Continuous time Markov Chains: construction and basic tools

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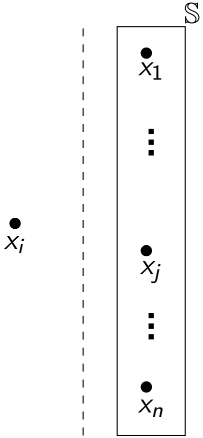
Outline

Finite state spaces

$\mathbb{S} = \{x_1, \ldots, x_n\}$ Time: t



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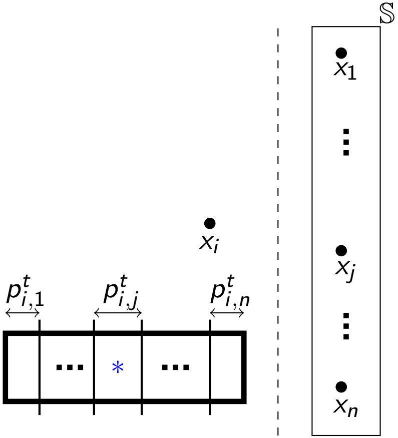

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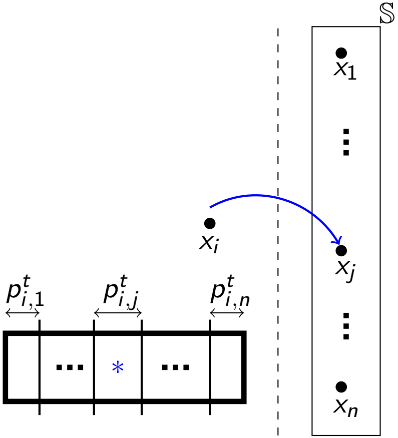

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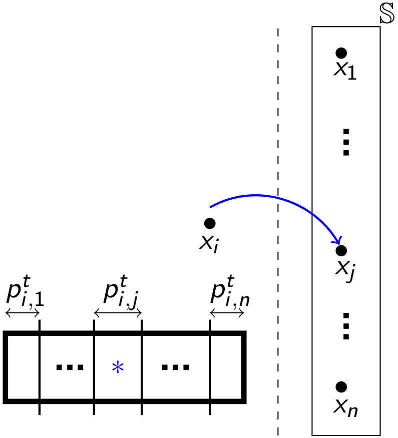

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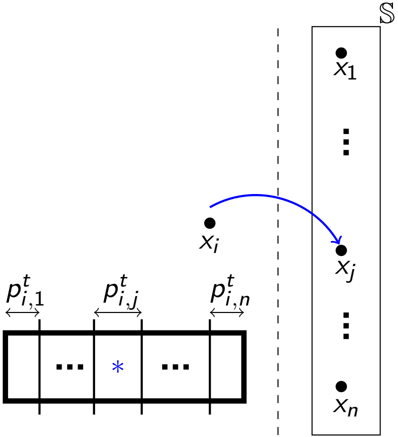

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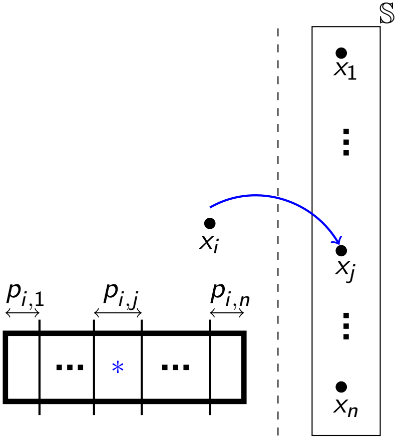

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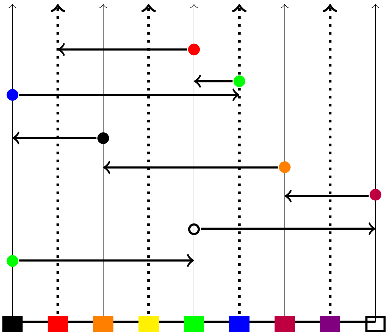
If $r_{xy} = p_{xy} r_x$ for all $x, y \in \mathbb{S}$ then $\mathbb{P}_x = \hat{\mathbb{P}}_x$

graphical construction

Poisson point processes + colouring = Graphical construction



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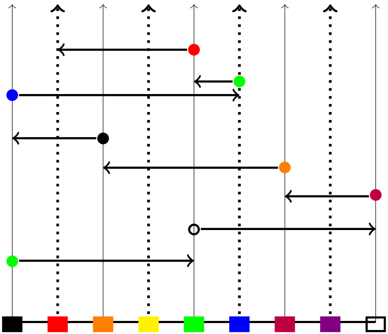

