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On eigenvalues of the Pauli and Magnetic Schrödinger operators.

$\boxed{0}$

$$A(x) = \left(-\frac{B_0}{2}x_2,\ \frac{B_0}{2}x_1\right) = (A_1, A_2)$$

$$x = (x_1, x_2) \in \mathbb{R}^2;\ x_1 + i x_2 = z$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli matrices

$$\mathcal{P} = (\sigma_1(-i\partial_1 + A_1)^2 + \sigma_2(-i\partial_2 + A_2))^2$$

$$B_0 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$ - const. magn. field

spectrum $(\mathcal{P}) = \{0, 2B_0, 4B_0, \dots\}$

infinitely degenerate eigenvalues

Landau levels

Questions: 1. What happens with the lowest eigenvalue 0 under general perturbation of the magn. field?

Question 2. What happens with other eigenvalues under VERY weak perturbations of the magnetic field?

On eigenvalues of Pauli and magnetic Schrödinger operators with variable magnetic fields

Part 1. Zero modes and 'infinity' case of Aharonov-Casher theorem

(joint with N. Shirokov
St. Petersburg, Russia)

Pauli operator

$$A(x) = (A_1(x), A_2(x))$$

$$x \in \mathbb{R}^2 \equiv \mathbb{C}^1$$

$$x = (x_1, x_2), \quad z = x_1 + i x_2$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \partial_j = -i \frac{\partial}{\partial x_j}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathcal{P} = \left(\sigma_1 (-i\partial_1 + A_1) + \sigma_2 (-i\partial_2 + A_2) \right)^2$$

spin $\frac{1}{2}$ particle in the magnetic field

$A(x)$ - magnetic potential

$$B(x) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

magnetic field

Find general conditions on B so that

1. 0 is an infinitely degenerate eigenv. (zero mode)
2. spectral gap.

$$Q_\pm = (\partial_1 + iA_1) \pm i(\partial_2 + iA_2)$$

$$\mathcal{P}\psi = \mathcal{P}\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$$

$$= \begin{pmatrix} Q_- Q_+ & 0 \\ 0 & Q_+ Q_- \end{pmatrix} \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$$

$$\mathcal{P}\psi = 0 \iff \begin{matrix} Q_+ \psi_+ = 0 \\ Q_- \psi_- = 0 \end{matrix}$$

So, the question on zero modes $\iff$ how many are there L_2- solutions of

Known:

1. $B =$ constant . Landau > 0

$\sigma(\mathcal{P}) = \{0, 2B, 4B, \dots$

all eigenvalues of inf. multiplic

2. $B \in L_\infty$, comp

Aharonov - Casher.

0 -eigenv. with finite multiplicity

$\left\{ \frac{|\Phi|}{2\pi} \right\}$ $\qquad \Phi = \int B$

largest integer < ; $\{0\} = 0$

3. Shigekawa.

inf. multiplicity.

$B\,|x|^2 \to \infty$

Reduction to PDE.

Ψ - scalar potential

$$\Delta \Psi = B$$

$$A = \operatorname{sgrad} \Psi = \left(-\frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_1}\right)$$

$$Q_+ u_+ = e^{\Psi} \frac{\partial}{\partial \bar{z}} \left(e^{-\Psi} u_+\right)$$

$$Q_- u_- = e^{-\Psi} \frac{\partial}{\partial z} \left(e^{\Psi} u_-\right)$$

$$2 \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}$$

$$2 \frac{\partial}{\partial z} = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}$$

B comp. support, bdd

$$\Psi(z) = \frac{1}{2\pi} \int \ln|z-\lambda|\, B(\lambda)\, d\lambda$$

$$Q_+ u_+ = 0 \Rightarrow \frac{\partial}{\partial \bar{z}} \left(e^{-\Psi} u_+ \right) = 0$$

$f_+ = e^{-\Psi} u_+$ - entire function of z

$$Q_- u_- = 0 \Rightarrow \frac{\partial}{\partial z} \left(e^{\Psi} u_- \right) = 0$$

