which you have (most likely) already seen in your SUSY lectures. This theory has a magnetic dual description consisting of an $SU(N_f - N_c)$ theory with N_f generations of fundamental and anti-fundamental, together with $N_f \times N_f$ meson singlets which are bound states $\tilde{Q}.Q$ of electric quarks. Here is another example due to Kutasov which I will refer to as the KSS model...

9.1 Anomaly matching in the KSS model

The model is based on an SU(N) gauge group with N_f flavours of quarks and anti-quarks, and an adjoint field of the SU(N) denoted by X. There is a superpotential of the form

$$W_{\mathrm{el}} = X^{k+1} \tag{134}$$

where X is an adjoint field of the SU(N) gauge group and k is an integer. (Note that normally you would think that $k = 2$ would be a marginal operator and $k > 2$ operators would be irrelevant – in both the technical and colloquial sense. However when the theory is strongly coupled anomalous dimensions are large and the operator can come to dominate the flow. How you can tell it is going to do this is a story in itself which I will not have time to get to in due course.)

The symmetry content is

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R\,. \tag{135}$$

The matter content is then summarised by Table 1.

	SU(N)	SU($N_f)_L$	SU($N_f)_R$	U(1)$_B$	U(1)$_R$
Q	$\square$	$\tilde{\square}$	1	$\frac{1}{N_c}$	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
$\tilde{Q}$	$\tilde{\square}$	**1**	$\square$	$-\frac{1}{N_c}$	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
X	adj	1	1	0	$\frac{2}{k+1}$

Tab. 1: *The matter content of the electric theory in the KSS model.*

The R-charges are determined precisely as for the SQCD model above and are completely fixed by the requirement that X^{k+1} has R-charge 2. (As a check note that when $k = 1$ the superpotential is a mass for the adjoint which can then be integrated out - the $R-$charges of normal SQCD are recovered.)

The F-term equation for the adjoint can easily be solved;

$$W' = 0 \equiv X^k. \tag{136}$$

To get the corresponding magnetic theory we need to include set of elementary meson fields associated with composite operators of the electric model. A crucial aspect of the superpotential is that it truncates the chiral ring; that is the equation of motion for X sets $X^k = 0$ along the F-flat directions. This means that when matching the moduli spaces, one need only consider operators upto X^{k-1}. Thus there are k types of meson operator that we denote m_j;

$$m_j = \tilde{Q} X^{j-1} Q, \qquad j = 1 \ldots k. \tag{137}$$

The $j = 1$ object is the meson of usual Seiberg duality. (Again the $k = 1$ model is just the original Seiberg SQCD model if one integrates out the adjoint field.) The field content of the magnetic theory is q, $\tilde{q}$, m_j and x, where x is an adjoint in the magnetic gauge group, and where the elementary magnetic mesons are directly and unambiguously identified with the composite operator m_j. Baryon matching implies that the gauge group of the full (unbroken) magnetic theory is

$$\mathrm{SU}(n) = \mathrm{SU}(kN_f - N_c)\,. \tag{138}$$

The matter content of the magnetic theory is summarised in Table 2.

	$\mathrm{SU}(n)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
q	$\square$	$\square$	1	$\frac{1}{n}$	$1 - \frac{2}{k+1}\frac{n}{N_f}$
$\tilde{q}$	$\tilde{\square}$	$\mathbf{1}$	$\tilde{\square}$	$-\frac{1}{n}$	$1 - \frac{2}{k+1}\frac{n}{N_f}$
x	adj	1	1	0	$\frac{2}{k+1}$
m_j	1	$\tilde{\square}$	$\square$	0	$2 - \frac{4}{k+1}\frac{N_c}{N_f} + \frac{2(j-1)}{k+1}$

Tab. 2: *The matter content of the magnetic theory in the KSS model;* $n = kN_f - N_c$.

