Inverse spectral problems for Bessel operators*

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The problem

We consider a Schrödinger operator

$$(Sy)(\mathbf{x}) = -\Delta y(\mathbf{x}) + Q(\mathbf{x})y(\mathbf{x})$$

in the unit ball of $\mathbb{R}^3$, with a spherically symmetric distributional potential $Q(\mathbf{x}) = q(|\mathbf{x}|)$, $q \in W_2^{-1}(0,1)$. Rotational symmetry allows a decomposition of S via the spherical harmonics, which leads to Bessel operators

$$S(q,l,\theta)y(x) := -y''(x) + \frac{m(m+1)}{x^2}y(x) + q(x),$$

$m \in \mathbb{Z}_+$, subject to $\sin\theta y^{[1]}(x) = \cos\theta y(1)$, $\theta \in [0,\pi)$. $S(q,m,\theta)$ has a simple discrete spectrum $\lambda_1(q,m,\theta) < \lambda_2(q,m,\theta) < \dots$.

Que: Does the spectrum $(\lambda_n(q,m,\theta))$ determine q,m,θ? **No!** E.g., for $m=0$, extra information is needed (e.g. spectrum for $\theta_1 \neq \theta$, or norming constants α_n).

The **inverse spectral problem (ISP)** is to reconstruct a Bessel operator $S(q,m,\theta)$ from the spectral data (SD)

Aim:
- find the algorithm of solution of ISP;
- give an explicit and complete description of SD

Known results: $m = 0$

Borg (1946), Levinson (1949): two spectra determine q uniquely

Gelfand & Levitan, Krein, Marchenko (1950-ies) treated the regular case $q \in L_1(0,1)$, found sufficient conditions and necessary conditions on the SD and solved the ISP.

Zhikov (1967): $q = F'$ with $F \in \mathrm{BV}[0,1]$; $Tu = f$ is defined through the corresponding integral equation; necessary and sufficient conditions on the SD found, the ISP solved.

Ben Amor & Remling (2003): $q = F'$ on $(0,\infty)$ with F locally of bounded variation; applied de Branges space method to solve ISP on $[0,N]$ for arbitrary $N \in \mathbb{R}_+$. "Spectral data" used is $\phi(x) := \int \cos\sqrt{\lambda}x\, d(\rho_N - \rho_0)(\lambda)$.

Andersson (1988) considered a SL operator in impedance form $Su = \frac{1}{a}(au')'$ in $L_2((0,1);a)$ with $a \in W_p^1[0,1]$, $p \geq 1$, or $a \in \mathrm{BV}[0,1]$ and established local solvability of the ISP.

Rundell & Sacks (1992) studied the case $a \in W_2^1(0,1)$. With the help of transformation operators they found necessary conditions on the SD, solved the ISP, and suggested a numerical algorithm.

Coleman & McLaughlin (1993) treated the case $a \in W_2^1(0,1)$ by recasting $Su = \lambda u$ as $v'' + bv' + \lambda^2 v = 0$ with $b := a'/a$; studied in detail the mapping $b \mapsto$SD; generalized the approach of Pöschel & Trubowitz (1987).

Observe that S is similar to $Tu = -u'' + qu$ with $q = \frac{(\sqrt{a})''}{\sqrt{a}}$. In particular, for $a \in W_2^1(0,1)$ we get $q \in W_2^{-1}(0,1)$.

The case of a generic $q \in W_2^{-1}(0,1)$ was treated by Shkalikov a.o. (99–05); T is defined by the regularisation method, its spectral properties studied in detail.

ISP (in different settings) for SL operator with such q is completely solved by Albeverio, H., Mykytyuk (03–05)

Some other types of singularities were treated by Carlsson, Hald, Freiling, McLaughlin, Yurko a.o.

Known results: $m > 0$, $q \in L_2$

Gulliot, Ralston (88): studied the map from q to SD for $m = 1$, generalised the approach by Pöschel–Trubowitz, proved that the map is $1-1$, described the isospectral sets

Carlson (93) completely described the possible spectra for arbitrary $m \in \mathbb{N}$ using the Darboux–Crum transformation and studied the isospectral sets

Carlson (97) studied the map q to SD for $m \geq -\frac{1}{2}$, proved several results on unique reconstruction of $S(q,m,\theta)$ from the spectral data, without characterising the spectral data

Gasymov (65) claimed a complete solution for $q \in L_2(0,1)$ and $m \in \mathbb{N}$ without proof

Another setting: reconstruct q from the spectra of $S(q,m_1,0)$ and $S(q,m_2,0)$ for two different angular momenta m_1 and m_2; even uniqueness is not proved!

Carlson, Shubin (94): isospectral set is of finite dimension if $m_1 - m_2$ odd;

Rundell, Sacks (01): local uniqueness in a linearised sense for $m_1, m_2 = 0, 1, 2, 3$.