$f_- = e^{\Psi} u_-$ - entire function of $\bar{z}$

Equivalent formulation:
how many entire functions $f_\pm$ such that $u_\pm = e^{\pm\Psi} f_\pm \in L_2$

<u>A-Casher:</u> $B \in L_\infty$, comp.

$$\Psi(z) \sim \frac{\int B}{2\pi} \ln |z| \quad , z \to \infty$$

$\int B = \Phi$

$$e^{\pm\Psi} \sim |z|^{\frac{\Phi}{2\pi}}$$

how many entire functions $f_\pm$ so that

$$e^{\pm \frac{\Phi}{2\pi} \ln|z|} f_\pm \in L_2 \quad (\text{at } \infty).$$

$\Phi > 0$ $\qquad f_+ = 0$

f_- polynomial degree $\leq \frac{\Phi}{2\pi} - 1$.

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so, for general B,

two ways of proving infiniteness

1 Direct, finding estimates from below for the potential Ψ, $\Delta\Psi = B$

2. indirect, without estimates for the potential.

Question (Simon , 1982)

$$\int B = \infty$$

is it correct that there are infinitely many zero modes ?

Theorem 1.

$B \geq 0$, $\int_{\mathbb{R}^2} B = \infty$ (total flux is infinite.

$\Rightarrow$ infinitely many zero modes.

How to prove?

Direct. try $B_j \nearrow B$

$\Psi_j \nearrow \Psi$ (??)

if Ψ grows fast $\Rightarrow$ there are ∞ entire functions $f(z)$ $f e^{-\Psi} \in L_2$

False: Example (Simon, Fefferman)

$B > 0$, $B(\mathbb{R}^2) = \infty$

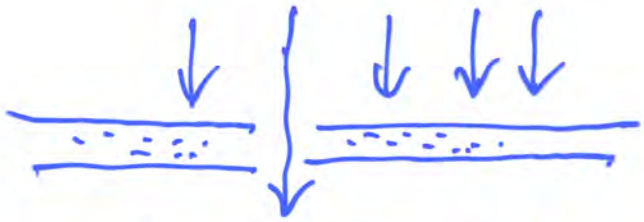
$\Delta \Psi = \mu$ has $\underline{\underline{\text{no}}}$ solutions bdd from below

Possibly singular B:

B - a signed Borel measure

the discrete part of B:

$$\alpha\, \delta(x-\lambda)$$



- Aharonov - Bohm

$\lambda \in \Lambda$ - discrete set

$\alpha \in (0, 2\pi)$

- gauge invariance

$$B = \mu = \mu_{disc} + \mu_{cont}.$$

[self-adjointness]

Idea of the proof:

th. $\cancel{B} \geq 0$, $\cancel{\ldots}$, $\Delta\psi = \cancel{\ldots}\, B$

$\int B = \infty$

$\Rightarrow$ for n, any points

$z_1, \ldots, z_n$, $w_1, \ldots, w_n$

$\exists$ an entire $f(z)$:

