Embedded eigenvalues of the Laplacian for domains with cylindrical ends

(naïve approach)

Michael Levitin (Heriot-Watt)

with

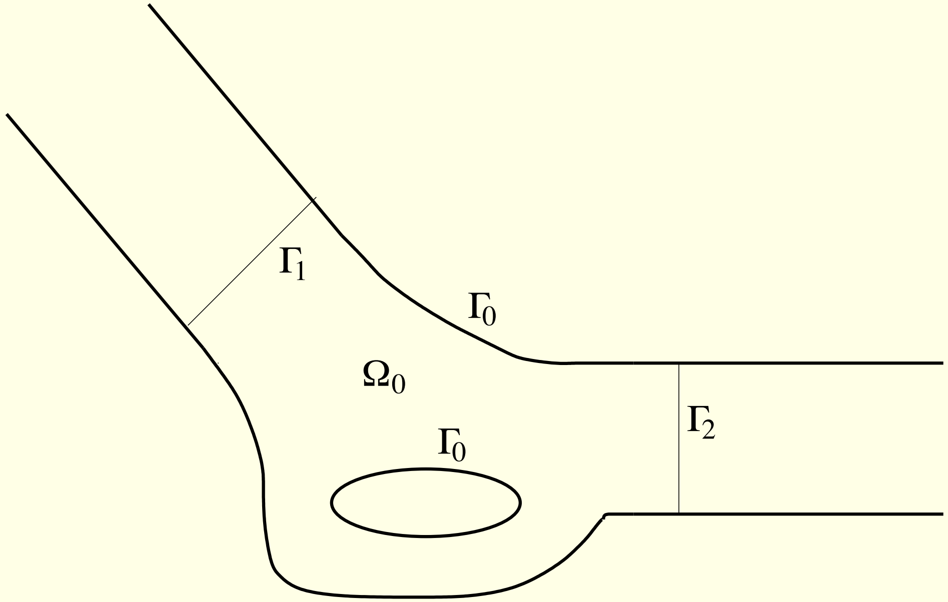
Marco Marletta (Cardiff)

LMS Durham Symposium, 2–12 August 2005.

Precise title: Some spectral characteristics of some operators for domains with some regular outlets

Model geometry:

$\Omega \subset \mathbb{R}^d$, $d \geq 2$ — an unbounded connected domain with a finite number of cylindrical ends: $\Omega \setminus B(R_0) = \bigcup_{n=1}^N \mathcal{C}_n =: \mathcal{C}$ for some $R_0 > 0$.



$R > R_0$, $\Omega_0 := \Omega \cap B(R)$ — the "centre", $\Gamma_0 := \partial\Omega_0 \cap \partial\Omega$, $\Gamma_n := \mathcal{C}_n \cap S(R)$ — cross-sections, $\Gamma := \bigcup_{n=1}^N \Gamma_n$. Boundary $\partial\Omega$ is piecewise C^2.

Model spectral problem:

$$\begin{cases} -\Delta u &= \lambda u \qquad \text{in} \quad \Omega\,, \\ \mathbf{B}(\mathbf{x})u &= 0\,, \qquad \mathbf{x} \in \partial\Omega\,. \end{cases} \tag{1}$$

Conditions on boundary operators $\mathbf{B}$:

- are some appropriate mix of Dirichlet and generalized Neumann conditions, allowing the construction of the corresponding operator $-\Delta_{\Omega,\mathbf{B}}$ in the weak sense via the Friedrichs extensions, with domains $\subseteq H^1(\Omega)$;
- the resulting operator $-\Delta_{\Omega,\mathbf{B}}$ is self-adjoint and semi-bounded below;
- the operators $\mathbf{B}$ do not depend on the longitudinal coordinate x for each particular cylindrical end C_n (but may differ from one cylinder to another and on connected components of each cylindrical boundary). Abusing notation, we would also allow periodic conditions where appropriate.

Main question: How to compute the eigenvalues of (1) (i.e. find λ s.t. there exists a solution u bounded in $H^1(\Omega)$ norm)?

