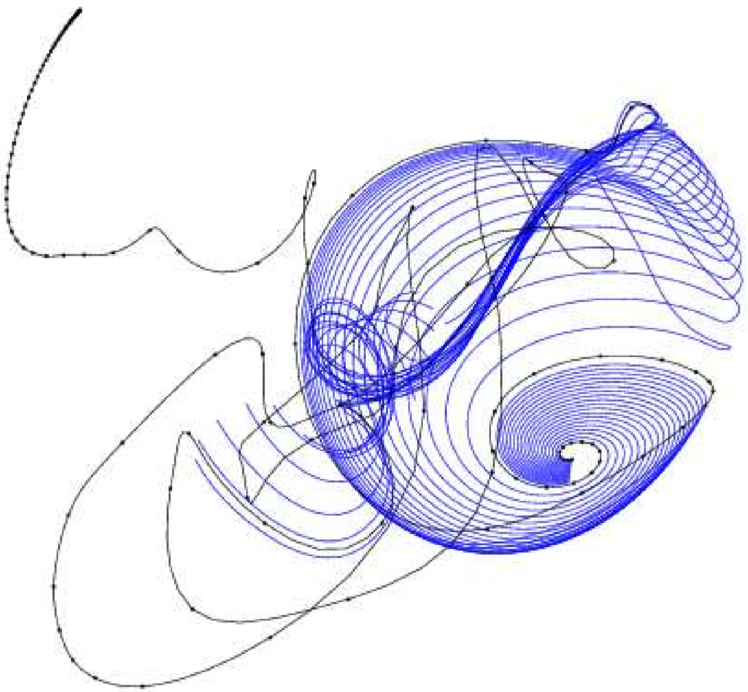
LMS Symposium

Dynamical Systems and Statistical Mechanics

Durham, 3–13 July 2006

Exact recurrent structures in shear flow turbulence



www.nongnu.org/channelflow

ChaosBook.org

Predrag Cvitanović

John F. Gibson, Yueheng Lan,

Evangelos Siminos

Physics, Georgia Tech, Atlanta

Fabian Waleffe

Mathematics & Engineering Physics

U. Wisconsin, Madison

Ruslan L. Davidchack

Mathematics, U. Leicester, UK

Turbulence: A walk through a repertoire of unstable recurrent patterns?

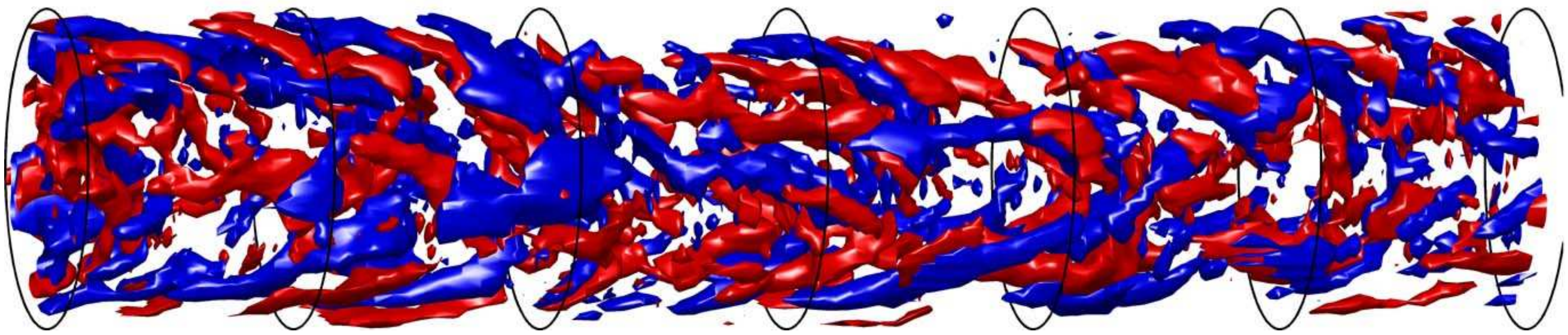
As a turbulent flow evolves, every so often we catch a glimpse of a familiar pattern:

$\Longrightarrow$ other swirls $\Longrightarrow$

For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns. The long term dynamics = a walk through the space of such unstable patterns.