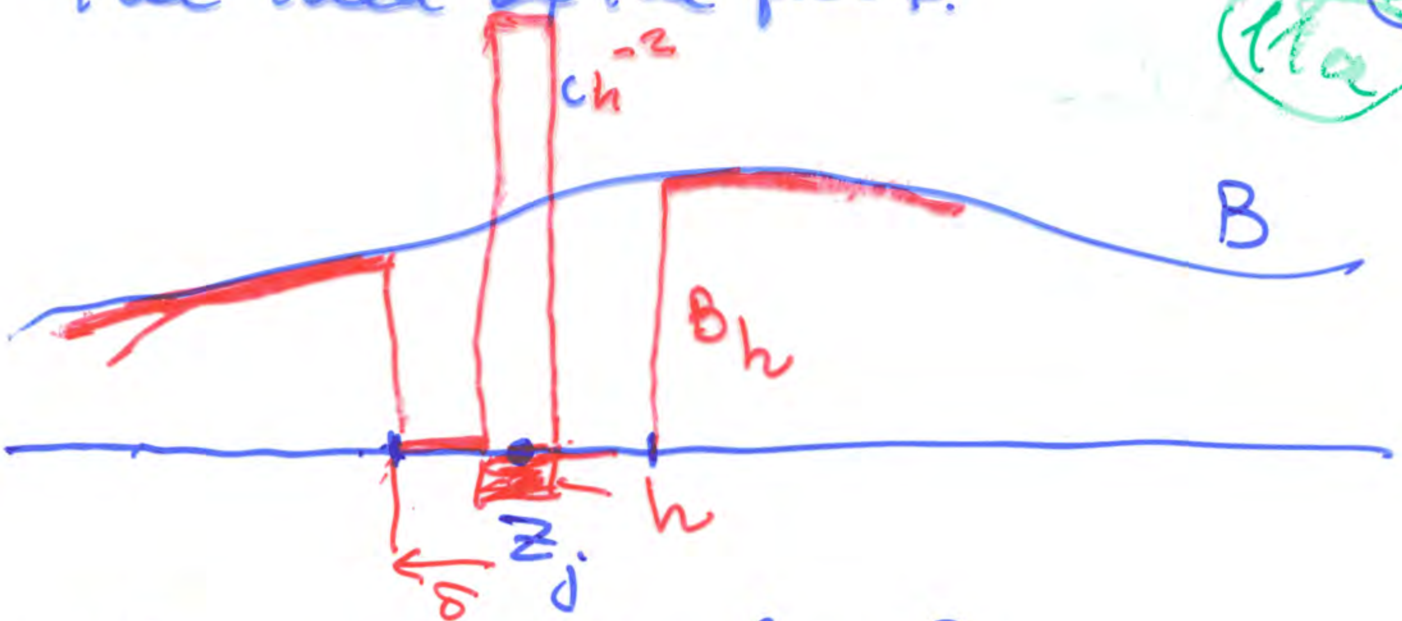
$$f(z_k) = w_k \;, \; k = 1, \ldots, n$$

$$\int |f(z)|^2 e^{-2\psi} \, dx < \infty.$$

estimates for solutions of

$\bar{\partial}$-equation.

The idea of the proof.



Ψ_h - potential for B_h

Ψ_h - subharmonic

$\Psi - \Psi_h$ is controllable for large $|z|$
but can be very large for small $|z|$

take $\varphi(z) = w_j$ in δ-neighb. of z_j

equation $\boxed{\bar\partial g = \bar\partial \varphi}$

Hörmander:

$$g = g_h : \int \frac{|g_h|^2 e^{-2\Psi_h}}{(1+|z|^2)^2} \leq C \int |\bar\partial \varphi|^2 e^{-2\Psi_h}$$

$$\int_{R<|z|<2R} |g_h|^2 e^{-2\Psi_h} \leq C_R$$

can be replaced by Ψ

$$f_h = g_h - \varphi$$

7

116

$$\int_{R<|z|<2R} |f_h|^2 e^{-2\psi} < \infty$$

$$\bar\partial f_h = \bar\partial g_h - \bar\partial \varphi = 0$$

f_h - analyt.

$\Rightarrow \{f_h\}$ is compact w.r.to unif convergence on compacts

a subseq. $f_h \to f$

$g_h \to g$ | unif. on compacts

$\int_{|z|<R} |g_h|^2 e^{-2\psi_h}$ can be bdd only if $g(z_j)=0$

since $\psi_h \to -\infty$ very fast near z_j.

$g(z_j)=0 \Rightarrow f(z_j) = \varphi(z_j) = w_j$.

Relaxing conditions

$\mu = B$ of variable sign ??

Example (Simon)

$$\int_{|z|<R} B \to +\infty, \quad R \to \infty$$

BUT NO ZERO MODES!

