MNS ALGEBRA: PROBLEMS

Group Theory

1. Which of the following algebraic systems form a group? Give reasons for your answers.
(a) The integers under subtraction.
(b) The reals under multiplication.
(c) The even integers under addition.
(d) The positive integers under multiplication.
(e) The rationals with odd denominator under addition.
(f) The integers 1, 2, 3, 4 under multiplication mod 5.
(g) The integers 1, 2, 3, 4, 5 under multiplication mod 6.

2. Show that each of the following sets of numbers is an infinite abelian group under multiplication:

$$\{2^n \mid n \in \mathbf{Z}\}; \qquad \{(1+2m)/(1+2n) \mid m, n \in \mathbf{Z}\}; \qquad \{\cos\theta + i\sin\theta \mid \theta \in \mathbf{Q}\}.$$

3. Which of the following sets of $n \times n$ matrices (with real entries) form groups under matrix multiplication?
(a) All symmetric matrices.
(b) All non-singular symmetric matrices.
(c) All diagonal matrices with non-zero diagonal entries.
(d) All matrices with 1's in the leading diagonal and 0's below.
(e) All non-singular matrices with integer entries.
(f) All non-singular matrices with rational entries.
(g) All matrices with integer entries and determinant ± 1.

4. Show that each of the following collections of numbers forms a group under addition.
(a) All real numbers of the form $a + b\sqrt{2}$ where a, $b \in \mathbf{Z}$.
(b) All real numbers of the form $a + b\sqrt{2}$ where a, $b \in \mathbf{Q}$.
(c) All complex numbers of the form $a + bi$ where a, $b \in \mathbf{Z}$.

5. Let $\mathbf{Q}[\sqrt{2}]$ denote the set described in part (b) of the previous problem. Given a non-zero element $a+b\sqrt{2}$ of $\mathbf{Q}[\sqrt{2}]$, express $1/(a+b\sqrt{2})$ in the form $c+d\sqrt{2}$ where c, $d \in \mathbf{Q}$. Prove that multiplication makes $\mathbf{Q}[\sqrt{2}] - \{0\}$ into a group.

6. Which of the following sets form groups under multiplication mod 14?

$$\{1,3,5\}; \qquad \{1,3,5,7\}; \qquad \{1,7,13\}; \qquad \{1,9,11\}.$$

7. Evaluate the following products, expressing the results as products of disjoint cycles:

$$(12)(23); \quad (12)(1234); \quad (1234)(12); \quad (145)(3524); \quad (123\ldots r)(r\ r+1).$$

8. Express each of the following permutations as (i) a product of disjoint cycles and (ii) a product of transpositions:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 4 & 1 & 8 & 2 & 3 & 5 \end{pmatrix}; \qquad (4568)(1245); \qquad (624)(253)(876)(45).$$

9. Given real numbers a, b with $a \neq 0$ define a function $T_b^a : \mathbf{R} \to \mathbf{R}$ by $T_b^a(x) = ax + b$. Show that the set of all such functions forms a group under composition of functions.

10. Show that the elements of finite order in an abelian group form a subgroup.

11. Write down the order of each element of (i) $\mathbf{Z}_9$; (ii) $\mathbf{Z}_8$; (iii) D_6; (iv) $\mathbf{C}^*$.

12. Find all subgroups of $\mathbf{Z}_{12}$, V, D_4 and S_3.

13. Let A, B be the matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Find the orders of A, B, AB and BA in the group $GL_2(\mathbf{R})$. What are their orders in the group $M_2(\mathbf{R})$ (2×2 real matrices under addition)?

14. (i) Show that if x and y are elements of finite order of a group G, and $xy = yx$, then xy is also an element of finite order. What can you say about the order of xy in terms of the orders of x and y?
(ii) Find a group G and elements x, y of G such that x and y have finite order yet xy has infinite order.

15. Decompose A_4 into left cosets with respect to the subgroup

$$V = \{e, (12)(34), (13)(24), (14)(23)\}.$$

Verify that each left coset is also a right coset.

16. Decompose D_6 into left cosets with respect to the subgroup $\{e, r^3, s, sr^3\}$. Is every left coset also a right coset?

17. Which of the following functions are homomorphisms from the multiplicative group of non-zero real numbers to itself?

(a) $x \mapsto |x|$; (b) $x \mapsto -x$; (c) $x \mapsto 2x$; (d) $x \mapsto x^2$; (e) $x \mapsto -1/x$.