The superpotential in the magnetic theory is of the form

$$W_{\mathrm{mag}} = x^{k+1} + \frac{1}{\mu^2} \sum_{j=1}^{k} m_j \tilde{q} x^{k-j} q \tag{139}$$

All that remains is to determine the $\mathrm{U}(1)_B$ charges of the magnetic quarks. They can be fixed by demanding that the $\mathrm{U}(1)_B$ charges of the electric and magnetic baryons match up:

$$B_q = -B_{\tilde{q}} = \frac{N}{n} B_Q, \tag{140}$$

hence the charges $1/n$ for the magnetic quarks.

We may now proceed to anomaly matching. The mixed anomalies are found to be

$$\begin{aligned}
\mathrm{U}(1)_B \times \mathrm{SU}(N_f)_L^2 &: \; 1 \\
\mathrm{U}(1)_B \times \mathrm{SU}(N_f)_R^2 &: \; -1 \\
\mathrm{U}(1)_R \times \mathrm{SU}(N_f)_L^2 &: \; -\frac{2N_c^2}{N_f(1+k)} \\
\mathrm{U}(1)_R \times \mathrm{SU}(N_f)_R^2 &: \; -\frac{2N_c^2}{N_f(1+k)} \\
\mathrm{U}(1)_B \times \mathrm{U}(1)_R^2 &: \; 0 \\
\mathrm{U}(1)_R \times \mathrm{U}(1)_B^2 &: \; -\frac{4}{(1+k)} \\
\mathrm{U}(1)_B &: \; 0 \\
\mathrm{U}(1)_R &: \; -\frac{(2N_c^2+k+3)}{(1+k)}, \qquad (141)
\end{aligned}$$

in both theories. The last two correspond to mixed U(1)-gravity anomalies – there is a contribution from the diagrams with gravitinos in the loop that is obviously universal and has been omitted. The cubic anomalies also match

$$\begin{aligned}
\mathrm{SU}(N_f)_L^3 &: \; N_c \\
\mathrm{SU}(N_f)_R^3 &: \; -N_c \\
\mathrm{U}(1)_B^3 &: \; 0 \qquad (142) \\
\mathrm{U}(1)_R^3 &: \; N_c^2 - 2 - \frac{16N_c^4}{N_f^2(1+k)^3} + (N_c^2-1)\left(\frac{2}{1+k}-1\right)^3 . \qquad (143)
\end{aligned}$$

Clearly this is a very stringent test!! Moreover there are other tests that all confirm that these two theories are dual to each other. One important test is that one can deform the superpotential to break the original SU(N_c) as

$$\mathrm{SU}(N) \to \mathrm{SU}(r_1) \times \mathrm{SU}(r_2) \ldots \mathrm{SU}(r_k) \times \mathrm{U}(1)^{k-1}. \qquad (144)$$

The broken model in the magnetic theory becomes

$$\mathrm{SU}(n) \to \mathrm{SU}(\bar{r}_1) \times \mathrm{SU}(\bar{r}_2) \ldots \mathrm{SU}(\bar{r}_k) \times \mathrm{U}(1)^{k-1} \qquad (145)$$

where

$$\bar{r}_i = F_Q - r_i. \tag{146}$$

Note that the matching of the sub-theories $\mathrm{SU}(r_i) \leftrightarrow \mathrm{SU}(\bar{r}_i)$ is the usual Seiberg duality: the original theory is a GUT for a product of standard SQCD duals!!

10 Topology, instantons and the θ-vacuum

10.1 The SU(2) instanton

To this point we have been discussing anomalous violation of current conservation in a gauge background, with the vague notion that there are *some* gauge configurations that exist that can be responsible for, for example, changing the axial charge. We now study these configurations and discuss their connection with topology.