Theor. 2.

$$\mu = \mu_+ - \mu_-$$

a). $\mu_-(\mathbb{R}^2) < \infty$

$\mu_+(\mathbb{R}^2) = \infty$

$\Rightarrow$ inf. many zero modes

b) $R > 0$

$$B_R(x) = \mu(D(x,R))$$

if B_R satisfies part (a),

$\Rightarrow$ inf. many zero modes

Spectral gap.

Location of the rest of the spectrum.

0 - an isolated point in the spectrum ??

Theorem 3.

$B_R(x) = \mu(D(x, R)) \geq C > 0$

$\Rightarrow$ zero is an isolated point in the spectrum

+ minor regularity

[if this condition fails, not isolated]

estimates for $\bar{\partial}$ equations.

Examples

1. $\mu = \mu_{disc}$,
   Λ suff. dense

2. μ periodic
   Λ - lattice

Γ - cell

$\mu(\Gamma) \neq 0$

~~Rem~~ 3. μ - quasi - periodic

$\mu = \sum \mu_j$

μ_j is periodic w. respect to its own lattice Λ_j

$$\sum \frac{\mu_j(\Gamma_j)}{|\Gamma_j|} \neq 0$$

Part 2.

Splitting of Landau levels under the weak perturbation of the magnetic field.

(joint with G. Taschian
St. Petersburg, Russia)

$$B = B_0 = \text{const.}$$

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{pmatrix}$$

$A = (A_1, A_2)$ linear funct.

$$B_0 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$

Landau levels:

$$0,\ 2B_0,\ 4B_0,\ \ldots$$

$\sigma(\mathcal{P}$ [redacted] $) = 0, 2B_0, 4B_0.$

$$\sigma(\mathcal{P}_+) = 2B_0, 4B_0, \ldots$$

Commutation relations

$$Q_\pm = (\partial_1 + iA_1) \pm i(\partial_2 + iA_2)$$

$$\mathcal{P}_+ = Q_- Q_+ = Q_+ Q_- + 2B_0 = \mathcal{P}_- + 2B_0$$

$$\operatorname{Ker} \mathcal{P}_- = \ker Q_- = \{ f(z), \text{ entire};\ f(z) e^{-B_0 |z|^2} \in L_2 \}$$

Segal-Bargmann space

$$Q_- Q_+ = Q_+ Q_- + 2B_0$$

Q_+, Q_- - creation and annihilation operators

$Q_- = Q_+^*$

$$\sigma(Q_- Q_+) \setminus \{0\}$$
$$= \sigma(Q_+ Q_-) \setminus \{0\}$$

$Q_+ : \mathcal{H}_q \longmapsto \mathcal{H}_{q+1}$; $Q_- : \mathcal{H}_q \to \mathcal{H}_{q-1}$

<u>Question</u>: What happens with Landau levels under perturbation (weak) of the magnetic field and/or under the perturbation by an electric field

$B = B_0 + b$, $b \to 0$ at ∞

$\mathcal{P} \longmapsto \mathcal{P} + V$

$V \to 0$ at ∞ fast!!

Along with $\mathcal{P}$,

H - the magnetic Schrödinger

$$H = -(\nabla + iA)^2$$

$$H = \mathcal{P}_+ - B = \mathcal{P}_- + B$$

constant field:

$$\sigma(H) = B_0,\ 3B_0,\ 5B_0, \dots$$

Perturbation by an electric field only:

$$H+V$$

Raikov, 1999

slowly decaying field

$$V \sim |x|^{-\beta}$$

$$\Lambda_q = (2q+1)B$$

$$\lambda^{q}_{n,\pm} \longrightarrow \Lambda_q$$

semi-classical behaviour

$$\# \lambda^{q}_{n\pm} \in \begin{matrix} (\Lambda_q + t, \ \Lambda_q + t_0) \\ (\Lambda_q - t_0, \ \Lambda_q + t) \end{matrix}$$

$$\sim \{x : \pm V(x) > t\} \asymp t^{-\frac{2}{\beta}}$$

V decaying very fast
(or even with compact support)

methods of microlocal analysis
do not work

2003 : Raikov + Warzel
Melgaard + R.