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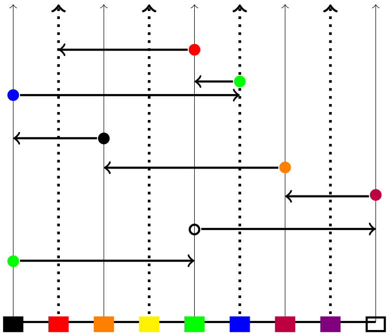

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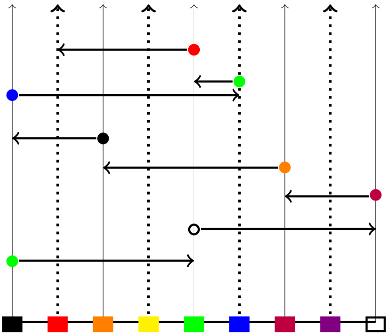

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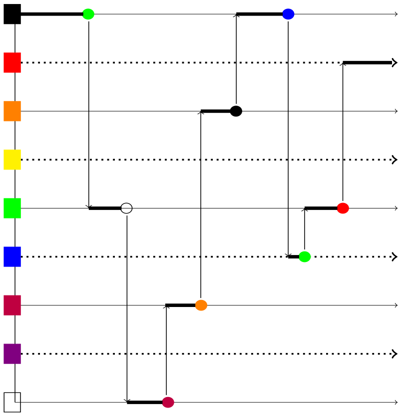

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Operator on $C_b(\mathbb{S})$: $Lf(x) = \sum_{y \in \mathbb{S}} r_{xy}[f(y) - f(x)].$

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To complete the construction: extension and regularization.

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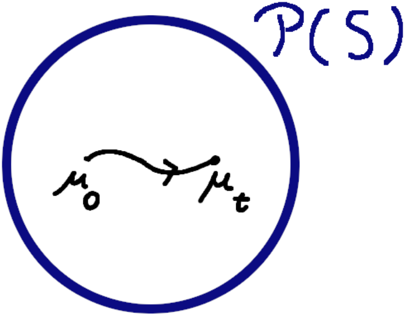

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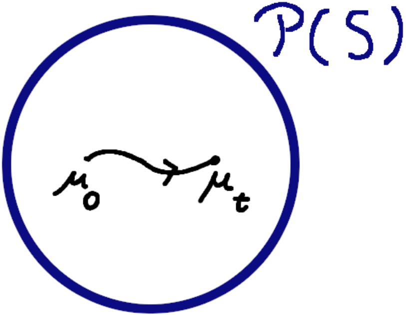

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Goal: To understand the behavior of μ_t in the relevant scales.

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Dynkin Martingales

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$$F : \mathbb{R}_+ \times S \to \mathbb{R} \quad \sup_{(s,x)} \left| \partial_s^j F(s,x) \right| < C$$

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Tightness criterion

Family of processes $(X^N_\cdot, N \in \mathbb{N})$

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Then there is a subsequence $\{N_k, k \in \mathbb{N}\}$ and $X_\cdot^* \in D$ s.t.

$$X_\cdot^{N_k} \to X_\cdot^*$$

Martingale problems

Martingale problems

A continuous adapted process $(M_t, t \geq 0)$ is a d-dimensional Brownian Motion if and only if

$$f(M_t) - f(M_0) - \frac{1}{2}\int_0^t \Delta f(M_s)\, ds$$

is a local martingale for all $f \in C^2(\mathbb{R})$.

Martingale problems

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Theorem:(Lévy) If $(M_t, t \geq 0)$ is a continuous real valued local martingale with

$$\langle M^k, M^j \rangle_t = t\delta_{k,t}$$

then M_t is a Brownian motion.