Motivation: waveguide problems in 2D. Here $\Omega = \mathbb{R} \times [-1, 1] \setminus \mathcal{O}$, where the obstacle $\mathcal{O}$ is a (not necessarily simply connected) compact subset of the strip which may be one-dimensional (e.g. an interval). Dirichlet boundary conditions $=$ a quantum waveguide, Neumann conditions $=$ an acoustic waveguide.

10^∞ papers but mainly:

- constructing sufficient conditions for existence or non-existence of the eigenvalues in lower parts of the spectrum, or below the continuous spectrum;
- procedures for finding embedded eigenvalues for obstacles of simple shapes (rectangles, circles, etc.) using special function expansions;
- asymptotic results in relation to various additional parameters, or asymptotic distribution of the counting function of the discrete and continuous spectrum;

For actual computations, difficult numerics involving two major steps:

1) Find money for a PhD student;

2) Wait three years and hope for the best.

Side track (a bottle of wine challenge a la Shargorodsky): find sufficient conditions for absolute continuity of the spectrum. Known in two cases and their variations: a straight strip (Dirichlet or Neumann) and Rellich's semi-strip bounded by a graph of a function (Dirichlet).

Back to our problem: our approach is "constructive mathematics" and we shall show that one in fact needs only the following:

- ability to solve numerically spectral problems in bounded domains;
- numerical integration;
- finding the roots of monotone functions.

Continuous spectrum

Transversal problem: cross-section $\Gamma \subset \mathbb{R}^{d-1}$ is a bounded (not necessarily connected) domain with sufficiently smooth boundary $g = \partial\Gamma$. Let $\mathbf{x}'$ be coordinates in $\mathbb{R}^{d-1}$. Let $-\Delta_{\Gamma;\mathbf{B}}$ be a Laplacian on Γ subject to the boundary conditions $\mathbf{B}(\mathbf{x}')u = 0$ on g, where $\mathbf{B}$ as above. Let $(0 \le)\kappa_1 \le \kappa_2 \le \dots \le \kappa_j \le \dots$ be its eigenvalues (called thresholds), and $\chi_j(\mathbf{x}')$ be the corresponding normalised orthogonal eigenfunctions.

Then the continuous spectrum of (1) is $[\kappa_1, +\infty]$. Moreover, at any point $\lambda \in [\kappa_1, \infty)$, the multiplicity of the continuous spectrum is $\#\{\kappa_j \le \lambda\}$.

Set $Q_j = \operatorname{span}\{\chi_\ell\}_{\ell=j}^{\infty}$, and denote by $\mathcal{Q}_j$ the projector onto Q_j. Set $\kappa_0 = -\infty$, $\mathcal{Q}_0 = I$ and $\mathcal{P}_j = I - \mathcal{Q}_j$, $j \ge 0$.

Cylindrical ends:

Let $u \in H^1(\Omega)$ be an eigenfunction of (1) corresponding to an eigenvalue $\lambda \in [\kappa_J, \kappa_{J+1})$. Then on the cylindrical ends, u can be found by separation of variables if we know, say, $h := \frac{\partial u}{\partial x}\big|_{x=0} \in Q_J(\Gamma)$:

$$u(x, \mathbf{x}') = -\sum_{j=J+1}^{\infty} \frac{1}{\eta_j(\lambda)} \langle h, \chi_j \rangle_\Gamma \chi_j(\mathbf{x}') e^{-\eta_j(\lambda) x} ,$$

where x is a longitudinal coordinate, $\eta_j(\lambda) := \sqrt{\kappa_j - \lambda}$, and $\langle \cdot, \cdot \rangle_\Gamma$ denotes a scalar product in $L^2(\Gamma)$. Then $f := u|_\Gamma$ is given by $f = -\sum_{j=J+1}^{\infty} \frac{1}{\eta_j(\lambda)} \langle h, \chi_j \rangle_\Gamma \chi_j(\mathbf{x}')$.

