QM Homework Problems 2 – Solutions

1 From the definition of the adjoint of an operator we have, for all states $|\alpha\rangle$ and $|\beta\rangle$:

$$\langle\alpha|\left(\widehat{A}\widehat{B}\right)^{\dagger}|\beta\rangle = \left(\langle\beta|\widehat{A}\widehat{B}|\alpha\rangle\right)^{*}$$

We now use the 'trick' of inserting a complete set of states, i.e. we use the fact that we can write $\widehat{I} = \sum_n |n\rangle\langle n|$ where $\{|n\rangle\}$ are a set of orthonormal basis states. So we see that:

$$\begin{aligned}
\left(\langle\beta|\widehat{A}\widehat{B}|\alpha\rangle\right)^{*} &= \left(\langle\beta|\widehat{A}\widehat{I}\widehat{B}|\alpha\rangle\right)^{*} = \sum_n \left(\langle\beta|\widehat{A}|n\rangle\langle n|\widehat{B}|\alpha\rangle\right)^{*} \\
&= \sum_n \langle n|\widehat{A}^{\dagger}|\beta\rangle\langle\alpha|\widehat{B}^{\dagger}|n\rangle = \sum_n \langle\alpha|\widehat{B}^{\dagger}|n\rangle\langle n|\widehat{A}^{\dagger}|\beta\rangle \\
&= \langle\alpha|\widehat{B}^{\dagger}\widehat{I}\widehat{A}^{\dagger}|\beta\rangle = \langle\alpha|\widehat{B}^{\dagger}\widehat{A}^{\dagger}|\beta\rangle
\end{aligned}$$

So we see that for all states $|\alpha\rangle$ and $|\beta\rangle$:

$$\langle\alpha|\left(\widehat{A}\widehat{B}\right)^{\dagger}|\beta\rangle = \langle\alpha|\widehat{B}^{\dagger}\widehat{A}^{\dagger}|\beta\rangle$$

which is equivalent to the statement:

$$\left(\widehat{A}\widehat{B}\right)^{\dagger} = \widehat{B}^{\dagger}\widehat{A}^{\dagger}$$

2 Recall that an eigenstate of a self-adjoint operator $\widehat{A}$ corresponds to a state with a definite value of the corresponding observable A. So in this case a measurement of E on a state $|n\rangle$ will give the result $E = E_n$. If we consider a linear combination of these eigenstates, say

$$|\alpha\rangle = \sum_{i=0}^{\infty} c_i|i\rangle$$

then a measurement of E on the state $|\alpha\rangle$ will give the result $E = E_n$ with probability $|c_n|^2$. Note that we are assuming that all states are normalised so that the probability of any outcome, i.e. the sum over the probabilities of each distinct outcome, is 1:

$$\sum_{i=0}^{\infty} |c_i|^2 = \langle\alpha|\alpha\rangle = 1$$

If $|\alpha\rangle$ is not normalised then we must first normalise it by dividing by some normalisation constant N where $\langle\alpha|\alpha\rangle = |N|^2$. We can then calculate the probabilities from the coefficients of this normalised state. So it is clear that the probability of outcome E_n is $|c_n|^2/|N|^2$.

For a given state the possible outcomes are simply those with non-zero probability. So for the state $|0\rangle$ the only possible outcome is E_0 with probability 1. Similarly for the state $|1\rangle$ the only possible outcome is E_1 with probability 1. For the other four states $|0\rangle+|1\rangle$; $|0\rangle-|1\rangle$; $|0\rangle+3|1\rangle$; $e^{i\theta}|0\rangle+3e^{i\phi}|1\rangle$ the only possible outcomes are E_0 or E_1 with probabilities $1/2, 1/2$; $1/2, 1/2$; $1/10, 9/10$; $1/10, 9/10$ respectively. Note that the phases of each coefficient don't alter the probabilities and it is also obvious that in each case the probabilities sum to 1 as required.

For a normalised state $|\alpha\rangle$ we define:

$$\langle E\rangle \equiv \langle\alpha|\widehat{H}|\alpha\rangle = \sum_{i=0}^{\infty} E_i |c_i|^2$$

so $\langle E\rangle$ is the average outcome of measuring E over a large number of identical systems. In the lectures we showed that the standard deviation for such a distribution of outcomes, ΔE is given by:

$$(\Delta E)^2 = \langle E^2\rangle - \langle E\rangle^2$$

so we can calculate it by also determining

$$\langle E^2\rangle = \sum_{i=0}^{\infty} E_i^2 |c_i|^2$$

Again if $|\alpha\rangle$ is not normalised we must first normalise it and this is equivalent to replacing $|c_i|^2$ by $|c_i|^2/|N|^2$ in the final expressions for $\langle E\rangle$ and $\langle E^2\rangle$. So for the six states, in the order given in the question we find:

$$\langle E\rangle = E_0,\ E_1,\ \frac{1}{2}(E_0+E_1),\ \frac{1}{2}(E_0+E_1),\ \frac{1}{10}(E_0+9E_1),\ \frac{1}{10}(E_0+9E_1)$$

$$\langle E^2\rangle = E_0^2,\ E_1^2,\ \frac{1}{2}(E_0^2+E_1^2),\ \frac{1}{2}(E_0^2+E_1^2),\ \frac{1}{10}(E_0^2+9E_1^2),\ \frac{1}{10}(E_0^2+9E_1^2)$$

So it is simple to calculate ΔE in each case:

$$\Delta E = 0,\ 0,\ \frac{1}{2}(E_1-E_0),\ \frac{1}{2}(E_1-E_0),\ \frac{3}{10}(E_1-E_0),\ \frac{3}{10}(E_1-E_0)$$

noting that ΔE is the positive square root of $\langle E^2\rangle - \langle E\rangle^2$.

