Quantum Mechanics – Problems Sheet (2007-2008)

1. Show that square matrices A and B satisfy

$$(A+B)(A-B) = A^2 - B^2,$$

if and only if they commute; *i.e.*

$$[A,\, B] = 0.$$

2. Two matrices U and H are related by

$$U = e^{ia\,H},$$

where a is real.

a. If H is Hermitian, show that U is unitary.

b. If U is unitary, show that H is Hermitian (to within terms of the form $\frac{2\pi n}{a}$)

3.

a. Show that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is symmetric (Hermitian). Determine its eigenvectors and eigenvalues. Show that the eigenvectors are mutually orthogonal.

b. Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

Note that the eigenvectors are **not** mutually orthogonal.

4. If $f_1(x)$ and $f_2(x)$ are complex-valued continuous functions of real $x \in [-1, 1]$ show that

$$\int_{-1}^{1} f_1^* f_2 \, dx$$

defines an inner product $f_1 \cdot f_2$. (You must check that $f_2 \cdot f_1 = (f_1 \cdot f_2)^*$, that $f_1 \cdot (\alpha f_2 + \beta f_3) = \alpha f_1 \cdot f_2 + \beta f_1 \cdot f_3$ for all complex α, β and that $f_1 \cdot f_1 \geq 0$ with equality iff $f_1 \equiv 0$). Give some examples of functions that are orthogonal with respect to this inner product. (Remember Fourier series, *etc.*?) Determine constants

A_r so that $f_r = A_r \sin(r\pi x), r = 1, 2, \ldots$ are normalised. Normalise the Legendre polynomials similarly.

5. If $|a\rangle$ is a column matrix with complex entries, what is $\langle a|$? What is the inner product? Check that it has all the required properties given above.

6. Show that $(\widehat{A}\widehat{B})^\dagger = \widehat{B}^\dagger \widehat{A}^\dagger$ and generalise to the product of any number of operators. Show that $(|a\rangle\langle b|)^\dagger = |b\rangle\langle a|$.

7. If $\widehat{A}$ and $\widehat{B}$ are self-adjoint prove that $\alpha\widehat{A} + \beta\widehat{B}$ is self-adjoint if α and β are real, prove that $\widehat{A}\widehat{B}$ is self-adjoint if $[\widehat{A}, \widehat{B}] = 0$ and, if $\widehat{C} = [\widehat{A}, \widehat{B}]$, prove that $\widehat{C}^\dagger = -\widehat{C}$.

8. Prove that $\langle\alpha|\widehat{A}|\alpha\rangle$ is real and $\langle\alpha|\widehat{A}^2|\alpha\rangle \geq 0$ for all $|\alpha\rangle$ if $\widehat{A} = \widehat{A}^\dagger$. What about the converses?

9. Let the f's in qu.4 now be $\mathbf{C}^\infty$ and vanish at $x = \pm 1$. Determine constants a, b so that

$$a\frac{d}{dx} \qquad \text{and} \qquad b\frac{d^2}{dx^2}$$

are self-adjoint with respect to the integral inner product. With a, b so determined find other possible conditions on the f's at ± 1 that make these two operators self-adjoint. In each case solve the eigenvalue problem for each operator.

10. Show that $u(x) = \exp(-x^2/2)$ is an eigenvector of

$$\widehat{A} = \frac{d^2}{dx^2} - x^2$$

and find the corresponding eigenvalue.

11. If $\widehat{A}$ and $\widehat{B}$ both commute with $[\widehat{A}, \widehat{B}]$, show that

$$[\widehat{A}, \widehat{B}^n] = n\widehat{B}^{n-1}[\widehat{A}, \widehat{B}] \qquad \text{and} \qquad [\widehat{A}, e^{\widehat{B}}] = e^{\widehat{B}}[\widehat{A}, \widehat{B}].$$

12. Show that $[d/dx, x^n] = nx^{n-1}$ and evaluate

$$[x\frac{d}{dx}, x\frac{d^2}{dx^2}] \qquad \text{and} \qquad [-i\hbar\frac{d}{dx}, f(x)].$$

13. The scalar product of states $\psi(r, p)$ and $\phi(r, p)$ is defined to be

$$\int_0^\infty dp \int_0^\infty r\psi^*(r, p)\phi(r, p)\, dr.$$

For what values of the complex constants a, b is the operator $a(p\,\partial/\partial p - br\,\partial/\partial r)$ self-adjoint?

14. If $\widehat{U}$ and $\widehat{V}$ are unitary, what about $\widehat{U}+\widehat{V}$, $\quad \widehat{U}\widehat{V}$, $\quad 2\widehat{U}$, $\quad i\widehat{U}$?

