Resurgence and the Physics of Divergence

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Young Theorists' Forum, IPPP Durham, December 17, 2014

GD & M. Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

GD, lectures at CERN 2014 Winter School

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108, 1405.0302

Physical Motivation

- strongly interacting/correlated systems
- non-perturbative definition of non-trivial QFT in continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- uncover hidden 'magic' in perturbation theory
- "exact" asymptotics in QM, QFT and string theory

Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$

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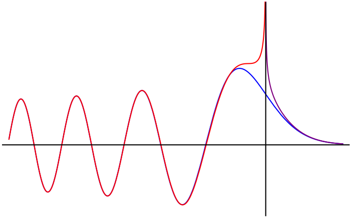


$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3+x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x\to+\infty \\[2ex] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x\to-\infty \end{cases}$$

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view semiclassical expansions as potentially exact

Trans-series

No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, *Divergent Series*, 1949

- deep result: "this is all we need" (J. Écalle, 1980)
- trans-series in many physics applications:

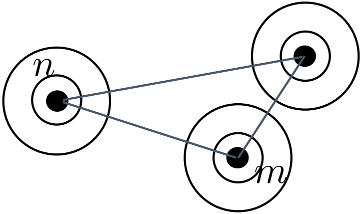
$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} c_{n,k,l}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\log\left(-\frac{1}{g^2}\right)\right]^l$$

- *trans-monomials*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$: familiar in physics

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: analytic continuation under control
- new: exponentially improved asymptotics

Perturbation theory

- hard problem = easy problem + "small" correction
- perturbation theory generally → divergent series

e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

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e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

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- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients

Perturbation theory

but it works ...

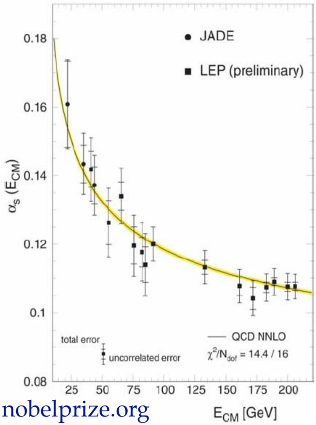
Perturbation theory works

QED perturbation theory:

$$\frac{1}{2}(g-2) = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - (1.7283(35))\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,184\,42$$



nobelprize.org

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$

Perturbation theory

but it is divergent ...

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever ... That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

N. Abel, 1802 – 1829

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The series is divergent; therefore we may be able to do something with it

O. Heaviside, 1850 – 1925

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

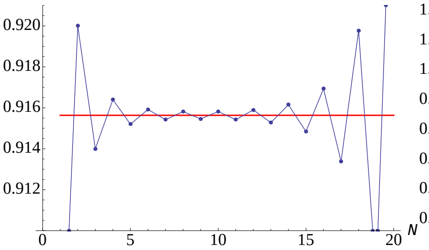
$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

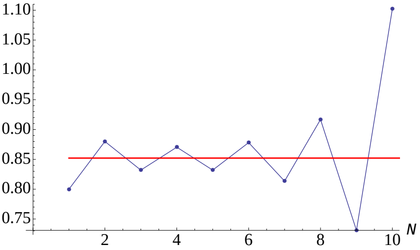
truncate just before least term (x dependent!)

Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, x^n \sim \frac{1}{x} \, e^{\frac{1}{x}} \, E_1\left(\frac{1}{x}\right)$$



$x = 0.1$



$x = 0.2$

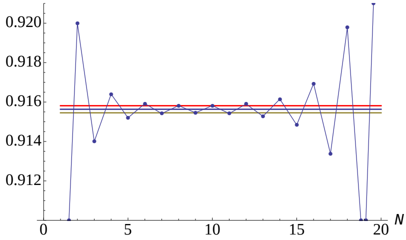
optimal order depends on x: $\quad N \approx \frac{1}{x}$

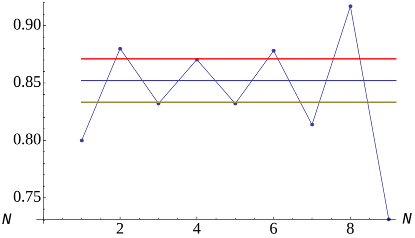
Asymptotic Series: exponential precision