Consider violation of axial current J_5 in a gauge background. Recall that the anomaly was given by

$$\partial.J_5 = \partial.K \tag{147}$$

where

$$K^\alpha = \frac{1}{4\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\mathrm{tr}\left[A_\beta\partial_\gamma A_\delta + \frac{2}{3}A_\beta A_\delta A_\gamma\right], \tag{148}$$

and

$$\partial K = \varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}. \tag{149}$$

In the language of forms, we would write $dK = \mathbf{tr}F\wedge F$ and $K = \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)$. In order to simplify notation I will often drop the wedge where there is no possibilty for confusion. For definitiness and simplicity we will consider the SU(2) case.

Now when quantizing the theory, one expects that finite classical actions might give additional contributions to the partition function, whereas quantizing around infinite actions would give zero. Therefore we seek finite action non-trivial classical solutions to the equations of motion that can change J_5. What configurations could give finite action? We need $\int_V F\wedge F$ to be finite which means that $F_{\mu\nu}\to 0$ as $r\to\infty$ or equivalently $A\to$pure gauge:

$$\lim_{r\to\infty} A_\mu = U^{-1}\partial_\mu U. \tag{150}$$

Gauge transformations in SU(2) are represented by the 2x2 unitary matrices, $U = e^{i\alpha_a \tau^a} \equiv u_0 + iu_a\tau^a$. The unitarity condition $U^\dagger U = \mathbf{1} \implies u_0^2 + u_a u_a = 1$ which also defines S_3: hence SU(2) $\simeq S_3$. This makes SU(2) instantons especially simple to configure as working in Euclidean space we can map the gauge transformaiton directly onto the sphere at infinity.

Let us therefore try $A_\mu = f(r)U^{-1}\partial_\mu U$ where $\lim_{r\to\infty} f(r)$=1 and U is a transformation that is a function depending only on the coordinates in S_3 (i.e. it is independent of r). The nice thing about this ansatz is that the divergence theorem,

$$\int_V \partial K = \int_{S_3} K \quad \left[\equiv \int_{S_3} dS^\alpha K_\alpha\right]$$

involves an integral over S_3 of K_α projected along the area element vector: in other words we need only consider the component of K_α along the radial direction K_r: in particular $K_r \sim \varepsilon^{r\beta\gamma\delta}$... and the indices $\alpha\beta\gamma$ are orthogonal to the radial index. Therefore we can use antisymmetry of indices and $U^\dagger U = 1$ as follows:

$$\begin{aligned}
\varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[A_\beta \partial_\gamma A_\delta\right] &= f^2 \varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(U^\dagger\partial_\beta U\right)\partial_\gamma\left(U^\dagger\partial_\delta U\right)\right] \\
&= f^2 \varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(U^\dagger\partial_\beta U\right)\left(\partial_\gamma U^\dagger\right)UU^\dagger\left(\partial_\delta U\right)\right] \\
&= -f^2 \varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(U^\dagger\partial_\beta U\right)\left(U^\dagger\partial_\gamma U\right)\left(U^\dagger\partial_\delta U\right)\right]. \qquad (151)
\end{aligned}$$

Therefore

$$\begin{aligned}
4\pi^2 K_r &= f^2(\frac{2}{3}f - 1)\varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(U^\dagger\partial_\beta U\right)\left(U^\dagger\partial_\gamma U\right)\left(U^\dagger\partial_\delta U\right)\right] \\
&\to -\frac{1}{3}\varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(U^\dagger\partial_\beta U\right)\left(U^\dagger\partial_\gamma U\right)\left(U^\dagger\partial_\delta U\right)\right]. \qquad (152)
\end{aligned}$$

Actually a simpler way to do this follows from the fact that $F = dA + A^2$ (wedges implied) so that

$$K = \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right) = \mathrm{tr}\left(A(F - A^2) + \frac{2}{3}A^3\right) = -\frac{1}{3}\mathrm{tr}\left(A^3\right).$$

Whichever way you skin it, we have

$$4\pi^2 \int_{S_3} K = -\frac{1}{3}\int dS_3 \varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(U^\dagger\partial_\beta U\right)\left(U^\dagger\partial_\gamma U\right)\left(U^\dagger\partial_\delta U\right)\right]. \qquad (153)$$