Thus we constructed, in the basis χ_j, the representation of the Neumann-Dirichlet operator $\mathcal{R}_{\mathcal{C},\Gamma}(\lambda) : g \mapsto f$ for the cylinder as

$$\mathcal{T}_\lambda := -\operatorname{diag}\left(\frac{1}{\eta_j(\lambda)}\right)_{j=J+1}^{\infty} .$$

Central domain (Ω_0): where the difficulty lives

If we can construct, in the same basis Q_J, the interior Neumann-Dirichlet operator $\mathcal{R}_{\Omega_0,\Gamma}(\lambda) : h \mapsto u|_\Gamma$, where

$$-\Delta u = \lambda u \quad \text{in } \Omega_0 \,, \qquad \mathbf{B}u = 0 \quad \text{on } \Gamma_0 \,, \qquad \partial u / \partial n = h \quad \text{on } \Gamma \,, \tag{2}$$

(1) would then be reduced to the following problem on Γ:

$$\boxed{\begin{array}{c}\text{find all} \lambda \in [\kappa_J, \kappa_{J+1}) \text{ such that the pencil } \mathcal{A}_\lambda(\mu) := \mathcal{Q}_J \mathcal{R}_{\Omega_0,\Gamma}(\lambda) - \mu \mathcal{R}_{\mathcal{C},\Gamma}(\lambda) \\ \text{has an eigenvalue } \mu = 1\end{array}} \tag{3}$$

and the orthogonality condition

$$\boxed{\mathcal{P}_J(\mathcal{R}_{\Omega_0,\Gamma}(\lambda) h) = 0} \tag{4}$$

holds for the corresponding eigenfunction h of $\mathcal{A}_\lambda(\mu)$.

Notation: σ_k, Ψ_k are Neumann eigenvalues and eigenfunctions of (2), $h = 0$.

General properties of $\mathcal{R}_{\Omega_0,\Gamma}(\lambda)$ [we need $\ll$ e.g. Agranovich (this meeting), Safarov (2005, in progress)]: self-adjoint, the associated form is monotone decreasing in λ on intervals not containing σ_k.

Let ϕ_j be some basis in $L_2(\Gamma_0)$. How to compute the actions of $\mathcal{R}_{\Omega_0,\Gamma}(\lambda)$ is this basis? [Remember this ideal PhD student!]

Main trick: can get explicit λ dependence!

We have, by integration by parts

$$< \mathcal{R}_{\Omega_0,\Gamma}(\lambda)\phi_i, \phi_j >=< u_i, \partial u_j/\partial n >= (u_i, \Delta u_j) + (\nabla u, \nabla u_j) = -\lambda(u_i, u_j) + (\nabla u_i, \nabla u_j) =$$

and now by expanding in Neumann eigenfunctions

$$= \sum_k (\sigma_k - \lambda)(u_i, \Psi_k)(u_j, \Psi_k) = \sum_k \frac{1}{\sigma_k - \lambda} < \phi_i, \Psi_k|_\Gamma >< \phi_j, \Psi_k|_\Gamma > \ .$$

In matrix form $\mathcal{R}_{\Omega_0,\Gamma}(\lambda) = \mathcal{M}\,\mathcal{S}_\lambda\,\mathcal{M}^t$, with

$$\mathcal{M} := (< \phi_i, \Psi_k|_\Gamma >)_{i,k=1}^{\infty}\,, \qquad \mathcal{S}_\lambda := \operatorname{diag}\left(\frac{1}{\sigma_k - \lambda}\right)_{k=1}$$

Returning to the original problem (1), using the basis of transversal eigenfunctions $\{\chi_j\}$ and combining everything together, we get:

Find $\lambda \in [\kappa_J, \kappa_{J+1})$ such that the pencil $\boxed{\mathcal{M}\,\mathcal{S}_\lambda\,\mathcal{M}^t - \mu\mathcal{T}_\lambda}$ on Q_J has eigenvalue $\mu = 1$ and the orthogonality condition $\boxed{\mathcal{P}_J\mathcal{M}\,\mathcal{S}_\lambda\,\mathcal{M}^t h = 0}$ holds for the corresponding pencil eigenfunction h.