15. If $|1\rangle$ and $|2\rangle$ are degenerate eigenstates of a self-adjoint operator $\widehat{R}$ show that $\langle 1|\widehat{Q}|2\rangle = 0$ for any operator $\widehat{Q}$ where $\widehat{R}\widehat{Q}\widehat{R} = -\widehat{Q}$.

16. If $\widehat{H} = \widehat{p}^2/2m$ show that $[\widehat{H}, \widehat{p}] = 0$. Show that any linear combination of eigenvectors of $\widehat{p}$ with eigenvalues k and $-k$ is an eigenvector of $\widehat{H}$ with eigenvalue E if $E = k^2/2m$.

17. Prove that $(\Delta A)^2 {=} \langle (A - \langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$.

18. Prove that $\mathrm{Im}\langle A\rangle = 0$ iff $\widehat{A} = \widehat{A}^\dagger$.

19. Prove that if $\langle A\rangle = \langle\alpha|\widehat{A}|\alpha\rangle$ then $\Delta A = 0$ iff $|\alpha\rangle$ is an eigenvector of $\widehat{A}$.

20. What is meant by an *observable* in quantum mechanics? What is the condition that two observables are simultaneously measurable? The observables β_μ (where $\mu = 1, 2, \ldots, N$) satisfy

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\lambda} + \beta_\lambda\delta_{\nu\mu}.$$

Show that the N observables $\eta_\mu \equiv 2\beta_\mu^2 - 1$ are simultaneously measurable and take values ± 1.

21. Show that the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are real. What are they? Do the eigenvectors form a complete set? Could this matrix represent an observable in quantum mechanics? Why?

22. For a certain quantum system an observable quantity E is represented by the self-adjoint operator $\widehat{E}$ with eigenvalues $\{E_n; \quad n = 0, 1, \ldots\}$ and corresponding normalised eigenvectors $\{|n\rangle\}$. State the possible outcomes of measuring E, calculate their relative probabilities of realisation, and determine $\langle E\rangle$ and ΔE in each case when the state vector $|\alpha\rangle$ of the system is: $|0\rangle$; $|1\rangle$; $|0\rangle + |1\rangle$; $|0\rangle - |1\rangle$; $|0\rangle + 2|1\rangle$; $e^{i\theta}|0\rangle + 2e^{i\phi}|1\rangle$. Don't forget normalisation!

23. A particle with 'spin one-half' is assigned a state vector that is a complex two-component column vector (*i.e.* a complex 1×2 matrix). Suggest a scalar product in the space of such states. What are the linear operators in this space? How many different observables (*i.e.* linearly independent self-adjoint linear operators) are there for this system? Given that 'spin one-half' refers to the system's angular momentum, suggest what observable quantities may be identified with these operators.

24. What are the possible values of the observable $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$? What is the probability of measurement yielding each value on the state $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$? Answer the same question for

$$\begin{pmatrix} 1 & 3+4i \\ 3-4i & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

25. For one particle in a potential state, give reasons, whether or not each of the following pairs of observables is simultaneously measurable: position, momentum; potential energy, position; kinetic energy, momentum; total energy, momentum; total energy, position.

26. A particle is described by a state vector $|\psi\rangle$ that is a complex-valued differentiable function ψ of a real variable $x \in (-\infty, \infty)$. Verify that a suitable scalar product is

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\phi(x)\, dx$$

where $\psi(|x| \to \infty) \to 0$ fast enough. It is proposed that the momentum observable P be represented by the operator $\widehat{P} = -i\hbar\, d/dx$. Show that $\widehat{P}$ is self-adjoint. If the particle's position is represented by the linear operator $\widehat{X} = x$ check that $\widehat{X}$ is self-adjoint and that $[\widehat{X}, \widehat{P}] = i\hbar$. Consider a particle with state vector $\psi(x) = C\exp(-x^2/a^2)$ where C and a are constant and a is real. Normalise ψ to unity and calculate $\langle X\rangle$, $\langle P\rangle$, $\langle P^2\rangle$ and $\langle X^2\rangle$ for this particle. Verify agreement with the Uncertainty Relation.

27. State the commutation rules for position and conjugate momentum in quantum mechanics. Derive the Heisenberg Uncertainty Relation for Cartesian components of position and linear momentum, assuming the Schwarz Inequality. A particle of mass m moving on a line is known to be in a segment of length L. Prove that its kinetic energy is bounded below by $\hbar^2/8mL^2$.