The simplest non-trivial U to consider is the identity map

$$U = \frac{1}{r}\left(x^0\mathbf{1} - ix^a\tau^a\right). \tag{154}$$

Then $U^\dagger\partial_\alpha U = -\frac{x_\alpha}{r^2} - i\frac{\sigma_\alpha}{r}$ where $\sigma^\alpha = (i\mathbf{1}, \tau^a)$, and hence by the vanishing of τ^a traces and antisymmetry in indices

$$\begin{aligned}\varepsilon^{r\beta\gamma\delta}\mathrm{tr}\left[\left(-\frac{x_\beta}{r^2} - i\frac{\sigma_\beta}{r}\right)\left(-\frac{x_\gamma}{r^2} - i\frac{\sigma_\gamma}{r}\right)\left(-\frac{x_\delta}{r^2} - i\frac{\sigma_\delta}{r}\right)\right] &\equiv i\varepsilon^{\beta\gamma\delta}\mathrm{tr}\left[\frac{\sigma_\beta\sigma_\gamma\sigma_\delta}{r^3}\right] \\ &= i\frac{1}{2}\varepsilon^{\beta\gamma\delta}\mathrm{tr}\left[\frac{\sigma_\beta 2i\varepsilon_{\gamma\delta\beta'}\sigma_{\beta'}}{r^3}\right] \\ &= -2\mathrm{tr}\left[\frac{\sigma_\beta\sigma_\beta}{r^3}\right] = -12/r^3, \end{aligned} \tag{155}$$

where on the RHS I took the $\beta\gamma\delta$ indices to be the basis of defining indices of $S_3 \simeq \mathrm{SU}(2)$.

Thus this solution has

$$\int_{S_3} K = \frac{1}{12\pi^2}\mathrm{Vol}(S_3)12 = 2. \tag{156}$$

Topologically, transitions in J_5 can be arranged as taking the infinite R_4 volume and splitting it into 2 regions. By adding the point at infinity we can map it to S_4, with the origin at the north pole and $t \to \infty$ at the south pole. We will consider a case where $A = 0$ in the southern hemisphere $V_S = \{x \in R_4; |x| \geq L - \varepsilon\}$, with the interesting configuration being in the northern hemisphere, $V_N = \{x \in R_4; |x| \leq L + \varepsilon\}$. All the interesting topological information is then contained in the transition function between northern and southern hemisphere on the equator which is topologically equivalent to S_3. The additional piece in the action is $\int d^4x\partial K = \int_{V_N} F \wedge F = \int_{S_3} K = \int_{S_3} K$. We would then compute the Q_5 change between the north and south poles as $\Delta Q_5 = \Delta\int d^3xJ^0 = \Delta\int d^4x\partial_0 J^0 \equiv \int d^4x\partial K = 2$. In other words the single (anti) instanton configuration eats (vomits) a single quark anti-quark pair. If there had been N_f fermions flavours we would have found $\Delta Q_5 = 2N_f$.

These numbers are topological. In fact the Pontryagin number is defined as

$$\nu = \frac{1}{32\pi^2}\int d^4x\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{157}$$

In this example $\nu = 1$. (In the forms language $\nu = \frac{1}{8\pi^2}\int F \wedge F$). It is simple to find higher instanton numbers by starting with U_1 in the northern hemisphere and U_1 in the southern.

This configuration clearly has $\nu = 2$ but can be smoothly deformed so that the southern hemisphere is empty and the northern one has $U_2 = U_1 U_1$. General ν configurations are of the form

$$U_\nu = \frac{1}{r^\nu}\left(x^0 \mathbf{1} - i x^a \tau^a\right)^\nu . \tag{158}$$

The index labels the elements of the homotopy group $\pi_3(SU(2)) = \mathbf{Z}$: that is ν labels the homotopy classes of the mapping $g : S_3 \to \mathrm{SU}(2)$. (In other words how many times $SU(2)$ is wrapped over S_3.) Clearly since $\pi_3(S_1)$ is trivial, abelian groups cannot have such configurations. But for other simple groups, G, there is a theorem due to Bott that says a continuous mapping of S_3 into G can be deformed into a mapping of S_3 into an $SU(2)$ subgroup of G. Since $\pi_3(SU(N)) = \mathbf{Z}$ we conclude that similar instanton configurations exist for them as well.