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$

optimal truncation: error term is <u>exponentially small</u>

$$|R_N(x)|_{N\approx 1/x} \approx N!\, x^N\Big|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$

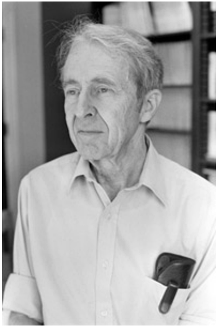




$x = 0.1$ $\qquad$ $x = 0.2$

Divergent series converge faster than convergent series because they don't have to converge

G. F. Carrier, 1918 – 2002



Perturbation theory

QED: fine-structure constant is small:

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137.0360037\ldots}$$

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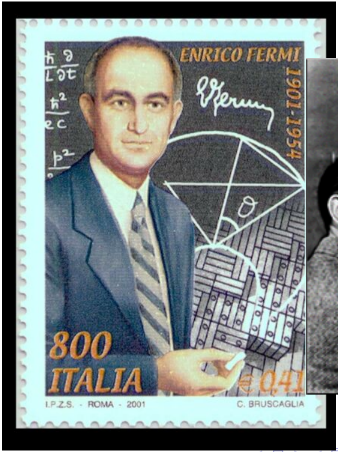

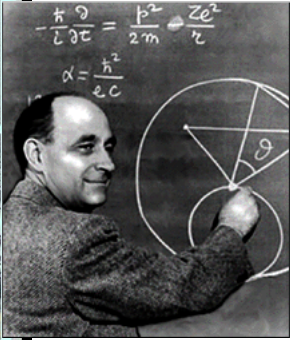

Borel summation: basic idea

Emile Borel

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

alternating factorially divergent series:

$$\sum_{n=0}^{\infty} (-1)^n\, n!\, g^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1+g\,t} \qquad (?)$$

integral convergent for all $g > 0$: "Borel sum" of the series

Borel Summation: basic idea

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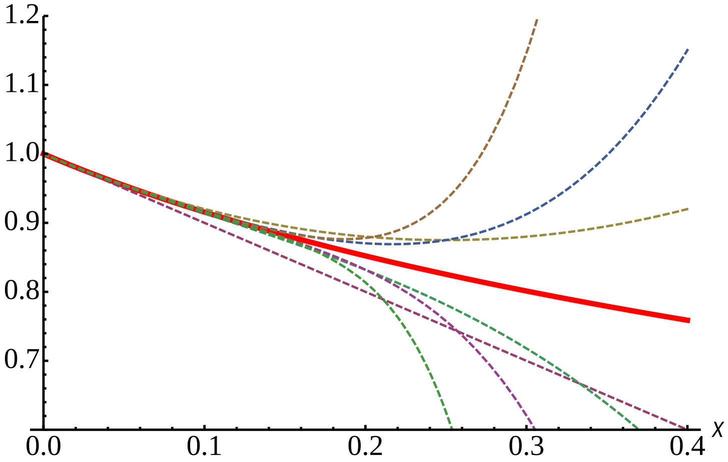

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pole on the Borel axis!

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pole on the Borel axis!

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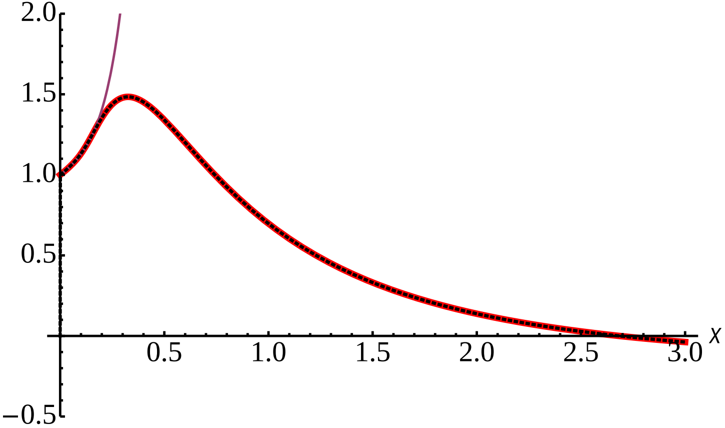
$\Rightarrow$ non-perturbative imaginary part

$$\pm \frac{i\,\pi}{g}\, e^{-\frac{1}{g}}$$

but every term in the series is real !?!