28. If $\widehat{H} = \widehat{p}^2/2M + \widehat{V}(\widehat{x})$ where $[\widehat{x}, \widehat{p}] = i\hbar$ then by considering a suitable expectation value of the double commutator $[[\widehat{x}, \widehat{H}], \widehat{x}]$ obtain the 'sum rule'

$$\sum_n (E_n - E_m)|\langle n|\widehat{x}|m\rangle|^2 = \frac{\hbar^2}{2M}$$

where $|n\rangle$ is the normalised eigenstate of $\widehat{H}$ belonging to eigenvalue E_n.

29. If $\widehat{L}_k = \epsilon_{kmn}\widehat{r}_m\widehat{p}_n$ and $[\widehat{r}_m, \widehat{p}_n] = i\hbar\delta_{mn}$ then prove that

$$[\widehat{L}_m, \widehat{L}_n] = i\hbar\epsilon_{mnk}\widehat{L}_k.$$

Calculate $[\widehat{r}_m, \widehat{L}_n]$, $[\widehat{p}_m, \widehat{L}_n]$, $[\widehat{r}_m, \widehat{L}^2]$ and $[\widehat{p}_m, \widehat{L}^2]$. Prove that

$$[\widehat{r}^2, \widehat{L}_k] = [\widehat{p}^2, \widehat{L}_k] = 0.$$

30. If $\widehat{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$ and $\widehat{J}_z|j,m\rangle = m\hbar|j,m\rangle$ calculate

$$\langle j,m|\widehat{J}^2|j',m'\rangle, \qquad \langle j,m|\widehat{J}_\pm|j',m'\rangle \qquad \text{and} \qquad \langle j,m|\widehat{J}_{x,y,z}|j',m'\rangle.$$

For $j=\frac{1}{2}$ and $j=1$ write out the matrices whose elements, indexed by m,m', are these numbers. (The usual convention is to place elements with $m=m'=j$ in the top left corner and index rows and columns with m & m' decreasing).

31. Evaluate $[\widehat{J}_z, \widehat{J}_x\widehat{J}_y + \widehat{J}_y\widehat{J}_x]$ and hence show that in the state in which $\widehat{J}_z$ has eigenvalue b and $\widehat{J}^2$ has eigenvalue a the expectation values of $\widehat{J}_x^2$ and $\widehat{J}_y^2$ are each $(a-b^2)/2$. Calculate also $\Delta J_{x,y}$.

32. A spin-$\frac{1}{2}$ particle is in a state where a measurement of its z-component of spin is equally likely to yield results $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$. Find $\langle J_x\rangle$ and $\langle J_y\rangle$.

33. If $\widehat{J}_1$, $\widehat{J}_2$ and $\widehat{J}_3$ are angular momentum operators and $\widehat{A}_1$, $\widehat{A}_2$ and $\widehat{A}_3$ are operators obeying

$$[\widehat{J}_j, \widehat{A}_k] = i\epsilon_{jkl}\hbar\widehat{A}_l$$

show that $\widehat{A}_+ \equiv \widehat{A}_1 + i\widehat{A}_2$ satisfies both

$$[\widehat{J}_3, \widehat{A}_+] = \hbar\widehat{A}_+ \qquad \text{and} \qquad [\widehat{J}^2, \widehat{A}_+] = 2\hbar(\widehat{A}_+\widehat{J}_3 - \widehat{A}_3\widehat{J}_+) + 2\hbar^2\widehat{A}_+,$$

where as usual $\widehat{J}^2 = \widehat{J}_1^2 + \widehat{J}_2^2 + \widehat{J}_3^2$ and $\widehat{J}_+ = \widehat{J}_1 + i\widehat{J}_2$. Hence show that if $|j,m\rangle$ is a conventional eigenstate of $\widehat{J}^2$ and $\widehat{J}_3$ then

$$(\widehat{A}_1 + i\widehat{A}_2)|j,j\rangle = \lambda|j+1,j+1\rangle$$

for some number λ. Evaluate likewise $(\widehat{A}_1 - i\widehat{A}_2)|j,-j\rangle$.

34. Three operators $\widehat{A}_{1,2,3}$ satisfy $[\widehat{A}_j, \widehat{A}_k] = i\epsilon_{jkl}\widehat{A}_l$, and the sum of their squares is twice the unit operator. Show, by using the quantum-mechanical theory of angular momentum or otherwise, that $\widehat{A}_j^3 = \widehat{A}_j$ for $j=1,2,3$.