It is interesting to find possible forms for the instanton – i.e. to find the solutions to the equation of motion and in particular the precise form of $f(r)$. (We will focus on the one instanton solution.) An interesting class of solutions to study are the self-dual (or anti self-dual) ones, satisfying $F_{\mu\nu} = *F_{\mu\nu}$. This is because the action is related to the Pontryagin number as

$$S = \frac{1}{4}\int F_{\mu\nu} F^{\mu\nu} = \frac{1}{4}\int F_{\mu\nu} * F^{\mu\nu} = \frac{\nu}{8\pi^2}. \tag{159}$$

In fact this saturates a bound (the BPS bound) similar to the Bogomolny bound since

$$\frac{1}{4}\int (F_{\mu\nu} - *F_{\mu\nu})^2 \geq 0$$

and therefore expanding (and using $*F * F = FF$) we find

$$S \geq \frac{\nu}{8\pi^2}. \tag{160}$$

Hence the self-dual solutions are a local minimum of the action. The equation $F_{\mu\nu} = *F_{\mu\nu}$ is a little easier to solve because it is only first order. With our ansatz one finds

$$r\frac{df}{dr} = 2f(1-f)$$

which has solutions

$$f = \frac{r^2}{r^2+\rho^2}, \tag{161}$$

where the free parameter ρ is the size of the instanton. Substituting in one finds

$$F_{\mu\nu} = \frac{4}{r^2/\rho^2 + 1}\sigma_{\mu\nu}, \tag{162}$$

where $\sigma_{\mu\nu} = \frac{1}{4}[\sigma_\mu, \sigma_\nu]$. Returning to our earlier discussion about the sorts of A^μ that can contribute to $\dot{Q}_5$, we see that while $F \sim 1/r^2$ the crucial part of the solution is that the gauge field tends to an $A^\alpha \sim -i\sigma^\alpha/r$ pure gauge configuration. (If we'd been smart – or pretentious – we could have guessed the action back then).

10.2 The theta vacuum

Finally we briefly discuss a crucial and puzzling feature of the standard model which is the θ-vacuum and theta parameter. From the fact that $\pi_3(SU(3)) = Z$ we conclude that instantons can change the vacuum: in other words the classical empty vacuum on which we were so confidently building our quantum theory is not even gauge invariant!

Indeed suppose we perform a large single instanton gauge transformation in the QCD gauge fields. This then results in a shift of the effective Lagrangian as

$$\mathcal{L} \to \mathcal{L} + \frac{1}{32\pi^2}\mathrm{tr}F * F. \tag{163}$$

The extra piece $\frac{1}{32\pi^2}\mathrm{tr}F * F$ is reinterpreted as a local term in the Lagrangian that was generated by a large (i.e. topologically non-trivial) gauge transformation.

In order to build a gauge covariant vacuum for QCD we need to start with a fiducial vacuum $|0\rangle$ and collect all the possible vacua related to it by large ν instanton gauge transformations $|\nu\rangle$. A gauge invariant (up to a phase) vacuum would then look like

$$|\theta\rangle = \sum_\nu e^{-i\nu\theta}|\nu\rangle, \tag{164}$$

where $0 < \theta < 2\pi$ is an arbitrary parameter. A single instanton gauge transformation (call it U_1) now sends $\nu \to \nu + 1$ but of course because we have an infinite sum the vacuum transforms into itself modulo a phase

$$U_1|\theta\rangle = e^{i\theta}|\theta\rangle. \tag{165}$$

This phase can be written into the action as a local term in the Lagrangian

$$\mathcal{L} = \mathcal{L}_{classical} + \frac{\theta}{32\pi^2}\mathrm{tr}F * F. \tag{166}$$