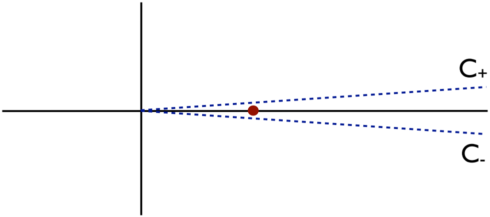
$$\text{Borel} \quad \Rightarrow \quad \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^{\infty} dt\, e^{-t}\,\frac{1}{1-x\,t} = \frac{1}{x}\, e^{-\frac{1}{x}}\, Ei\left(\frac{1}{x}\right)$$



Borel singularities

avoid singularities on $\mathbb{R}^+$: lateral Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



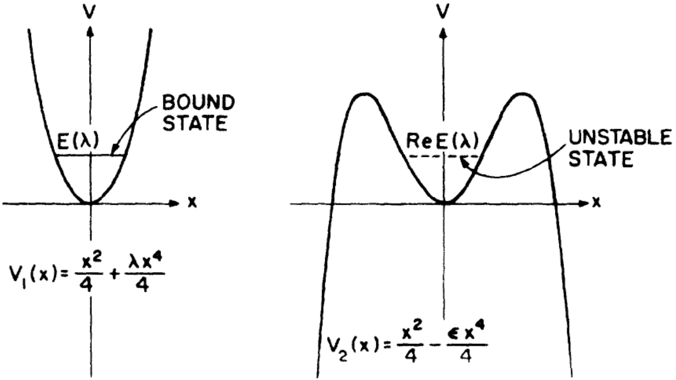
go above/below the singularity: $\theta = 0^\pm$

$\longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{S}_0 f(g)]$

challenge: use physical input to resolve ambiguity

Bender/Wu, 1969

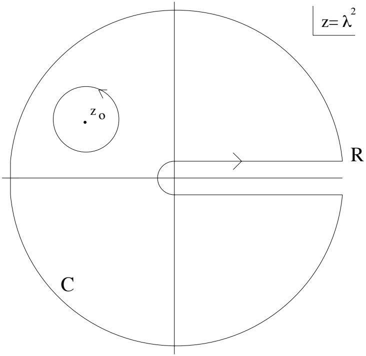
$$V(x) = \frac{x^2}{4} + \lambda \frac{x^4}{4}$$



Borel Summation and Dispersion Relations

cubic oscillator: $V = x^2 + \lambda\, x^3$

A. Vainshtein, 1964



$$E(z_0) = \frac{1}{2\pi i} \oint_C dz\, \frac{E(z)}{z - z_0}$$

$$= \frac{1}{\pi} \int_0^R dz\, \frac{Im\, E(z)}{z - z_0}$$

$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi} \int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}} \right)$$

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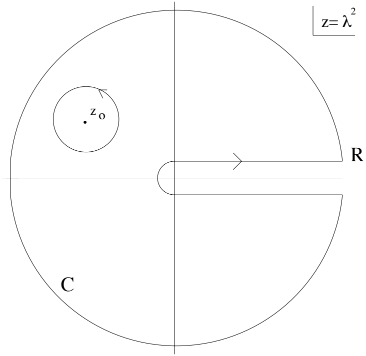

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$$\text{WKB} \Rightarrow \mathrm{Im}\, E(z) \sim \frac{a}{\sqrt{z}}\, e^{-b/z} \quad , \quad z \to 0$$

$$\Rightarrow c_n \sim \frac{a}{\pi}\int_0^\infty dz\, \frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}}$$

Borel summation in practice (physical applications)

direct quantitative correspondence between:

rate of growth $\leftrightarrow$ Borel poles $\leftrightarrow$ non-perturbative exponent

non-alternating factorial growth: $c_n \sim \beta^n \, \Gamma(\gamma\, n + \delta)$

positive Borel singularity: $t_c = \left(\dfrac{1}{\beta\, g}\right)^{1/\gamma}$

non-perturbative exponent: $\pm i\, \dfrac{\pi}{\gamma} \left(\dfrac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\dfrac{1}{\beta g}\right)^{1/\gamma}\right]$

Divergence of perturbation theory

an important part of the story ...