V of constant sign
(say, $V > 0$), compact support

$\lambda^q_{n,+} \to \Lambda_q$ super-exponentially. $\lambda^q_{n,+} - \Lambda_q \sim \frac{1}{n!}$

$$-\ln | \Lambda_q - \lambda^q_{n,+} | \sim c\, n \ln n$$

2005 Filonov - Pushnitsky
found the next term
in the expansion

Toeplitz operators:

P_q : Projection onto $\mathcal{H}_q$, eigenspace corresponding to $\Lambda_q = (2q+1) B_0$

$$T_q(V) = P_q V P_q : \mathcal{H}_q \to \mathcal{H}_q$$

$q=0$: Berezin-Toeplitz operator

$$\mathcal{H}_0 = \left\{ e^{-\frac{B_0 |z|^2}{2}} f(z) \right\}$$

f analytical, $\int |f(z)|^2 e^{-B_0 |z|^2} dz < \infty \}$

f belongs to $\boxed{\text{Segal-Bargmann - Fock}}$ space

$$H+V = \underbrace{P_q (H+V) P_q}_{= P_q + P_q V P_q} + \boxed{\begin{array}{l} P_q (H+V)(1-P_q) \\ + (1-P_q)(H+V) \end{array}}$$

do not influence the spectrum near Λ_q

$= P_q + P_q V P_q$

spectrum:

$\Lambda_q + \sigma(T_q(V))$

$\sigma\left(T_q(V)\right)$

Theorem (Raikov, Warzel, Melgaard, G.R., Filonov, Pushnitsky)

1. V with comp. support, $V \geq 0$
$V > 0$ on an open set

$$s_n\left(T_q(V)\right)^{-\frac{1}{n}} \sim c_1 n + c_2$$

2. Drop the condition $V \geq 0$:

$$s_n\left(T_q(v)\right)^{-\frac{1}{n}} \geq c_1 n + c_2$$

$$n(\lambda) \sim c_1' \frac{|\ln \lambda|}{\ln |\ln \lambda|} + c_2' \frac{|\ln \lambda|}{(\ln |\ln \lambda|)^2}$$

$\boxed{\text{we return!}}$

Perturbation of the magnetic field:

$B = B_0 + b \quad , \quad b \in C_0^{\ell}$

Complications: not a relatively compact perturbation of the operator:

$$A_1^0 = -\frac{B_0}{2} x_2 \; ; \; A_2^0 = \frac{B_0}{2} x_1$$

$$a_1, a_2 : \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} = b$$

If $\int b \neq 0$,

$$a_1, a_2 \sim |x|^{-1}$$

$$\int \left| \sigma_1 (-i \partial_1 + A_1^0 + a_1) + \sigma_2 (-i \partial_2 + A_2^0 + a_2) \right) u \Big|^2$$

product of the terms

A_1 and a_1

gives something not tending to 0 at ∞

!!!

$$P_q \; \mathcal{P} \; P_q \qquad ?$$

not a perturbation structure

no Toeplitz ....

no part with known spectrum

no monotonicity in b

Folkknowledge:

$$\delta b = 0 \Rightarrow$$

$$\sigma_{ess}(\mathcal{P}) = \sigma_{ess}(\mathcal{P}_0)$$

$$= \{0, 2B_0, \ldots$$

1980 Iwatsuka

$b(x)$ bdd, $\longrightarrow 0$, $x \to \infty$

$$\sigma_{ess}(\mathcal{P}) = \sigma_{ess}(\mathcal{P}_0)$$

Approximate creation - annihilation operators.