New experiments: Unstable Coherent Structures

Stereoscopic Particle Image Velocimetry → 3-d velocity field over the entire pipe[1]



Observed structures resemble numerically computed traveling waves

What lies beyond?

[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004); Hof et al., Science (Sep 10, 2004)

Theory: 3-d Navier-Stokes steady solutions

Unstable 3D steady state and traveling wave solutions of the Navier-Stokes equations

in plane Couette: first discovered by Nagata[2]

in plane shear flows: Exact Coherent Structures by Waleffe[3]

(+ many more recent numerical results)

[2] M. Nagata, "Three-dimensional finite-amplitude solutions in plane Couette flow: bifurcation from infinity. ", J. Fluid Mech. 217, 519 (1990)

[3] F. Waleffe, "3-D Coherent States in Plane Shear Flows", Phys. Rev. Lett. 81, 4140 (1998)

2001: 3-d Navier-Stokes periodic solution

Kawahara and Kida[4]

the first demonstration of existence of an unstable recurrent pattern in a turbulent hydrodynamic flow.

full numerical dynamicals simulation, a 15,422-dimensional discretization of the 3-d Plane Couette turbulence at Re = 400.

[4] G. Kawahara and S. Kida, "Periodic motion embedded in plane Couette turbulence: regeneration cycle and burst", J. Fluid Mech. 449, 291 (2001)

Found: an important unstable spatio-temporally periodic (?) solution. A 9 consecutive snapshots of a periodic video:

colored: high vorticity regions - look like steady turbulent state snapshots (but these are periodic)

Theory: 3-d Navier-Stokes relative periodic solutions

Unstable 3D relative periodic solutions of the Navier-Stokes equations

in plane Couette: several computed by Viswanath[5]

[5] D. Viswanath, "Recurrent motions within plane Couette turbulence", arXiv.org:physics/0604062

Turbulence = geometry of the phase space

Three examples, in order of increasing complexity

1. Rössler ``chaos"

3-d state space

2. Kuramoto-Sivashinsky ``turbulence''

∞-d state space

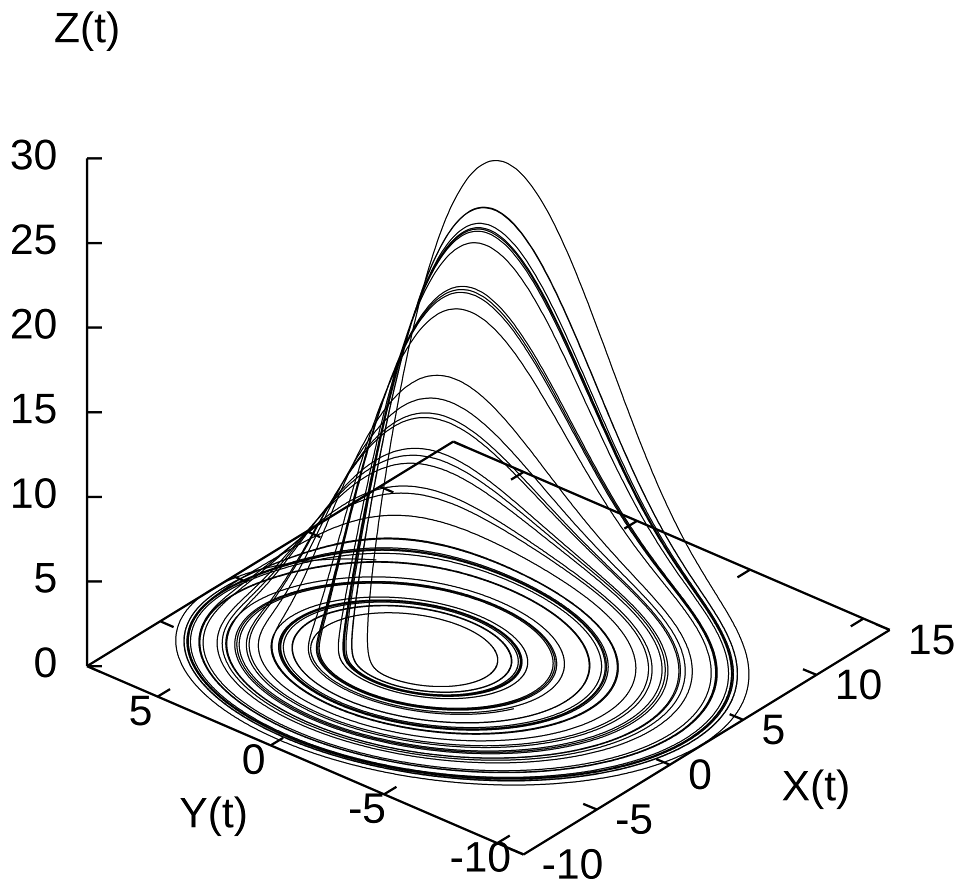
3. Navier-Stokes ``turbulence''