35. For the quantum one-dimensional simple-harmonic oscillator calculate

$$\langle r|\widehat{p}|s\rangle, \qquad \langle r|\widehat{x}|s\rangle, \qquad \langle r|\widehat{p}^2|s\rangle/2m, \qquad \tfrac{1}{2}m\omega^2\langle r|\widehat{x}^2|s\rangle \qquad \text{and} \qquad \langle r|\widehat{x}^3|s\rangle,$$

where $\widehat{H}|r\rangle = (r+\frac{1}{2})\hbar\omega|r\rangle$. (Relate $\widehat{x}$ & $\widehat{p}$ to $\widehat{a}$ & $\widehat{a}^\dagger$).

36. An oscillator has Hamiltonian $\widehat{H} = \widehat{p}^2 + \widehat{x}^2$ where $[\widehat{x},\widehat{p}] = i$ and $\widehat{x} = \widehat{x}^\dagger$, $\widehat{p} = \widehat{p}^\dagger$. Show that an operator $\widehat{A}$ can be defined (up to a constant factor) so that the normalised

n-th energy eigenstate $|n\rangle$ can be obtained from the normalised ground state $|0\rangle$ in the form $|n\rangle = c_n \widehat{A}^n|0\rangle$ for some real number c_n. Determine c_n for your choice of $\widehat{A}$.

37. Calculate by time-averaging over one cycle the mean values of position, momentum, kinetic energy and potential energy for a classical simple-harmonic oscillator of total energy $(n+\frac{1}{2})\hbar\omega$. Compare with the corresponding results in quantum mechanics.

38. Calculate Δp and Δx for a quantum-mechanical one-dimensional simple-harmonic oscillator of definite energy $(n+\frac{1}{2})\hbar\omega$. Check satisfaction of the Uncertainty Relation. What happens for a classical oscillator of the same energy?

39. Find an operator $\widehat{C}(f)$ such that $|f\rangle = \widehat{C}|0\rangle$ is a normalised eigenstate of the annihilation operator $\widehat{a}^\dagger$ with eigenvalue f. Here $|0\rangle$ is the ground state of the simple-harmonic oscillator and start by expressing $|f\rangle$ as a sum of simple-harmonic oscillator energy eigenstates. Hence calculate $\langle f|f'\rangle$. Can you find a similar construction for eigenstates of $\widehat{a}$?

40. A quantum one-dimensional simple-harmonic oscillator is in a state where an energy measurement would yield either $\frac{1}{2}\hbar\omega$ or $\frac{3}{2}\hbar\omega$ with equal likelihood but nothing else. At time $t=0$ the mean value of momentum is as large as is possible in these circumstances. Calculate $\langle E\rangle$, $\langle T\rangle$, $\langle V\rangle$, $\langle p\rangle$ and $\langle x\rangle$ as functions of time t and compare with the corresponding classical values for a oscillator of energy $\hbar\omega$.

41. A system has Hamiltonian

$$\widehat{H} = \tfrac{1}{2}\hbar \begin{pmatrix} 3 & i \\ -i & 3 \end{pmatrix}$$

and at time $t=0$ has state vector

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If a measurement of

$$\widehat{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

were made at any time t what values might be found, and with what probabilities? Calculate $\langle Q\rangle$ and $\langle[\widehat{H},\widehat{Q}]\rangle$ and verify the relation between them.

42. Write down the Schrödinger Equation for the time evolution of a state vector in quantum mechanics and give a solution as an expansion in the energy eigenbasis. A certain system has an observable B that can take values 1, 2 or 3 and its Hamiltonian operator acts on the corresponding normalised state vectors as follows:

$$\begin{aligned} \widehat{H}|1\rangle &= |1\rangle + 2i|3\rangle, \\ \widehat{H}|2\rangle &= |2\rangle, \\ \widehat{H}|3\rangle &= -2|3\rangle - 2i|1\rangle. \end{aligned}$$

At one instant B is observed to have value 1. The system is left undisturbed for a time t and then B is re-measured. What values may then be found, with what probability, and what is the expectation value of B? When, if ever, would such a re-measurement certainly give value 3? What are the corresponding results if the original observation of B had given value 2 instead?

43. Derive the equation of motion in the Heisenberg Picture.

44. Write down the most general 2×2 Hermitian matrix and hence show that there are only four such that are linearly independent over $\mathbf{R}$. What about 3×3,... $n \times n$?

45. Show that the Pauli matrices σ obey the SU_2 algebra $[\sigma_k, \sigma_l] = 2i\epsilon_{klm}\sigma_m$.

46. The states $\{|n\rangle; n = 1, 2, \ldots\}$ are eigenstates of the observable $\widehat{G}$ with distinct eigenvalues g_n respectively. They form an orthonormal basis for the quantum system's state space. For $n > 2$, $|n\rangle$ is also an eigenstate of the system's Hamiltonian, $\widehat{H}$. Furthermore $\langle 1|\widehat{H}|1\rangle = 0$, $\langle 1|\widehat{H}|2\rangle = \sqrt{3}v$ and $\langle 2|\widehat{H}|2\rangle = 2v$. Here v is a real number. Two measurements are made of G, a time t apart. Calculate the probabilities of the possible results of the second measurement if the first gave g_1. What if the first gave g_4?