Our case: $m \in \mathbb{Z}_+$ and $q \in W_2^{-1}(0,1)$

$m = 0$: Definition

For real-valued $q \in W_2^{-1}(0,1)$ define the SL operator T by **regularisation** method:

take $\sigma \in L_2(0,1)$ s. t. $q = \sigma'$, (e.g., with $\int \sigma = 0$) and put

$$Tu = T_\sigma u = l_\sigma(u) := -(u' - \sigma u)' - \sigma u'$$

$$\operatorname{dom} T_\sigma = \{u \in W_2^1 \mid u' - \sigma u \in W_1^1,\ l_\sigma(u) \in L_2,$$
$$u(0) = u(1) = 0\}.$$

T_σ is a self-adjoint bounded below operator with discrete spectrum $\{\lambda_k\}$; we may assume $\lambda_k > 0$.

Example 1: $q = \alpha\delta(\cdot - \frac{1}{2})$. Take

$$\sigma(x) = 0 \quad \text{for } x \le \tfrac{1}{2}, \qquad \sigma(x) = \alpha \quad \text{for } x > \tfrac{1}{2}$$

then $l_\sigma(u) = -u''$ if $x \ne \frac{1}{2}$ and $u \in \operatorname{dom} T_\sigma$ means u is continuous at $x = \frac{1}{2}$ and $u'(\frac{1}{2}+) - u'(\frac{1}{2}-) = \alpha u(\frac{1}{2})$.

Example 2: $q = (x - \frac{1}{2})^{-1}$. Restriction-extension theory defines the corresponding (non-s.a.) operators $T(\gamma)$, $\gamma \in \mathbb{C} \cup \{\infty\}$ by the interface conditions $y(\frac{1}{2}+) = y(\frac{1}{2}-) =: y(\frac{1}{2})$, $y'(\frac{1}{2}+) - y'(\frac{1}{2}-) = \gamma y(\frac{1}{2})$; cf. Kurasov (1996), Bodenstorfer a.o. (2000). This corresponds to

$$\sigma(x) = \begin{cases} \log(\frac{1}{2} - x) & \text{for } x \le \frac{1}{2}, \\ \log(x - \frac{1}{2}) + \gamma & \text{for } x > \frac{1}{2}. \end{cases}$$

$m = 0$: ISP

[H.&Mykytyuk'04]: There is a **transformation** operator $I + K_\sigma$ s. t. $K_\sigma u(x) = \int_0^x k(x,t)u(t)\,dt, \quad k(x,\cdot) \in L_2$, and

$$y(x,\lambda) := (I + K_\sigma)\sin\sqrt{\lambda}x$$

solves the equation $l_\sigma(u) = \lambda u$, $u(0) = 0$.

Spectral asymptotics:

(A1) $\sqrt{\lambda_k} = \pi k + \tilde{\lambda}_k$ for some $(\tilde{\lambda}_k) \in \ell_2$;

(A2) $\alpha_k^{-1} := 2\|y(\cdot,\lambda_k)\|^2 = 1 + \beta_k$ for some $(\beta_k) \in \ell_2$.

Reconstruction of σ. *Assume that $\{(\lambda_k),(\alpha_k)\}$ satisfy (A1)–(A2), α_k are positive, and λ_k are pairwise distinct. Put $\phi(s) := \sum_{k\in\mathbb{N}}\Big(\cos\pi ks - \alpha_k\cos\lambda_k s\Big) \in L_2(0,2)$, $f(x,t) := \phi(x-t) - \phi(x+t)$, and consider the* ***Gelfand-Levitan-Marchenko*** *(GLM) equation:*

$$k(x,t) + f(x,t) + \int_0^x k(x,s)f(s,t)\,ds = 0, \quad x > t.$$

Then:

(1) GLM is soluble, and the integral operator K with kernel k coincides with K_σ for

$$\sigma(x) := -2\phi(2x) - 2\int_0^x k(x,t)f(t,x)\,dt \in L_2(0,1);$$

(2) the sequence $\{(\lambda_k),(\alpha_k)\}$ is the SD for the Sturm–Liouville operator T_σ with σ found.

Reconstruction by two specra. *Assume that sequences (λ_k) and (μ_k) interlace, λ_k satisfy (A1), and μ_k are such that*

(A3) $\sqrt{\mu_k} = \pi(k - \frac{1}{2}) + \tilde{\mu}_k$ *for some* $(\tilde{\mu}_k) \in \ell_2$.

Then there is a unique $\sigma \in L_2$ such that λ_n (resp. μ_n) are Dirichlet (resp. Dirichlet–Neumann) eigenvalues of l_σ.

An analogue of *Marchenko's theorem* for $q \in L_2(0,1)$: interlacing and correct asymptotics suffice!