∞-d state space

Rössler flow

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c),$$
$$a = b = 0.2, \quad c = 5.7.$$

A typical numerically integrated long-time trajectory



1-d "Navier-Stokes" equation

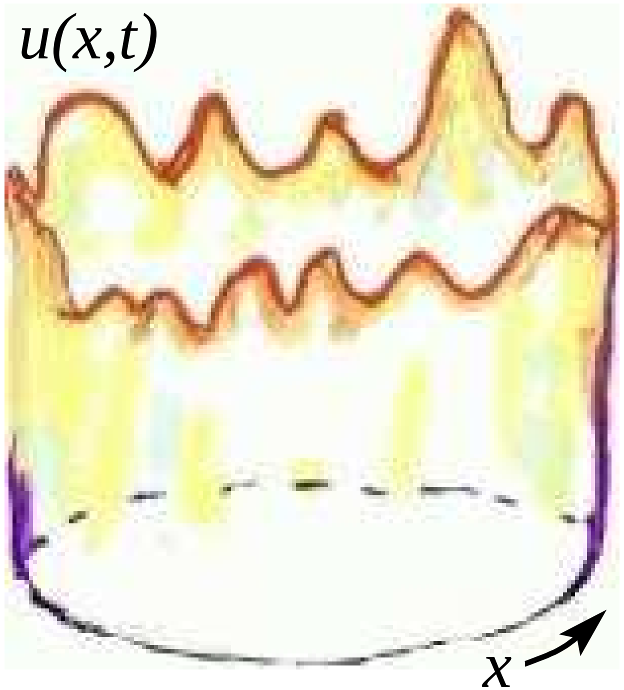
Navier-Stokes $\rightarrow$

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}$$

- "inertial" term $u\partial_x u$; nonlinear
- "diffusive" terms $\partial_x^2 u$, $\partial_x^4 u$
- "viscosity" ν – suppresses fast spatial variations