To see this consider the partition function:

$$\langle\theta|e^{iS}|\theta\rangle = \int \mathcal{D}A e^{iS[A]}$$

The path integral over A contains contributions from each instanton sector (i.e. there is a contribution from quantizing around each instanton background). We can split A into the classical instanton background piece and the quantum fluctuations, and write the sum over instanton sectors where now $\mathcal{D}A$ is over small fluctuations only, at the same time allowing an arbitrary phase to appear proportional to the instanton number:

$$\begin{aligned}\langle\theta|e^{-iHt}|\theta\rangle &= \sum_{\nu\nu'}\langle\nu|e^{i(\nu-\nu')\theta}e^{iS[A_{\nu-\nu'}]}|\nu'\rangle \\ &\propto \sum_n \int \mathcal{D}A_n e^{in\theta}e^{iS[A_n]} \\ &= \sum_n \int \mathcal{D}A e^{iS[A]+\frac{i\theta}{32\pi^2}\int \mathrm{tr}F*F}. \end{aligned} \tag{167}$$

This term is not constrained, is not CP invariant and is measurable in neutron electric dipole moment experiments: it is found to be less than 10^{-9} and is consistent with zero. But no known principles (even anthropic ones) say that it should be small. Various ideas, such as the Peccei-Quinn mechanism have been put forward to explain why it is zero.

Extra Questions

1. Prove
$$\delta\varphi_i = i[Q,\varphi_i]. \tag{168}$$
for external symmetries.

2. Show that $\delta S = 0$ and Noether's theorem imply covariant current conservation of the form
$$(D.J)^a = 0 \tag{169}$$

where

$$D^b_{\mu a} = \delta^b_a \partial_\mu + f^b_{ac} A^c_\mu . \tag{170}$$

3. The anomaly we derived in lectures for QED was

$$\partial . J_5 = \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} . \tag{171}$$

Consider an abelian gauge theory in which only left-handed fermions couple to the gauge fields $\mathcal{L} = i\bar{\psi}_R \not{\partial} \psi_R + i\bar{\psi}_L \not{D} \psi_L - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. Derive the anomalous Ward identities for this case. Use them to show that the anomaly of the left-handed current $J_L = \bar{\psi}_L \gamma^\mu \psi_L$ is

$$\partial . J_L = -\frac{g^2}{96\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} . \tag{172}$$

[Hint: There is a J_L current at each vertex so the symmetry factors are different. Also you may use the diagrammatic results from lectures.]

4. Use the power counting argument and the fact that the chiral Ward identity has a single J_5^μ vertex and interchangeable J^μ vertices, to show that the chiral anomaly in any (even) number of dimensions is finite and appears at one-loop. What is the diagram in 6 dimensions?

5. The Lagrangian for quarks is of the form $\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi$ where D is the covariant derivative for colour and ψ stands for the three flavours of Dirac fermions, u, d or s. Construct the currents and conserved charges for the global flavour symmetry $SU(3)_L \times SU(3)_R$. Show that the Noether charges are generators for this symmetry. *[Hint: you may use the equal time commutators* $\{\psi_i(\mathbf{x}), \psi_j^\dagger(\mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y})\delta_{ij}$*].*

6. For some reason I wish to construct a consistent $SU(N)$ theory where each generation contains a some antifundamentals, plus a single symmetric and antisymmetric. The contributions of fundamentals, symmetrics and antisymmetrics to the cubic $SU(N) - SU(N) - SU(N)$ anomaly are 1, $N + 4$, $N - 4$ respectively. How many antifundamentals do I need? If I also allow antisymmetrics in the conjugate representation, find a single combination of fields that will cancel anomalies for any N.

7. **Homework question: Use the Ward identity to derive the non-Abelian**