$m > 0$: idea

For $q \in L_2$, Carlson (93) showed that the eigenvalues of $S(q, m, 0)$ satisfy

$$\lambda_n(q, m, 0) = \pi^2(n + \tfrac{m}{2})^2 + C + c_n \text{ with } (c_n) \in \ell_2.$$

In particular, $S(q, 2, 0)$ has 1 EV less than $S(q, 0, 0)$!
Idea: take $\lambda_0 < \lambda_1(q, 2, 0)$, find a SL operator with potential $\tilde{q}$, whose Dirichlet spectrum is λ_0, $\lambda_1(q, 2, 0), \lambda_2(q, 2, 0), \ldots$, and then determine q from $\tilde{q}$

Realisation via the transformation operators: take $\{(\lambda_k)_{k\in\mathbb{N}}, (\alpha_k)_{k\in\mathbb{N}}\}$, $0 < \lambda_0 < \lambda_1$, $\alpha_0 > 0$, and let

$$\begin{aligned} I + F_j &:= \operatorname*{s-lim}_{n\to\infty} \sum_{k=j}^{n} \alpha_k(\cdot, \cos\sqrt{\lambda_k}t)\cos\sqrt{\lambda_k}x \\ &= (I + K_j)^{-1}(I + K_j^*)^{-1} \end{aligned}$$

Lemma: $K := (I + K_1)(I + K_0)^{-1} - I$ has kernel

$$k(x, t) = \frac{\alpha_0 y(x, \lambda_0) y(t, \lambda_0)}{1 - \alpha_0 \int_0^x y^2(s, \lambda_0)\, ds},$$

with $y(x, \lambda_0) := (I + K_0)\cos\sqrt{\lambda_0}x$

In particular: (1) $\quad I + K_1 = (I + K)(I + K_0)$,

(2) $$\sigma_1(x) - \sigma_0(x) = 2k(x, x) + \alpha_0,$$

(3) $$k(x, x) \sim 3x^{-1} \text{ as } x \to 0.$$

Spectral transformations

Let $q \in W_2^{-1}(0,1)$, $m \in \mathbb{Z}_+$, and $y(\cdot,\lambda)$ be a solution to

$$-y''(x) + \frac{m(m+1)}{x^2} y(x) + q(x)y(x) = \lambda y(x)$$

subject to $y(1) = 0$; then either $y(x,\lambda) \sim x^{-m}$ or $y(x,\lambda) \sim x^{m+1}$ as $x \to 0$, in the latter case λ is an EV.

Lemma: Take λ_0 not an EV and $\alpha_0 > 0$, and put

$$\beta(x,\lambda) := \alpha_0 \int_x^1 y(t,\lambda)y(t,\lambda_0)\,dt$$

and $V(x) := 1 + \beta(x,\lambda_0)$; then $\exists q_0 \in W_2^{-1}(0,1)$ s. t.

$$u(x,\lambda) := y(x,\lambda) - y(x,\lambda_0)\frac{\beta(x,\lambda)}{V(x)}$$

solves the equation

$$-y''(x) + \frac{(m-2)(m-1)}{x^2} y(x) + q_0(x)y(x) = \lambda y(x).$$

Rem: In fact, $V(x) = x^{-2m+1}v(x)$ and $q_0 := q - 2\frac{d^2}{dx^2}\log v(x)$

Thm: *The spectrum of the operator $S(q_0, m-2, 0)$ consists of λ_0 and $\lambda_k(q,m,0)$, $k \in \mathbb{N}$; moreover, $\|u(\cdot,\lambda_0)\| = \alpha_0$ and $\|u(\cdot,\lambda_k)\| = \|y(\cdot,\lambda_k)\|$, $k \in \mathbb{N}$.*

There is an analogous transformation removing one EV of $S(q,m,0)$ and changing neither the others nor the corresponding norming constants; this produces an operator $S(q_1, m+2, 0)$ for some $q_1 \in W_2^{-1}(0,1)$

ISP for Bessel operators

Reconstruction from one spectrum.

Assume that sequences (λ_n) and (α_n) of real numbers are such that

(B1) the λ_n strictly increase and satisfy the asymptotics $\lambda_n = \left[\pi(n + \frac{m}{2}) + \tilde{\lambda}_n\right]^2$ with $(\tilde{\lambda}_n) \in \ell_2$

(B2) the α_n are positive and satisfy the asymptotics $\alpha_n = 1 + \tilde{\alpha}_n$ with $(\tilde{\alpha}_n) \in \ell_2$.

Then there exists a unique real-valued $q \in W_2^{-1}(0,1)$ such that λ_n and α_n are respectively the eigenvalues and the norming constants of the Bessel operator $S(q,m,0)$.

Reconstruction from two spectra.

In order that two strictly increasing sequences (λ_n) and (μ_n) be the spectra of the operators $S(q,m,0)$ and $S(q,m,\theta)$ for some real-valued $q \in W_2^{-1}(0,1)$, $m \in \mathbb{N}$, and $\theta \in (0,\pi)$, it is necessary and sufficient that these sequences interlace, i.e., that $\mu_n < \lambda_n < \mu_{n+1}$ for all $n \in \mathbb{N}$, that λ_n satisfy the asymptotics of (B1) and that

(B3) $\mu_n = \left[\pi(n + \frac{m-1}{2}) + \tilde{\mu}_n\right]^2$ with $(\tilde{\mu}_n) \in \ell_2$.

In this case $q \in W_2^{-1}(0,1)$ and $\theta \in (0,\pi)$ are unique and are effectively reconstructed from the two spectra.

Idea: two spectra determine the norming constants!