Recall that $\mathcal{T}_\lambda := -\operatorname{diag}\left(\frac{1}{\sqrt{\kappa_j - \lambda}}\right)_{j=J+1}^{\infty}$.

Procedure: compute $\mu_n(\lambda)$ and solve for $\mu = 1$. But how many do we need? Two results help.

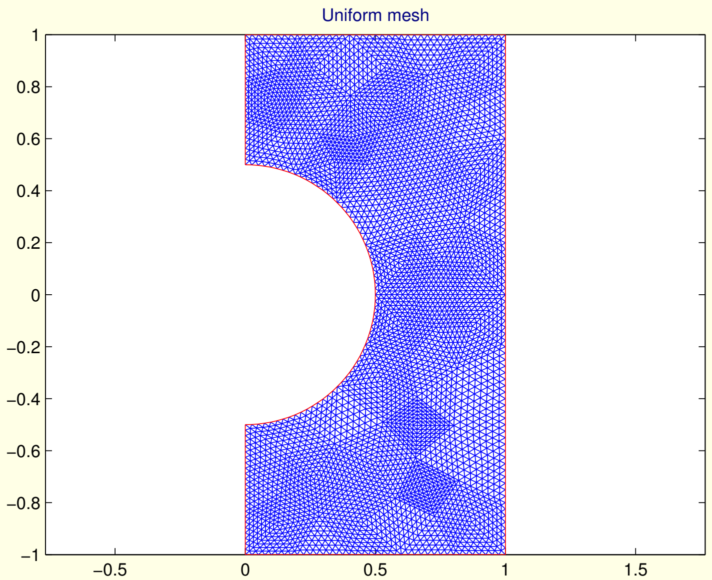
Monotonicity in λ:

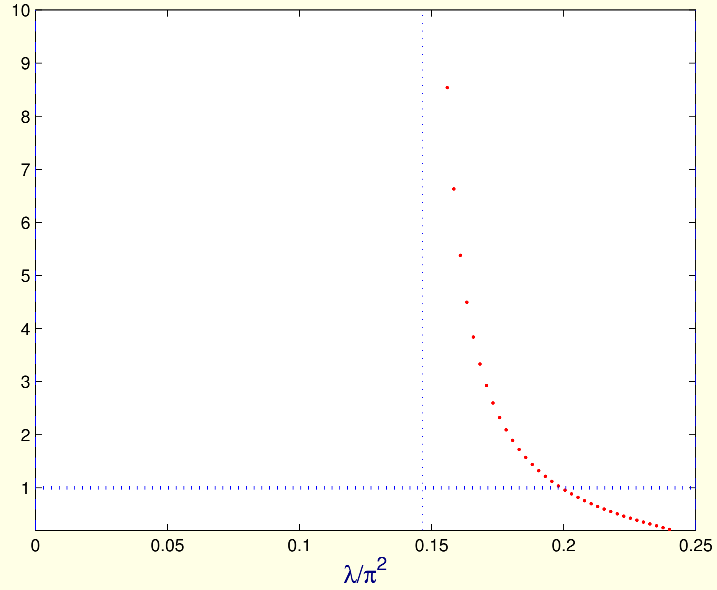
Proposition 1. *Pencil eigenvalues $\mu_n(\lambda)$ are monotone functions of λ in intervals not containing σ_k.*

Estimates of the counting function:

Proposition 2. *If $b \in [\kappa_J, \kappa_{J+1})$,*

$$\mathcal{N}(-\Delta_\Omega; b) \le \mathcal{N}(-\Delta_{\Omega_0, \partial/\partial n + \sqrt{\kappa_{j+1} - b}}; b) - \mathcal{N}(-\Delta_{\Omega_0, \mathrm{Dir}}; b) + \mathcal{N}(-\Delta_\Gamma; b)$$





Example of a half-strip with an obstacle (boundary condition is Neumann everywhere) and a corresponding curve $\mu(\lambda)$ — all other eigenvalues are below 1 and are not shown. The point $\lambda \approx 0.2\pi^2$ is indeed an embedded eigenvalue — we know that the orthogonality condition is automatically satisfied due to the symmetry of the problem.