47. Prove that if

$$\langle x|\widehat{p}|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\alpha\rangle$$

then

$$\langle x|\widehat{p}^2|\alpha\rangle = -\hbar^2\frac{\partial^2}{\partial x^2}\langle x|\alpha\rangle \qquad \text{and} \qquad \langle \mathbf{r}|\widehat{L}|\alpha\rangle = -i\hbar\mathbf{r} \times \nabla\langle \mathbf{r}|\alpha\rangle.$$

48. Prove that $\widehat{p} = -i\hbar\partial/\partial x$ is self-adjoint in a space of wavefunctions obeying periodic boundary conditions.

49. Construct the energy eigenfunctions $\phi_1(x)$ and $\phi_2(x)$ for the one-dimensional simple-harmonic oscillator by applying $\widehat{a}$ to the ground-state eigenfunction $\phi_0(x)$. Check orthonormality explicitly.

50. Work out the details of the series solution for the energy eigenvalue differential equation for the one-dimensional simple-harmonic oscillator (see *e.g.* Schiff *pps.* 66–76) and check agreement with the results of the preceding question for $\phi_{0,1,2}$.

51. Determine the probability that a quantum particle undergoing simple-harmonic motion on a line, in its ground state, will be found outside its classical limits.

52. For a quantum particle moving freely on the x-axis between impenetrable walls at $x = 0$ and $x = a$ write down energy eigenfunctions and eigenvalues. Then

if the particle is in the n^{th} energy state then find:

(i) $\langle x \rangle$ and Δx, and compare with corresponding classical results;

(ii) the momentum probability distribution function;

(iii) the average force exerted by the particle on the well's walls.

if the particle has, at a given instant, wavefunction $\psi(x) = Ax(a-x)$ where A is a constant, find the probability distribution for energy measurements and find $\langle E \rangle$ and ΔE.

if the particle is in its ground state when suddenly the well-width is doubled, calculate probabilities that its energy is thereby **(i)** unchanged; **(ii)** decreased; **(iii)** increased.

53. For a free particle on the x-axis confined between $x = 0$ and $x = a$ estimate its least energy using the Uncertainty Principle.

54. A particle is trapped in a one-dimensional box of length L. Its energy is $E = \pi \frac{\hbar^2}{2mL^2}$, where m is its mass. Where in the box is the particle the most likely to be found?

55. Find the energy levels of a particle of mass m in the one-dimensional potential: $V(x) = \frac{1}{2} m\omega^2 x^2$ for $x > 0$ and $V(x) = \infty$ for $x < 0$.

56. A particle of mass m moves on the x-axis with potential $V(x) = \infty$ for $x < 0$, $V(x) = 0$ for $0 < x < a$ and $V(x) = V_0 > 0$ for $x > a$. What continuity conditions must the wavefunction obey at $x = 0$ and at $x = a$?

Show that if the particle is incident from the right with energy $E > V_0$ then it is always reflected, and calculate the wavefunction in $0 < x < a$.

Find an equation determining possible bound states and show that none exist unless $V_0 a^2 > (\pi\hbar)^2/8m$. Write down a condition for exactly n bound states.

57. A classical particle of mass m and energy E (> 0) moves on the x-axis in a potential

$$\begin{aligned} V(x) \quad &= 0 \qquad x < 0, \quad x > a > 0; \\ &= V_0 \quad (> 0) \qquad 0 \le x \le a. \end{aligned}$$

Initially the particle is at $x < 0$ with positive momentum; describe briefly its subsequent motion according to classical mechanics. Formulate the corresponding problem in quantum mechanics and for $E > V_0$ derive the probability of transmission to $x > a$ as

$$P = (\cos^2 Ka + \lambda \sin^2 Ka)^{-1}$$

where $K^2 = 2m(E - V_0)/\hbar^2$, giving an expression for λ. What if $E < V_0$? Comment on the differences between classical and quantum mechanics that this example illustrates.

58. An electron of mass m moves in a one-dimensional potential

$$V(x) = -(g^2\hbar^2/m)\delta(x^2 - a^2)$$

where g and a are real constants. Discuss its bound states as a function of g.