The majority of nontrivial theories are seemingly unstable at some phase of the coupling constant, which leads to the asymptotic nature of the perturbative series

A. Vainshtein (1964)

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

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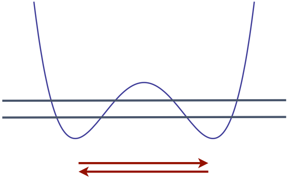
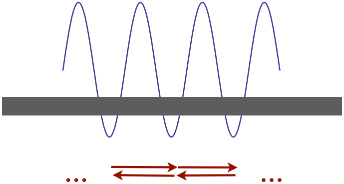
Bogomolny/Zinn-Justin mechanism in QM





- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

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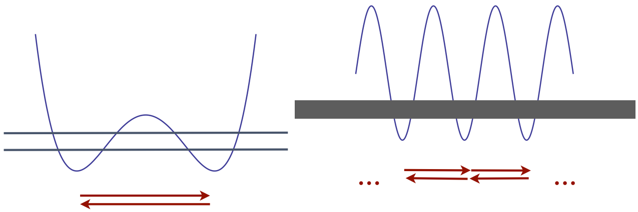

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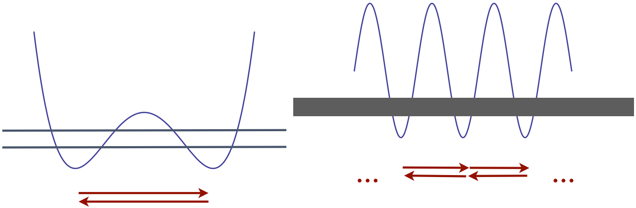
surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel summable:
     ill-defined/incomplete

  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

Decoding of Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q} \, g^{2n} \left[\exp\left(-\frac{S}{g^2} \right) \right]^k \left[\ln\left(-\frac{1}{g^2} \right) \right]^q$$

- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0} \, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i \, \pi \, \alpha \, e^{-2S/g^2}$

- but $c_{0,2,1} = -\alpha$: BZJ cancellation !

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pert flucs about instanton: $e^{-S/g^2}\, \left(1 + a_1 g^2 + a_2 g^4 + \dots\right)$

divergent:

$a_n \sim \frac{n!}{(2S)^n} \left(a\, \ln n + b\right) \Rightarrow \pm i\, \pi\, e^{-3S/g^2} \left(a \ln \frac{1}{g^2} + b\right)$

- 3-instanton: $e^{-3S/g^2} \left[\frac{a}{2}\left(\ln\left(-\frac{1}{g^2}\right)\right)^2 + b\, \ln\left(-\frac{1}{g^2}\right) + c\right]$

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resurgence: *ad infinitum*, also sub-leading large-order terms

Towards Resurgence in QFT

- resurgence $\equiv$ analytic continuation of trans-series
- effective actions, partition functions, ..., have natural integral representations with resurgent asymptotic expansions
- analytic continuation of external parameters: temperature, chemical potential, external fields, ...
- e.g., magnetic $\leftrightarrow$ electric; de Sitter $\leftrightarrow$ anti de Sitter, . . .
- matrix models, large N, strings, ... (Mariño, Schiappa, ...)
- soluble QFT: Chern-Simons, ABJM, $\rightarrow$ matrix integrals

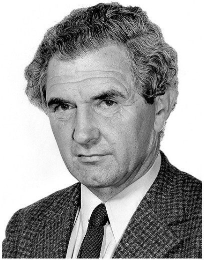
- asymptotically free QFT ?

Divergence of perturbation theory in QFT

- C. A. Hurst (1952):

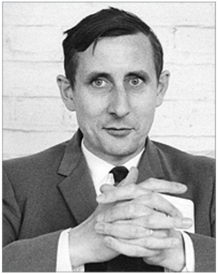
ϕ^4 perturbation theory is divergent:

(i) factorial growth of number of diagrams
(ii) explicit lower bounds on diagrams



If it be granted that the perturbation expansion does not lead to a convergent series in the coupling constant for all theories which can be renormalized, at least, then a reconciliation is needed between this and the excellent agreement found in electrodynamics between experimental results and low-order calculations. It is suggested that this agreement is due to the fact that the S-matrix expansion is to be interpreted as an asymptotic expansion in the fine-structure constant ...