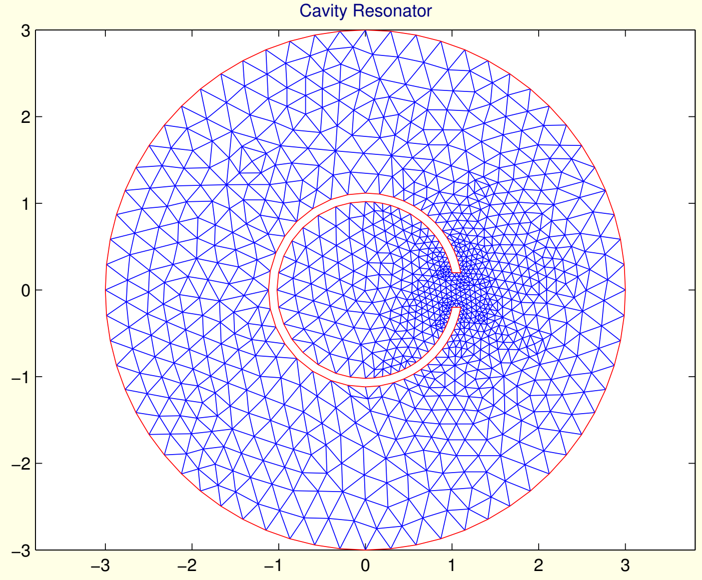
In general, embedded eigenvalues are very unstable — e.g. they may become resonances under small perturbations of geometry destroying symmetry [APV].

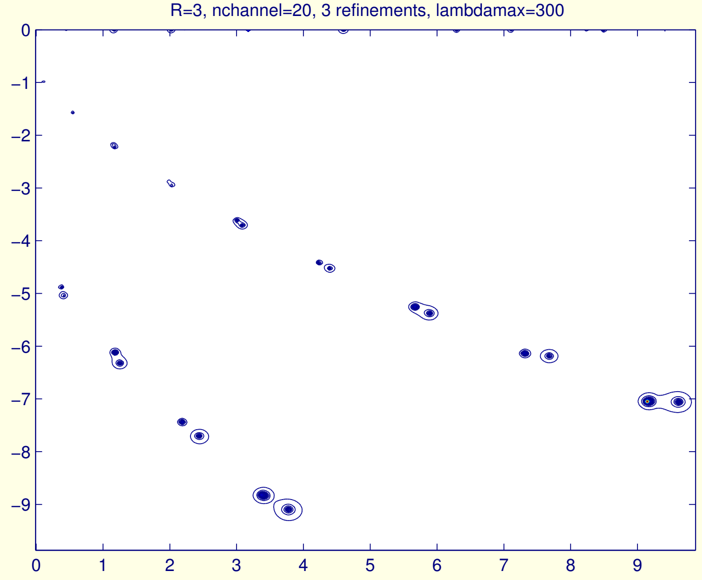
Also, numerically we can NEVER check that the orthogonality condition is exactly satisfied.

So, are we looking for embedded eigenvalues or resonances with small imaginary part? Let us just look for the latter...

Trivial modifications:

- omit all projectors;
- forget the orthogonality conditions;
- look for the $\lambda \in \mathbb{C}^-$ zeros of $\det(\mathcal{M}\,\mathcal{S}_\lambda\,\mathcal{M}^t + \mathcal{T}_\lambda)$ (note the change of sign in front of $\mathcal{T}_\lambda$) by your favourite method of finding complex zeros.





Some resonances for a spherical cavity ("acoustic resonator") — match reasonably well the asymptotic values

The method works equally well for manifolds with conical/spherical/other "regular" ends...

and for operators other than Laplacians

assuming that we can . . .

- separate variables at the ends. . .
- solve boundary value spectral problems for compact domains. . .
- integrate traces of eigenfunctions. . .
- find zeros of monotone real functions or zeros of complex functions as required