N.B. The states $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ as well as $|0\rangle + 3|1\rangle$ and $e^{i\theta}|0\rangle + 3e^{i\phi}|1\rangle$ (for $\theta \neq \phi$) are linearly independent states but the question shows that we cannot distinguish between them by measuring the energy. However, we could distinguish them by measuring other physical quantities instead.

3 First note that since $[\widehat{x}, \widehat{V}(\widehat{x})] = 0$ we can calculate the given double commutator:

$$\begin{aligned} [\widehat{x}, \widehat{H}] &= [\widehat{x}, \widehat{p}^2/2M] = \frac{1}{2M}(\widehat{x}\widehat{p}^2 - \widehat{p}^2\widehat{x}) = \frac{1}{2M}\Big((\widehat{p}\widehat{x} + i\hbar)\widehat{p} - \widehat{p}^2\widehat{x}\Big) \\ &= \frac{1}{2M}\Big(\widehat{p}(\widehat{p}\widehat{x} + i\hbar) + i\hbar\widehat{p} - \widehat{p}^2\widehat{x}\Big) = \frac{i\hbar}{M}\widehat{p} \end{aligned}$$

So we see that:

$$[[\widehat{x}, \widehat{H}], \widehat{x}] = [\frac{i\hbar}{M}\widehat{p}, \widehat{x}] = \frac{\hbar^2}{M}\widehat{I}$$

However, we can also simply expand the double commutator:

$$[[\widehat{x}, \widehat{H}], \widehat{x}] = (\widehat{x}\widehat{H} - \widehat{H}\widehat{x})\widehat{x} - \widehat{x}(\widehat{x}\widehat{H} - \widehat{H}\widehat{x}) = 2\widehat{x}\widehat{H}\widehat{x} - \widehat{H}\widehat{x}^2 - \widehat{x}^2\widehat{H}$$

Using this expansion, since we know that $\widehat{H}|m\rangle = E_m|m\rangle$ for any m, we can evaluate the expectation value of the double commutator for a state $|m\rangle$. In other words we can calculate:

$$\langle m|[[\widehat{x}, \widehat{H}], \widehat{x}]|m\rangle = 2\langle m|\widehat{x}\widehat{H}\widehat{x}|m\rangle - \langle m|\widehat{H}\widehat{x}^2|m\rangle - \langle m|\widehat{x}^2\widehat{H}|m\rangle$$

By definition $\widehat{H}|m\rangle = E_m|m\rangle$ means that $\langle m|\widehat{H}^\dagger = \langle m|E_m^*$ so since $\widehat{H}$ is self-adjoint (and so its eigenvalues are real) we have $\langle m|\widehat{H} = \langle m|E_m$. To further simplify the expressions we again insert a complete set of states. So we see that:

$$\langle m|\widehat{x}^2\widehat{H}|m\rangle = E_m\langle m|\widehat{x}^2|m\rangle = E_m\sum_n\langle m|\widehat{x}|n\rangle\langle n|\widehat{x}|m\rangle = E_m\sum_n|\langle n|\widehat{x}|m\rangle|^2$$

and similarly

$$\langle m|\widehat{H}\widehat{x}^2|m\rangle = E_m\sum_n|\langle n|\widehat{x}|m\rangle|^2$$

To evaluate the final expectation value we can insert a complete set of states immediately before and/or after the operator $\widehat{H}$. E.g.

$$\langle m|\widehat{x}\widehat{H}\widehat{x}|m\rangle = \sum_n\langle m|\widehat{x}\widehat{H}|n\rangle\langle n|\widehat{x}|m\rangle = \sum_n E_n|\langle n|\widehat{x}|m\rangle|^2$$

Note that we could also have simply replaced $\widehat{H}$ with its spectral representation $\sum_n E_n|n\rangle\langle n|$ to obtain the same result. So putting everything together we see that:

$$\langle m|[[\widehat{x},\widehat{H}],\widehat{x}]|m\rangle = 2\sum_n (E_n - E_m)|\langle n|\widehat{x}|m\rangle|^2$$

However, we also know that

$$\langle m|[[\widehat{x},\widehat{H}],\widehat{x}]|m\rangle = \langle m|\frac{\hbar^2}{M}\widehat{I}|m\rangle = \frac{\hbar^2}{M}$$

So we have the required result:

$$\sum_n (E_n - E_m)|\langle n|\widehat{x}|m\rangle|^2 = \frac{\hbar^2}{2M}$$

for any m.