Dyson's argument (QED)



- F. J. Dyson (1952):
physical argument for divergence of QED perturbation theory

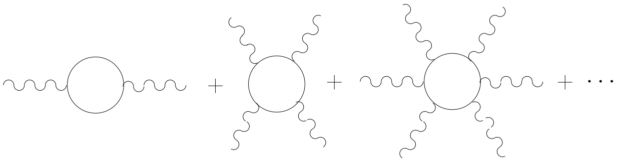
$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \ldots$$

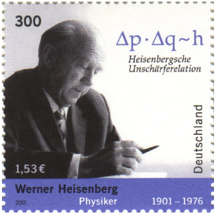
> *Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*

- *suggests* perturbative expansion cannot be convergent

Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216





- 1-loop QED effective action in uniform emag field
- e.g., constant B field:

$$S = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left(\coth s - \frac{1}{s} - \frac{s}{3} \right) \exp\left[-\frac{m^2 s}{eB} \right]$$

$$S = -\frac{e^2 B^2}{2\pi^2} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{2eB}{m^2} \right)^{2n+2}$$

Euler-Heisenberg Effective Action and Schwinger Effect

B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \mathrm{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

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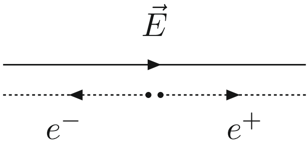
Schwinger effect:



$\mathrm{Im}\, S \to$ physical pair production rate

- suggests Euler-Heisenberg series must be divergent

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$
$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even

another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

Asymptotic Expansions & Analytic Continuation

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots + \frac{174611}{6600z^{20}} - \dots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$

formal series $\quad \Rightarrow \quad \mathrm{Im}\, \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$

- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \mathrm{Im}\, \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

"raw" asymptotics inconsistent with analytic continuation

QFT: Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

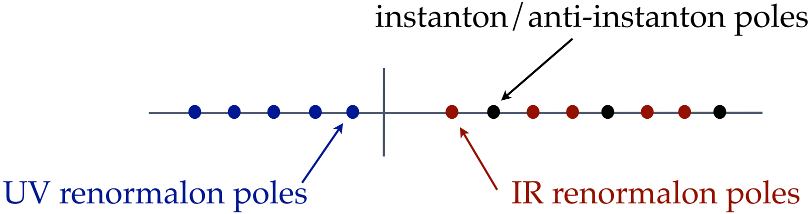
QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: "renormalons"
- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow$ $\pm\, i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow$ $\pm\, i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

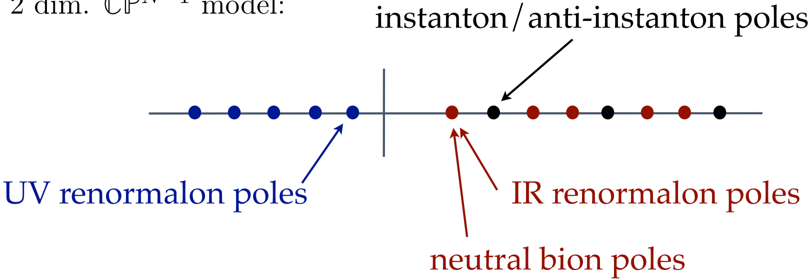
asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

The Bigger Picture

Q: should we expect resurgent behavior in QM and QFT ?

QM uniform WKB $\Rightarrow$

(i) trans-series structure is generic
(ii) all multi-instanton effects encoded in perturbation theory

(GD, Ünsal, 1306.4405, 1401.5202)

Q: what is behind this resurgent structure ?

- basic property of all-orders steepest descents integrals

Q: could this extend to (path) functional integrals ?