59. What boundary conditions are applicable to energy eigenfunctions in wave mechanics? A particle moves on the x-axis in a potential $V(x)$ where $V = \infty$ for $|x| > a+b$, V=0 for $a < |x| < a+b$ and $V = V_0 > 0$ for $0 < |x| < a$. Show how energy eigenvalues E can be found for $E < V_0$. Derive an approximate formula when V_0 is large and $E \ll V_0$, and show that then the spectrum contains close doublets.

60. A particle on the x-axis with mass $\frac{1}{2}\hbar^2$ experiences potential $V(x) = -2(\operatorname{sech} x)^2$. Show that there is a bound state with energy -1 and eigenfunction $\phi = \operatorname{sech} x$, and eigenstates with energy k^2 and $\phi = \exp(ikx)(-ik + \tanh x)$ for any real k. Show that the latter solutions imply that the potential is completely transparent at all energies.

61. At a given time the wavefunction of a quantum particle in Cartesian coordinates is

$$\psi = (x + y + z)\exp(-\sqrt{x^2 + y^2 + z^2}).$$

Show that $\ell = 1$ and calculate the probabilities of a measurement of L_z giving results 0 or $\pm\hbar$.

62. For the isotropic three-dimensional simple-harmonic oscillator construct annihilation and creation operators for each of the x, y and z components and write down their commutation rules. From them find expressions for $\widehat{H}$, $\widehat{L}^2$, $\widehat{L}_\pm$ and $\widehat{L}_z$, and verify the commutation rules for these operators. Express the quantum numbers ℓ and m in terms of the integers $n_{x,y,z}$ that count the Cartesian excitations and express energy and angular momentum eigenstates $|E\ell m\rangle$ in terms of states $|n_x, n_y, n_z\rangle$. (See *e.g.* Messiah *pps.* 451–460).

63. Show how to determine the energy eigenfunctions and eigenvalues of a quantum particle in a two-dimensional circular region with impenetrable walls.

64. For a quantum particle moving in the central potential $V(\mathbf{r}) = -V_0\exp(-r/b)$, where $r = |\mathbf{r}|$ and V_0 and b are real and positive, change variable in the radial equation to $x = \exp(-r/2b)$ and show that, for $l = 0$, Bessel's equation results. What boundary conditions must be imposed on the radial wavefunction as a function of x, and how can these be used to determine energy levels?

65. In one dimension a stream of particles of mass m and energy E is incident on a potential barrier $V(x)$; $V(x) = 0, \quad x < 0$; $V(x) = V_0, \quad 0 < x < a$; $V(x) = 0, \quad x > a$. For $E > V_0$ find the transmission and reflection coefficients. Defining

$$\alpha = \sqrt{\frac{2m}{\hbar^2}(E - V_0)}$$

find the condition on αa for the complete transmission through the barrier.

66. A particle of mass m moves in a potential

$$V(\mathbf{r}) = \frac{A}{r^2} - \frac{B}{r}$$

where $r \equiv |\mathbf{r}|$ and A and B are positive real constants. Show that bound states of angular momentum ℓ exist with energies

$$E = -\frac{mB^2}{2\hbar^2(N + \mu + 1)^2}, \qquad N = 0, 1, 2, \ldots,$$

where $\mu(\mu + 1) = \ell(\ell + 1) + 2mA/\hbar^2$. Compare the degeneracy of the spectrum with that of the spectrum of the simplest model of the hydrogen atom (case $A = 0$, $B = e^2$).

67. A dynamical system has an observable $\hat{A}$ that can take values 1,2,3 and its Hamiltonian $\hat{H}$ acts on the corresponding orthogonal eigenstates of $\hat{A}$ as follows

$$\begin{aligned} \hat{H}|1\rangle &= |1\rangle + i|2\rangle \\ \hat{H}|2\rangle &= |2\rangle - i|1\rangle \\ \hat{H}|3\rangle &= |3\rangle \end{aligned}$$

Find the eigenstates of the Hamiltonian.

At $t = 0$ $\hat{H}$ is measured and found to have value 1. The system is left undisturbed for a time t and $\hat{A}$ is remeasured. What values may there be found and with what probabilities? What is the expectation value of $\hat{A}$? When, if ever, would such a remeasurement certainly give the result 2?

68. A three dimensional oscillator has Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{\omega^2}{2}\hat{\mathbf{x}}^2,$$

where $[\hat{x}_k, \hat{p}_j] = i\hbar\delta_{jk} \quad k, j = 1, 2, 3$. Calculate $[[\hat{p}_k, \hat{H}], \hat{p}_j]$ and by considering its suitable expectation value derive

$$\omega^2\hbar^2\delta_{kj} = \sum_n (E_n - E_m)[F_k(n,m)F_j^*(n,m) + F_j(n,m)F_k^*(n,m)],$$

where $F_k(n,m) = \langle m|\hat{p}_k|n\rangle$ and $|n\rangle$ is the normalised eigenstate of $\hat{H}$ corresponding to eigenvalue E_n. * denotes complex conjugation.

