Dirichlet spaces with no reference measure

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Weak solutions

$$\sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|}\partial_\alpha\Big(a_{\alpha\beta}\partial_\beta u\Big) = F \text{ (a measure)}$$

$$\mathcal{E}[\mathbf{u},\phi] = \sum_{|\alpha|,|\beta|\leq m} \int a_{\alpha\beta}(\partial_\beta u)(\partial_\alpha \phi)dx = \mathbf{F}(\phi),$$

$$\forall \phi \in \mathcal{D} \text{ - test functions}$$

Green function $G : \mathcal{E}[G(\cdot,x),\phi] = \phi(x)$, $\forall \phi \in \mathcal{D}$

Super-harmonic $u : \mathcal{E}[u,\phi] \geq 0$, $\forall \phi \in \mathcal{D}^+$

The Fukushima construction

m - a full support measure

$(\mathcal{E}, \mathcal{D})$ - a closable Markov form in $L^2(m)$, associates Markov SG P_t on $L^p(m)$, $p \in [1, \infty]$

P_t is *transient*
$\Leftrightarrow \forall f \in L^1_+(m) : Gf = \int_0^\infty P_t f \, dt < \infty$ m-a.e.

$$m(\phi f) - m(\phi P_T f) \quad \Big(\to m(\phi f) \Big)$$
$$= \int_0^T \mathcal{E}[P_t f, \phi] dt \quad \Big(\to \mathcal{E}\Big[\int_0^\infty P_t f \, dt, \phi \Big] \Big)$$

$\exists g \in L^1(m)$, $g > 0$ m-a.e.: $\sqrt{\mathcal{E}[\phi]} \geq \int |\phi| dm$, $\forall \phi \in \mathcal{D}$

Fukushima: transience (recurrence) depends of measure m.

Examples

$\Omega \subset \mathbb{R}^N$ smooth bdd connected

$\mathcal{E}(u) := \int_\Omega |\nabla u|^2 dx$, $\mathcal{D}_0 = H_0^1(\Omega)$, $\mathcal{D}_1 = H^1(\Omega)$.

λ be the N-dim Lebesgue measure on Ω

$\Delta := \sum_{q \in \mathbb{Q}^N \cap \Omega} c_q \delta_q$.

H^1 is recurrent wr to any reference measure it is closable.

H_0^1 is transient wr to $m = \lambda$.

H_0^1 is recurrent wr to $m = \lambda + \Delta$.

H^1 and H_0^1 are not closable wr to $m = \Delta$, $N \geq 3$.

Philosophy: measure as a clocking device

Let $m_0 \longleftrightarrow \frac{du}{dt} = Au$.

Then $dm := \rho\, dm_0 \longleftrightarrow \frac{du}{d\tau} = \frac{1}{\rho}Au$, i.e., $t = \frac{\tau}{\rho}$.

Fukushima: for m not charging sets of zero capacity,

$t = T_\tau(\omega)$:

$$\frac{1}{\tau}\mathbb{E}_{m_0}\int_0^\tau f(X_\tau)dT_\tau \to m(f),\ \tau \to 0$$

$X_t(\omega)$:
$\mathbb{E}_x f(X_t) = P_t f(x)$, $P_t \longleftrightarrow \Big(\mathcal{E}, \mathcal{D}\Big)$ on $L^2(m_0)$.

Transient Dirichlet space $\Big(\mathcal{H}, [\cdot,\cdot]\Big)$

Given: state space Ω, $\mathcal{B}$ - Borel σ-algebra on Ω, $\mathcal{B}(\Omega)$ - $\mathcal{B}$-measurable functions of Ω

1. $\mathcal{H}$ is a separable Hilbert space.

2. $\mathcal{H}$ is a ordered vector space
   $\mathcal{H}^{+}$ closed, $\mathcal{H}^{+} \cap (-\mathcal{H}^{+}) = \{0\}$.

3. $\mathcal{H}$ is a *Stone lattice* i.e. a vector lattice with an order-convex sub-lattice $\mathcal{H}^{\wedge} \subset \mathcal{H}^{+}$ of "positive elements not exceeding the unit". $\mathcal{H}^{\wedge}$ is closed.

4. $\mathcal{H} \overset{\text{dense}}{\hookleftarrow} \mathcal{D} \subset \mathcal{B}(\Omega)$, a Stone sub-lattice in the pointwise order, generating $\mathcal{B}$.

5. For all $\in \mathcal{H}$: $\|(u^{+})^{\wedge}\|_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}$.

Stone lattice $\mathcal{V}$

- vector lattice (= ordered vector space with $\wedge, \vee$ operations);
- countable type (= a majorized family of disjoint elements is at most countable);
- $\exists$ order-convex sub-lattice $\mathcal{V}^\wedge \subset \mathcal{V}^+$ such that:
  $0 = \min \mathcal{V}^\wedge$;
  $\forall u \in \mathcal{H}^+ : \exists u^\wedge := \sup\{v \in \mathcal{H}^\wedge,\ v \leq u\}$;
  $\forall u \in \mathcal{H}^+ : \left(\forall \alpha \in \mathbb{R}^+ :\ \alpha u \in \mathcal{H}^\wedge\right) \Rightarrow u = 0.$

Daniell Stone integral

A Stone lattice allows for an abstract version of the Lebesgue (Daniell-Stone) integral:

- order completion $\widehat{\mathcal{V}}$ ($\widehat{\mathcal{V}}^+ =$ limits of increasing positive sequences) is an analog of the measurable functions space;

- $\sigma(\mathcal{V}) := \left\{ \sup_n[(nu)^\wedge] \middle| u \in \mathcal{V}^+ \right\} \subset \widehat{\mathcal{V}}$ is a (Boolean) σ-algebra of "(indicators of) supports of elements of $\mathcal{V}$"

- Daniell-Stone theorem: an order continuous positive linear functional on $\mathcal{V}$ is a positive measure on $\sigma(\mathcal{V})$.

Properties of a transient Dirichlet space

1. $\sigma(\mathcal{H}) \supset \mathcal{B}$.

2. $\mathcal{S}^+ := \left(\mathcal{H}^*\right)^+$ separates points on $\mathcal{H}$. They are positive measures on $\sigma(\mathcal{H})$ satisfying $\mu(u) \le c\|u\|_{\mathcal{H}},\ u \in \mathcal{D}^+$

3. $\exists m \in \mathcal{S}^+$ of a full support. $\left([,], \mathcal{H} \cap L^2(m)\right)$ is a transient Dirichlet form in $L^2(m)$ in the Fukushima sense.

4. The Green operator G is the Riesz isometry $\mathcal{H}^* \to \mathcal{H}$ restricted to (signed) measures on $\sigma(\mathcal{H})$.

Construction

$\mathcal{D} \subset C_c(\Omega)$

- Stone lattice with the pointwise order;
- dense in $C_c(\Omega)$;
- $\forall v \in \mathcal{D}^+ \; \exists u \in \mathcal{D}^\wedge$ such that $\forall \epsilon > 0$ $(u + \epsilon v)^\wedge = u$ ("$u = 1$ on supp v");
- $\|(u^+)^\wedge\|_\mathcal{H} \leq \|u\|_\mathcal{H}$;
- for any $\|u_n\|_\mathcal{H} \to 0$, $\sup_n \|v_n\|_\mathcal{H} < \infty$: $0 \leq v_n \leq u_n \Rightarrow v_n \to 0$ (weakly) in $\mathcal{H}$.