Uniform WKB and Resurgent Trans-Series for Eigenvalues

(GD, Ünsal, 1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi + \frac{V(g\,x)}{g^2}\psi = E\,\psi \rightarrow -g^4\,\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\,E\,\psi(y)$$

- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

Connecting Perturbative and Non-Perturbative Sector

Uniform WKB $\Rightarrow$ trans-series form for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions

generic and universal

Zinn-Justin/Jentschura: generate *entire trans-series* from

(i) perturbative expansion $E = E(N, g^2)$
(ii) single-instanton fluctuation function $\mathcal{F}(N, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

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(i) perturbative expansion $E = E(N, g^2)$
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(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

in fact ... (GD, Ünsal, 1306.4405, 1401.5202)

$$\mathcal{F}(N, g^2) = \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4}\left(\frac{\partial E(N, g^2)}{\partial N} - 1 + \frac{\left(N + \frac{1}{2}\right)\, g^2}{S}\right)\right]$$

implication: perturbation theory encodes everything !

Connecting Perturbative and Non-Perturbative Sector

e.g. double-well potential: $B \equiv N + \frac{1}{2}$

$$E(N, g^2) = B - g^2\left(3B^2 + \frac{1}{4}\right) - g^4\left(17B^3 + \frac{19}{4}B\right) - g^6\left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - \dots$$

- non-perturbative function $(\mathcal{F} \sim (\dots)\exp[-A/2])$:

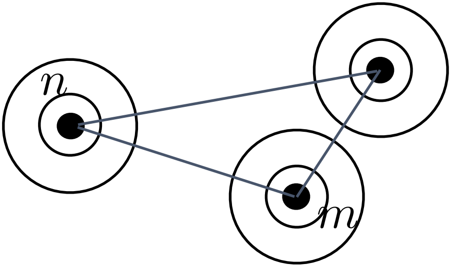
$$A(N, g^2) = \frac{1}{3g^2} + g^2\left(17B^2 + \frac{19}{12}\right) + g^4\left(125B^3 + \frac{153B}{4}\right) + g^6\left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) +$$

- simple relation:

$$\frac{\partial E}{\partial B} = -3g^2\left(2B - g^2\frac{\partial A}{\partial g^2}\right)$$

Connecting Perturbative and Non-Perturbative Sector

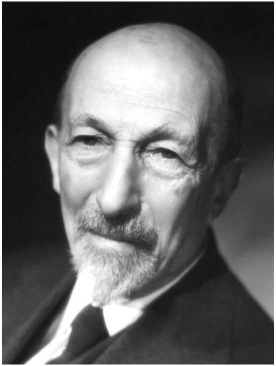
all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



why ? turn to path integrals ....

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard, 1865 - 1963



- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- exact resurgent relation between fluctuations about $n^{\rm th}$ saddle and about neighboring saddles m

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

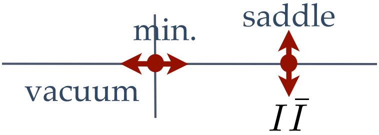
- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

All-Orders Steepest Descents: Darboux Theorem

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^{\pi} dz\, e^{-\frac{1}{g^2} \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$
\begin{aligned}
T_r^{(0)} &= \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)} \\
&\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1-\frac{\frac{1}{4}}{(r-1)}+\frac{\frac{9}{32}}{(r-1)(r-2)}-\frac{\frac{75}{128}}{(r-1)(r-2)(r-3)}+
\end{aligned}
$$

- low order coefficients about saddle z_1:

$$
T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1-\frac{1}{4}g^2+\frac{9}{32}g^4-\frac{75}{128}g^6+\dots\right)
$$

- fluctuations about the two saddles are explicitly related

Resurgence in Path Integrals: "Functional Darboux Theorem"

could something like this work for path integrals?

"functional Darboux theorem" ?

- multi-dimensional case is already non-trivial and interesting

Pham (1965); Delabaere/Howls (2002)

- Picard-Lefschetz theory

- do a computation to see what happens ...