69. A quantum one-dimensional single harmonic oscillator is described by the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}),$$

where

$$\hat{a} = \frac{1}{\sqrt{2m}}(\hat{p} + im\omega\hat{x}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m}}(\hat{p} - im\omega\hat{x}).$$

Show that $\hat{a}, \hat{a}^\dagger$ satisfy

$$[\hat{a}^\dagger, \hat{a}] = \hbar\omega.$$

The oscillator is in a state where an energy measurement would yield $\frac{3}{2}\hbar\omega$ or $\frac{5}{2}\hbar\omega$ with equal likelihood but no other value. At $t = 0$ the mean value of momentum is as large as possible in these circumstances. Calculate $\langle p\rangle$ and $\langle x\rangle$ and energies $\langle E\rangle$ and $\langle T\rangle$ as functions of t. Compare and contrast with the corresponding classical values of an oscillator of energy $2\hbar\omega$.

Note: You may use the fact that the energy eigenstates $|n\rangle$ satisfy

$$\hat{H}|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle;$$
$$\hat{a}|n\rangle = \sqrt{(n+1)\hbar\omega}|n+1\rangle; \quad \hat{a}^\dagger|n\rangle = \sqrt{n\hbar\omega}|n-1\rangle.$$

70. In quantum mechanics, in addition to "standard" Schrödinger and Heisenberg pictures, one often introduces various intermediate (interaction) pictures.

Splitting the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

define the states in the interaction picture as

$$|\Psi(t)\rangle_I = e^{\frac{i\hat{H}_0 t}{\hbar}}|\Psi(t)\rangle,$$

where $|\Psi(t)\rangle$ denotes the corresponding state in the Schrödinger picture.

Define the operators in the interaction picture by requiring the invariance of matrix elements of operators with respect to the change of picture.

Derive the equations which describe the time evolution of states and operators in the interaction picture.

Discuss what happens when either $\hat{H}_0$ or $\hat{H}_1$ vanish.

71. A particle of mass m moves in one dimension under the influence of a potential

$$V(x) = \begin{cases} V_0(\frac{a}{x} - \frac{x}{a})^2, & x > 0 \\ \infty & x < 0, \end{cases}$$

where $V_0 > 0$.

Find the energy levels and show that their spectrum coincides with the spectrum of a harmonic oscillator of frequency $\omega = \sqrt{\frac{8V_0}{ma^2}}$ (apart from an overall energy shift).

Note: You may use the fact that a non-singular (as $z \to 0$) solution of

$$z\ddot{f} + (b - z)\dot{f} - af = 0$$

is $f = A\,{}_1F_1(a; b; z) = A\big[1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + ...\big]$ if $1 - b < 0$.

72. In one dimension a stream of particles of mass m and energy E is incident on a potential "hole"

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & a > x > 0 \\ 0 & x > a, \end{cases}$$

with $V_0 > 0$.

Find the transmission and reflection coefficients. Show that when

$$2m(E + V_0) = \frac{n^2\pi^2\hbar^2}{a^2}, \qquad n = 1, 2, ...$$

there is no reflection.

73. A two-dimensional quantum mechanical system resembles a particle in a two-dimensional, infinite, square well; *i.e.* subject to the potential

$$V(r) = \begin{cases} 0 & r < a \\ \infty & r \geq a. \end{cases}$$

Estimate energies of the two lowest bound states of this system and give their (normalised) wave functions.

Note: The lowest zeros of Bessel's functions, which satisfy

$$x^2\ddot{J}_\nu(x) + x\dot{J}_\nu(x) + (x^2 - \nu^2)J_\nu(x) = 0$$

are $J_0(x) = 0$ for $x \sim 2.405..,\quad 5.520..,..$ $J_1(x) = 0$ for $x = 0$, $x \sim 3.832...,\quad 7.016...$ and Bessel's functions satisfy

$$\int_0^a \big[J_\nu\big(\alpha_{\nu m}\frac{\rho}{a}\big)\big]^2 \rho\, d\rho = \frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu m})]^2,$$

where $\alpha_{\nu m}$ is the m^{th} zero of J_ν.