Proof (Lévy)

Goal: to show that $M_t - M_s \sim N(0, t-s)$ and $M_t - M_s \perp \mathcal{F}_s$

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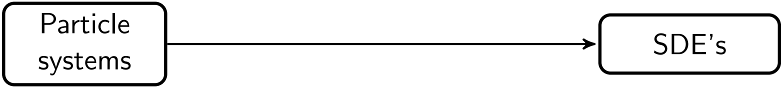
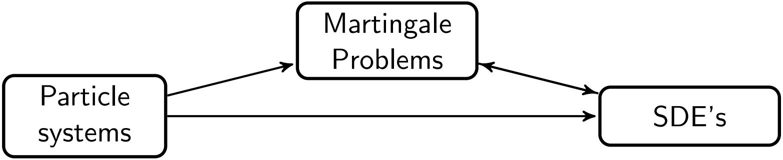
□

Panorama

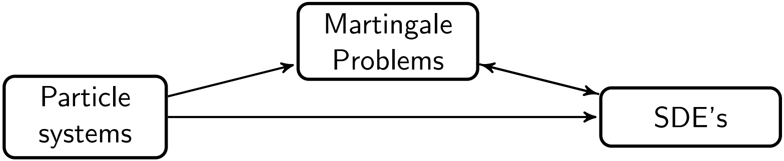


Panorama



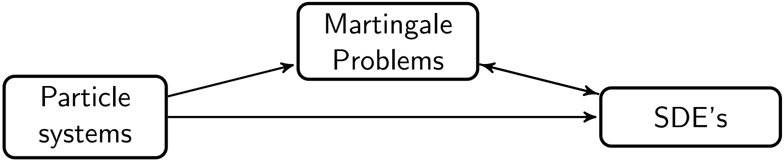
Panorama

$$X_t^n - X_0^n - \int_0^t L_n(X_s^n)\, ds = M_t^n$$



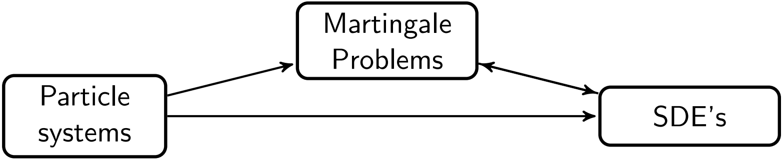
Panorama

$$X_t - X_0 - \int_0^t b(X_s)\, ds = M_t$$



Panorama

$$X_t - X_0 - \int_0^t b(X_s)\, ds = M_t \qquad M_t = \int_0^t \sigma(X_s)\, dB_s$$



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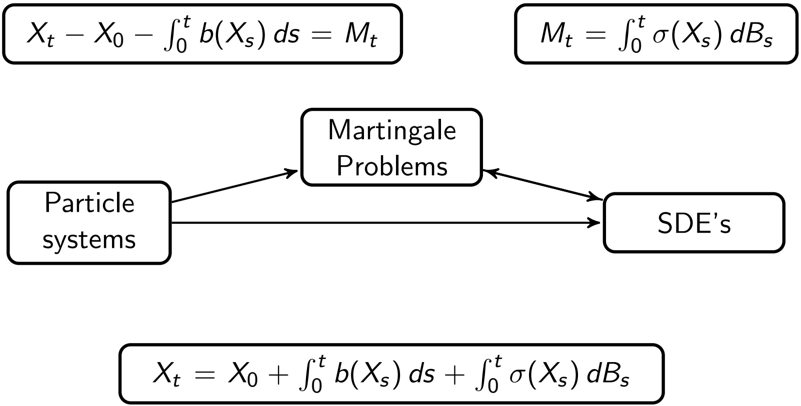
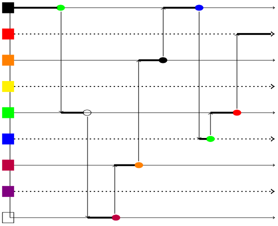
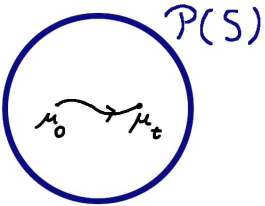

$X_t - X_0 - \int_0^t b(X_s)\,ds = M_t$
$M_t = \int_0^t \sigma(X_s)\,dB_s$
Martingale Problems
Particle systems
SDE's
$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s$





Thank you!

$$\mathbb{E}[e^{i\lambda(M_t - M_s)} 1_A] = \mathbb{P}(A) e^{-\lambda^2/2(t-s)}$$