only parameter: dimensionless length $\tilde{L} = \frac{L}{2\pi\sqrt{\nu}}$

Flame front flutter

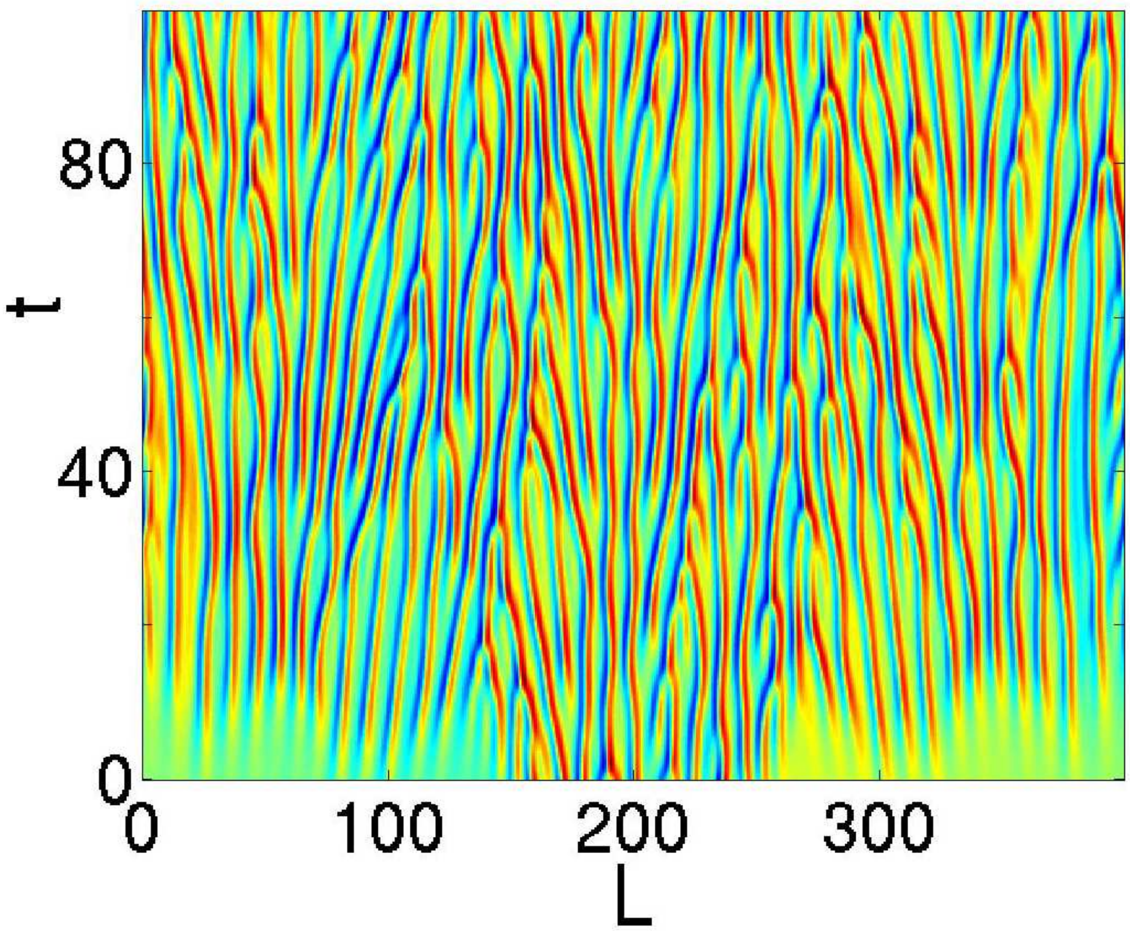


Bunsen burner[6]

Q : 1-d turbulence –
flutter of a flame front?

[6] R.W. Bunsen (1811–1899), Doctorate U. Göttingen, age 19

Kuramoto-Sivashinsky 45$\langle\lambda\rangle$ wide



the "Reynolds" parameter: dimensionless length

$$\tilde{L} = \frac{L}{2\pi\sqrt{\nu}}$$

spatial wavelength: $\langle\lambda\rangle = \sqrt{2}$ in units of $\tilde{L}$

A small Kuramoto-Sivashinsky system

(empirical: ``smallest'' cell that exhibits turbulence) [7], [8]

weakly turbulent regime:

$$L = 22$$

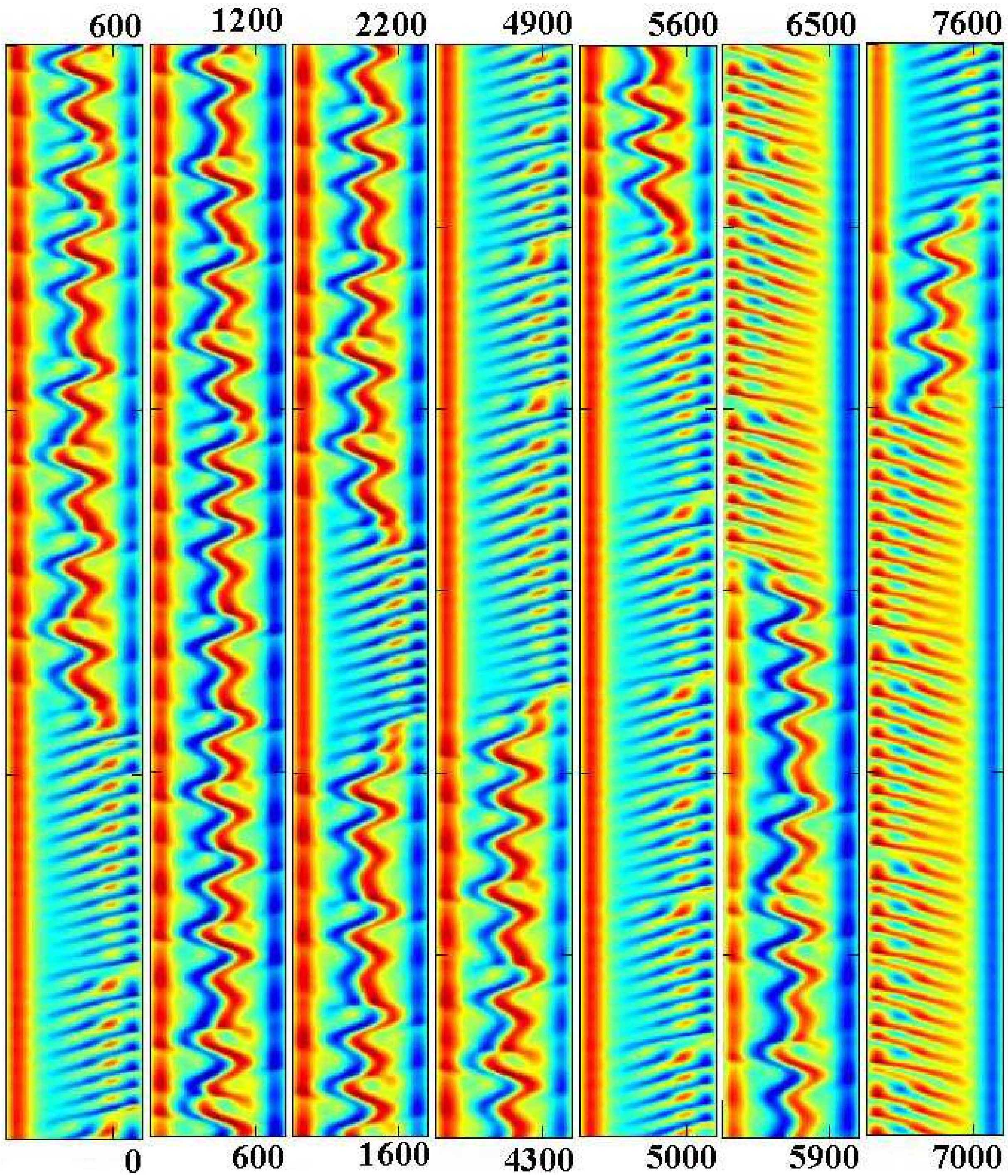
or

$$\approx 2.5 \langle \lambda \rangle \text{ mean spatial wavelengths}$$

[7] Y. Lan and P. Cvitanović, in preparation

[8] R.L. Davichack, in preparation

A long time series:
jumps between
center "wobble"
side "traveling waves"