74. **(i)** If $\hat{H}$ is a self-adjoint operator and $|\psi\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E show that

$$\langle\psi|e^{i\hat{H}} = e^{iE}\langle\psi|.$$

(ii) Calculate the commutator

$$[x^a \frac{d^2}{dx^2},\, x\frac{d}{dx}],$$

where a is an integer and find the value of a for which the commutator vanishes.

(iii) Operators $\hat{A}_i$ and $\hat{J}_k$ have the following commutation relations:

$$[\hat{J}_i,\, \hat{A}_k] = i\epsilon_{ikl}\hat{A}_l.$$

Calculate $[\hat{J}^2, \hat{A}_k]$, where $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$.

(iv) Describe the correspondence between the Poisson bracket in classical mechanics and the commutator in quantum mechanics. Use it to define

$$[\hat{p}_i,\, \hat{f}(\hat{\mathbf{r}})]$$

and calculate

$$[\hat{p}_i,\, \frac{1}{|\mathbf{r}|}].$$

75. **(i)** In a quantum mechanical system an observable A is represented by an operator $\hat{A}$ which has two normalised eigenvectors $|1\rangle$ and $|2\rangle$.

Calculate $\langle\psi|\widehat{A}|\psi\rangle$ and $\langle\psi|\widehat{A}^2|\psi\rangle$ for a general normalised state $|\psi\rangle$. Which of the two quantities

$$(\langle\psi|\widehat{A}|\psi\rangle)^2, \qquad \text{or} \qquad \langle\psi|\widehat{A}^2|\psi\rangle$$

is larger? What is the condition on $|\psi\rangle$ for these two quantities to be equal?

(ii) Which of the following four operators can represent observables in quantum mechanics and why?

$$(a)\begin{pmatrix} 1, & i \\ -i, & 2 \end{pmatrix}, \qquad (b)\begin{pmatrix} 1, & 1 \\ -1, & 2 \end{pmatrix}, \qquad (c)\,\frac{d^3}{dx^3}, \quad (d)\, i\frac{d^3}{dx^3} + f(x),$$

where $f(x)$ in (d) is real, and the space of functions on which operators act in (c) and (d) involves $\psi(x)$, $b \le x \le a$, $\psi(a) = \psi(b) = 0$.

76. A quantum system can exist in two states $|a_0\rangle$ and $|a_1\rangle$ which are normalised eigenstates of an observable A with eigenvalues 0 and 1 respectively. A second observable is represented by an operator $\hat{B}$ defined by

$$\begin{aligned} \hat{B}|a_0\rangle &= 7|a_0\rangle - 24i|a_1\rangle \\ \hat{B}|a_1\rangle &= 24i|a_0\rangle - 7|a_1\rangle. \end{aligned}$$

Find the eigenvalues and eigenstates of $\hat{B}$. The system is in the state $|a_1\rangle$ when B is measured and its value is lost. Immediately afterwards A is measured. Find the probability that the measurement of A gives the value 1.

77. For a particle moving in one dimension find the uncertainty Δx when the wave function is the Gaussian:

$$\psi(x) \;=\; \Big(\frac{2}{\pi a^2}\Big)^{\frac{1}{4}}\, exp\Big(-\frac{x^2}{a^2}\Big).$$

Show that

$$\tilde{\psi}(p) \;=\; \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \psi(x)\, exp\Big(-\frac{ipx}{\hbar}\Big)\, dx$$

has a similar shape and calculate $\Delta x \cdot \Delta p$.

Note: $\int_{-\infty}^{\infty} e^{-ax^2}\, dx \;=\; \sqrt{\frac{\pi}{a}}$.

78. A beam of particles of mass m and energy $\frac{4}{3}V_0$ ($V_0 > 0$) is incident from the right ($i.e.\ x = \infty$) on a potential barrier $V(x)$ given by

$$V(x) = \begin{cases} 0, & |x| > a \\ V_0, & |x| < a. \end{cases}$$

Find the reflection and transmission coefficients.

79. Determine the energy levels and the corresponding wavefunctions of a particle of mass m moving in a one-dimensional infinite square well potential

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < \alpha \\ \infty, & x > \alpha. \end{cases}$$

Show that for this particle, at the n^{th} energy level

$$< x > = \frac{\alpha}{2}, \qquad < (x- < x >)^2 > = \frac{\alpha^2}{12}\Big(1 - \frac{6}{n^2\pi^2}\Big).$$

80. The interaction between a proton and a neutron can be modelled by the potential

$$V(\mathbf{r}) \;=\; -A\, exp\Big(-\frac{r}{\alpha}\Big).$$

Find the wave function of the ground state ($l = 0$).

Hint: Change variables $r \to \xi = exp(-\frac{r}{2\alpha})$.

Note: $\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$