Resurgence in Path Integrals

- periodic potential: $V(x) = \frac{1}{g^2}\, \sin^2(g\, x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots \right)$$

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- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \dots \right)$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$Z = \int dx\, e^{-S(x)}$$

- critical points (saddle points): $\partial S/\partial z = 0$
- steepest descent contour: $\mathrm{Im}\, S(z) = \text{constant}$
- contour flow-time parameter t:

$$\frac{d}{dt}\mathrm{Im}\, S(z) = \frac{1}{2i}\left(\frac{\partial S}{\partial z}\,\dot{z} - \frac{\partial \bar{S}}{\partial \bar{z}}\,\dot{\bar{z}}\right) \quad , \quad \frac{d}{dt}\mathrm{Re}\, S(z) = \frac{1}{2}\left(\frac{\partial S}{\partial z}\,\dot{z} + \frac{\partial \bar{S}}{\partial \bar{z}}\,\dot{\bar{z}}\right)$$

- flow along a steepest decent path:

$$\dot{z} = \frac{\partial \bar{S}}{\partial \bar{z}} \qquad \Rightarrow \frac{d}{dt}\mathrm{Im}\, S(z) = 0 \quad , \quad \frac{d}{dt}\mathrm{Re}\, S(z) = \left|\frac{\partial S}{\partial z}\right|^2 > 0$$

- monotonic in real part

$$Z = e^{-i\, S_{\mathrm{imag}}(x)} \int_\Gamma dz\, e^{-S_{\mathrm{real}}(z)}$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\rm imag}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\rm real}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Non-perturbative Physics Without Instantons

e.g, 2d Principal Chiral Model: (Cherman, Dorigoni, GD, Ünsal, 1308.0127)

$$S = \frac{N}{2\lambda} \int d^2x \, \mathrm{tr}\, \partial_\mu U \partial^\mu U^\dagger \qquad , \qquad U \in SU(N)$$

- non-Borel-summable pert. theory: IR renomalons
- but, the theory has no instantons !

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- non-Borel-summable pert. theory: IR renomalons
- but, the theory has no instantons !

resolution: non-BPS saddle point solutions to 2nd-order classical Euclidean equations of motion: "unitons"

$$\partial_\mu \left(U^\dagger \partial_\mu U \right) = 0 \qquad \text{(Uhlenbeck 1985)}$$

- have negative fluctuation modes: saddles, not minima
- fractionalize on cylinder $\longrightarrow$ BZJ cancellation

$\mathbb{CP}^{N-1}$, PCM, Yang-Mills, ... all have finite action non-BPS solutions (Din/Zakrzewski 1980; Uhlenbeck 1985; Sibner/Sibner/Uhlenbeck 1989)

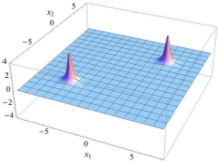
- "unstable": negative modes of fluctuation operator
- what do these mean ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

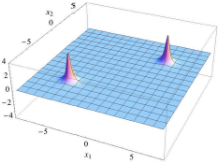
$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

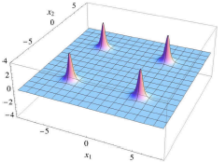
(Dabrowski, GD, arXiv:1306.0921)



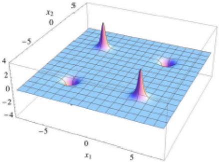
Action Density of $\omega_{(0)}$: ($S_{(0)} = 2$)



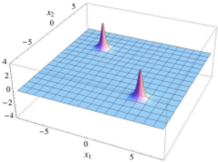
Charge Density of $\omega_{(0)}$: ($Q_{(0)} = 2$)



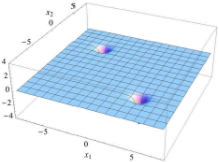
Action Density of $\omega_{(1)}$: ($S_{(1)} = 4$)



Charge Density of $\omega_{(1)}$: ($Q_{(1)} = 0$)



Action Density of $\omega_{(1)}$: ($S_{(2)} = 2$)



Charge Density of $\omega_{(2)}$: ($Q_{(2)} = -2$)

Conclusions

- perturbation theory is generically divergent
- resurgence systematically unifies perturbation theory and non-perturbative physics into a trans-series
- there is extra ‘magic’ in perturbation theory
- IR renormalon puzzle in asymptotically free QFT
- basic property of steepest descents expansions
- moral: consider all saddles, including non-BPS
- resurgence required for analytic continuation

Open Problems

- natural path integral construction
- analytic continuation of path integrals
- physics of QFT saddles/thimbles ?
- renormalization group flow ?
- strong- & weak-coupling expansions: dualities ?
- operator product expansion (OPE) ?
- SUSY and extended SUSY ?
- localization ?
- ...