Navier-Stokes equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \eta\nabla^2\mathbf{u} + \mathbf{f}\,.$$

requires at least 15,000 dimensional discretization,

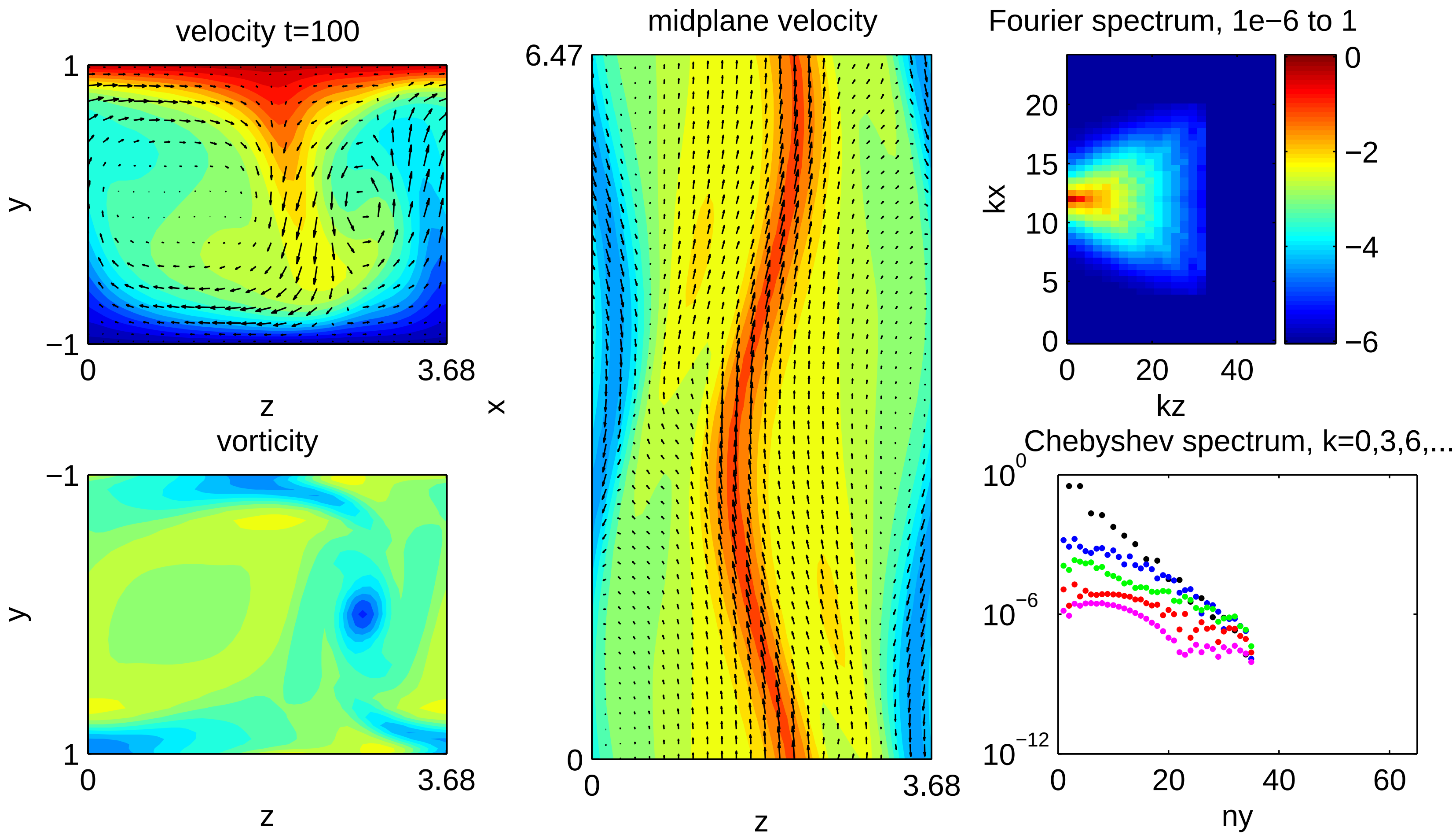
Plane Couette at Re = 400

a snapshot of a "typical" turbulent flow[9].

Periodic $[L_x = 2\pi/1.14,\ L_z = 2\pi/2.5]$ box
in x (streamwise) and z (spanwise),

Chebyshev wall normal.