$$Q_\pm = (\partial_1 + iA_1) \pm i(\partial_2 + iA_2)$$

$$A_j = A_j^0 + a_j$$

$$Q_+ = -Q_-^*$$

$$\mathcal{P}_+ = -Q_- Q_+ = \mathcal{P}_- + 2(B_0 + b)$$

$$\mathcal{P}_- = -Q_+ Q_-$$

$$\mathbf{H} = \mathcal{P}_- + (B_0 + b) = \mathcal{P}_+ - (B_0 + b)$$

b - relatively compact

$$\sigma_{ess}(\mathcal{P}_+) = \sigma_{ess}(\mathcal{P}_-) + 2B$$

$$\sigma(\mathcal{P}_+) \setminus \{0\} = \sigma(\mathcal{P}_-) \setminus \{0\}$$

we consider

$$\mathcal{P}_- + V$$

Appoximate Landau supspaces.

$$\mathcal{H}_0 = \ker Q_- = \ker \mathcal{P}_-$$

$$\mathcal{H}_q = Q_+^q \mathcal{H}_0$$

P_q - projection onto $\mathcal{H}_q$

- the replacement of Landau projection.

<u>Theorem.</u> Let $V, b \in C_0^{2q}$

then:

1. There exist: an operator

$$\mathcal{U} : \mathcal{H}_0 \to \mathcal{H}_q$$

$$\mathcal{U}^* \mathcal{U}, \ \mathcal{U}\mathcal{U}^* = I + \text{comp}$$

such that

$$\mathcal{U} P_q (\mathcal{P}_- + V) P_q \mathcal{U}^*$$

$$= P_0 \ Z_q (b, V) P_0 = T_{Z_q}$$

$Z_q(b, V)$ - effective weight

$$Z_q(b,V) = L_1[b] + L_2[b]\,V$$

L_1 - expression containing
b and its derivatives
up to q

$L_2[b]$

a linear different. operator (acting on V)
of order $2q$
with coeff. being polynomials
in b and its derivatives
up to q

$$Z_1(b,V) = c_1 b + c_2 b^2 + c_3 |\nabla b|^2 + (c_4 + c_5 b)V + c_6 \Delta V$$

2. $P_q - P_q^0$ is
compact for any q.

P_q^0 - the spectral projection
of $\mathcal{P}_- + V$ corresponding to
some small interval around Λ_q.

Part 1 reduces the study of the spectrum of

$$P_q(\mathcal{P}_- + V)\,P_q$$

to the study of the spectrum of the Toeplitz operator with the effective potential.

Part 2 helps to dispose of the 'nondiagonal' terms in the representation

$$\begin{aligned}\mathcal{P}_- + V &= P_q(\mathcal{P}_- + V)P_q \\ &+ (1-P_q)(\mathcal{P}_- + V)\,P_q \\ &+ (\mathcal{P}_- + V)(1-P_q)\end{aligned}$$

Spectral properties.

Theor.

The Landau levels split into sequences of eigenvalues converging superexponentially

$$\left|\lambda_n^q - \Lambda_q\right|^{\frac{1}{n}} \le (cn)^{-1}$$

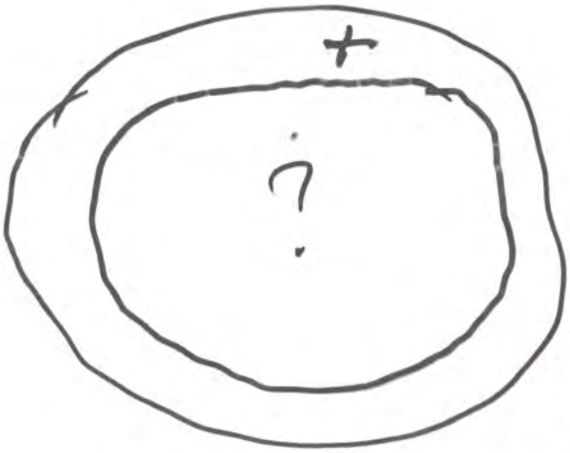
-estimate.

Asymptotics — Depends on the progress in Toeplitz - Berezin operators.

$$T_F = P_0 F P_0$$

$$(F = Z_q(b, V))$$

Th.



$\Rightarrow$ the negative spectrum of T_F · finite

For positive asymptotics