18. Which of the following functions are homomorphisms?
(i) $\phi : \mathbf{R} \to \mathbf{R}$, $\phi(x) = x^2$.
(ii) $\phi : \mathbf{C} \to \mathbf{R}$, $\phi(z) = |z|$.
(iii) $\phi : \mathbf{C}^* \to \mathbf{R}^*$, $\phi(z) = |z|$.
(iv) $\phi : \mathbf{R} \to GL_2(\mathbf{R})$ given by $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

19. Determine whether or not $\mathbf{Z} \times \mathbf{Z}$ is isomorphic to $\mathbf{Z}$.

20. Let G denote the full symmetry group of a regular tetrahedron T. Find a symmetry of T which induces a cyclic permutation of the four vertices. Prove that G is isomorphic to S_4.

21. Determine which of the following groups are isomorphic to one another:

$$\mathbf{Z}_8, \quad \mathbf{Z}_4 \times \mathbf{Z}_2, \quad D_4, \quad A_4$$

22. Prove that $\mathbf{C}$ is isomorphic to $\mathbf{R} \times \mathbf{R}$, and that $\mathbf{C}^*$ is isomorphic to $\mathbf{R}^{pos} \times C$.

23. Think of A_4 as the group of rotational symmetries of a regular tetrahedron T, and let $E_1, \ldots, E_6$ be the edges of T. Each element of A_4 permutes $E_1, \ldots, E_6$ and therefore gives us an element of S_6. What is the cycle structure of the element of S_6 corresponding to:

(a) a 3-cycle in A_4;
(b) the product of two disjoint transpositions in A_4?

24. Find a subgroup of S_4 which contains six elements. How many subgroups of order six are there in S_4?

25. Prove that a group G is abelian if and only if the correspondence $x \mapsto x^{-1}$ is an isomorphism from G to G.

26. Let G and H be groups. Show that $G \times H$ is abelian if and only if both G and H are abelian. If $G \times H$ is cyclic, prove that G and H are both cyclic.

27. Which of the following groups are isomorphic to one another?

$$\mathbf{Z}_4 \times \mathbf{Z}_3, \quad \mathbf{Z}_6 \times \mathbf{Z}_2, \quad V \times \mathbf{Z}_3, \quad S_3 \times \mathbf{Z}_2.$$

28. Which of the following groups are isomorphic to one another?

$$\mathbf{Z}_{24}, \quad D_4 \times \mathbf{Z}_3, \quad D_{12}, \quad A_4 \times \mathbf{Z}_2, \quad \mathbf{Z}_2 \times D_6, \quad S_4, \quad \mathbf{Z}_{12} \times \mathbf{Z}_2.$$

29. Carry out the procedure of Cayley's theorem to produce a subgroup of S_8 which is isomorphic to D_4.

30. The infinite dihedral group D_∞ is generated by the translation $t(x) = x + 1$ and the reflection $s(x) = -x$ of the real line. Its elements are the translations $\cdots t^{-2}, t^{-1}, e, t, t^2, t^3, \cdots$, and the reflections $\cdots t^{-2}s, t^{-1}s, s, ts, t^2s, t^3s, \cdots$. Work out the orbit and the stabilizer of each of the points 1, 1/2, 1/3.

31. Each edge of a cube is painted black or white. How many genuinely different decorated cubes result.

32. A cube has stripes drawn on its faces as shown below. Which rotational symmetries of the cube send stripes to stripes? To which subgroup of S_4 do these rotations correspond?

33. Imagine painting each half of the subdivided faces of the cube shown above. In how many different ways can this be done (up to rotational symmetry) if there are two colours available?

34. Show there are 9,099 essentially different ways of colouring the faces of a dodecahedron red, white or blue.

35. Take the cube shown in Problem 32 and paint each of the stripes which bisect the faces. In how many different ways can this be done if we have three colours available?

36. A bracelet is made from five beads mounted on a circular wire. How many different bracelets can we manufacture if we have red, blue and yellow beads at our disposal?

37. How many different ways are there of colouring the vertices and edges of a regular hexagon using red, blue or yellow for the edges and black or white for the vertices?

38. Square table cloths are manufactured by sewing together n^2 coloured squares (with plain backs). If q colours are available show that the total number of distinct $n \times n$ table cloths is

$$\frac{1}{4}\left(q^{n^2} + 2q^{[(n^2+3)/4]} + q^{[(n^2+1)/2]}\right).$$