[9] John F. Gibson – www.nongnu.org/channelflow



velocity t=100
y
z
vorticity
midplane velocity
x
Fourier spectrum, 1e−6 to 1
kx
kz
Chebyshev spectrum, k=0,3,6,...
ny

THE POINT OF THIS TALK

!!! THE POINT OF THIS TALK !!!

UNLEARN: 3-d VISUALIZATION

instant in turbulent evolution:

a 3-d video frame,

each pixel a 3-d velocity field

THINK: ∞-d PHASE SPACE

instant in turbulent evolution:

a unique point

theory of turbulence =

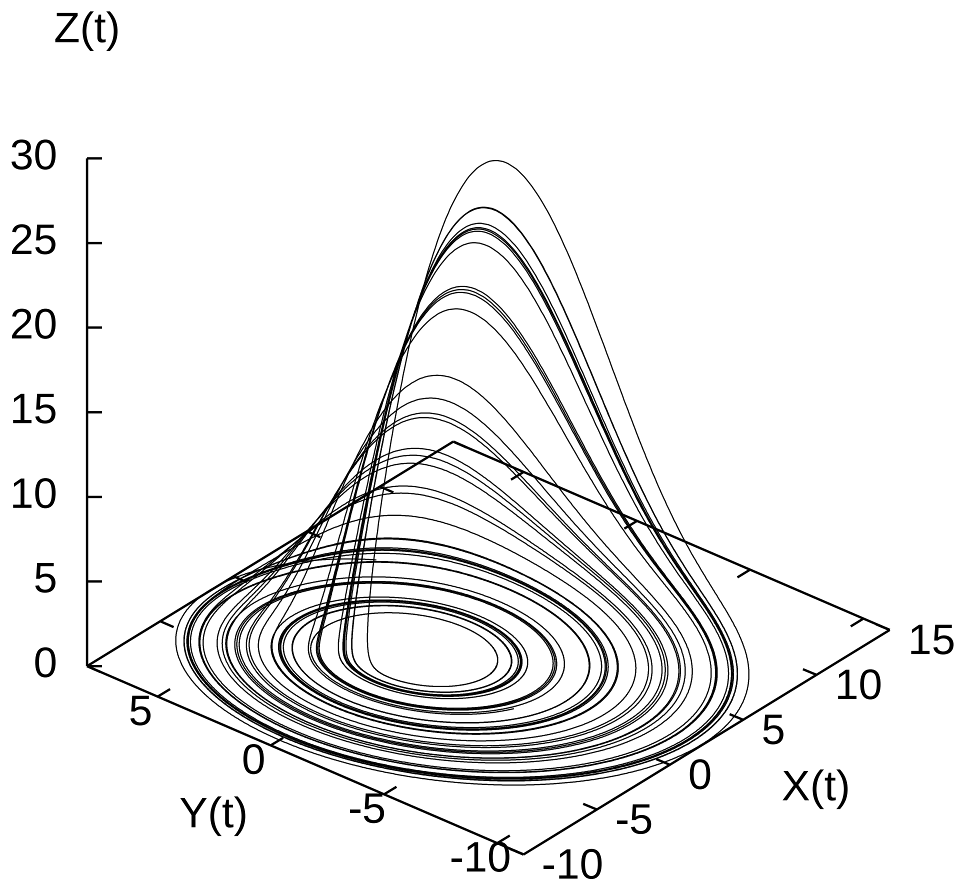
geometry of the phase space

[E. Hopf 1948]

Rössler flow

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c),$$
$$a = b = 0.2, \quad c = 5.7.$$

A typical numerically integrated long-time trajectory



THINK IN PHASE SPACE!

Fourier representation

spatial Fourier basis:

$$u(x,t) = i \sum_{k=-\infty}^{+\infty} a_k(t) e^{ikx} .$$

odd solutions subspace: $u(x,t) = -u(-x,t)$:

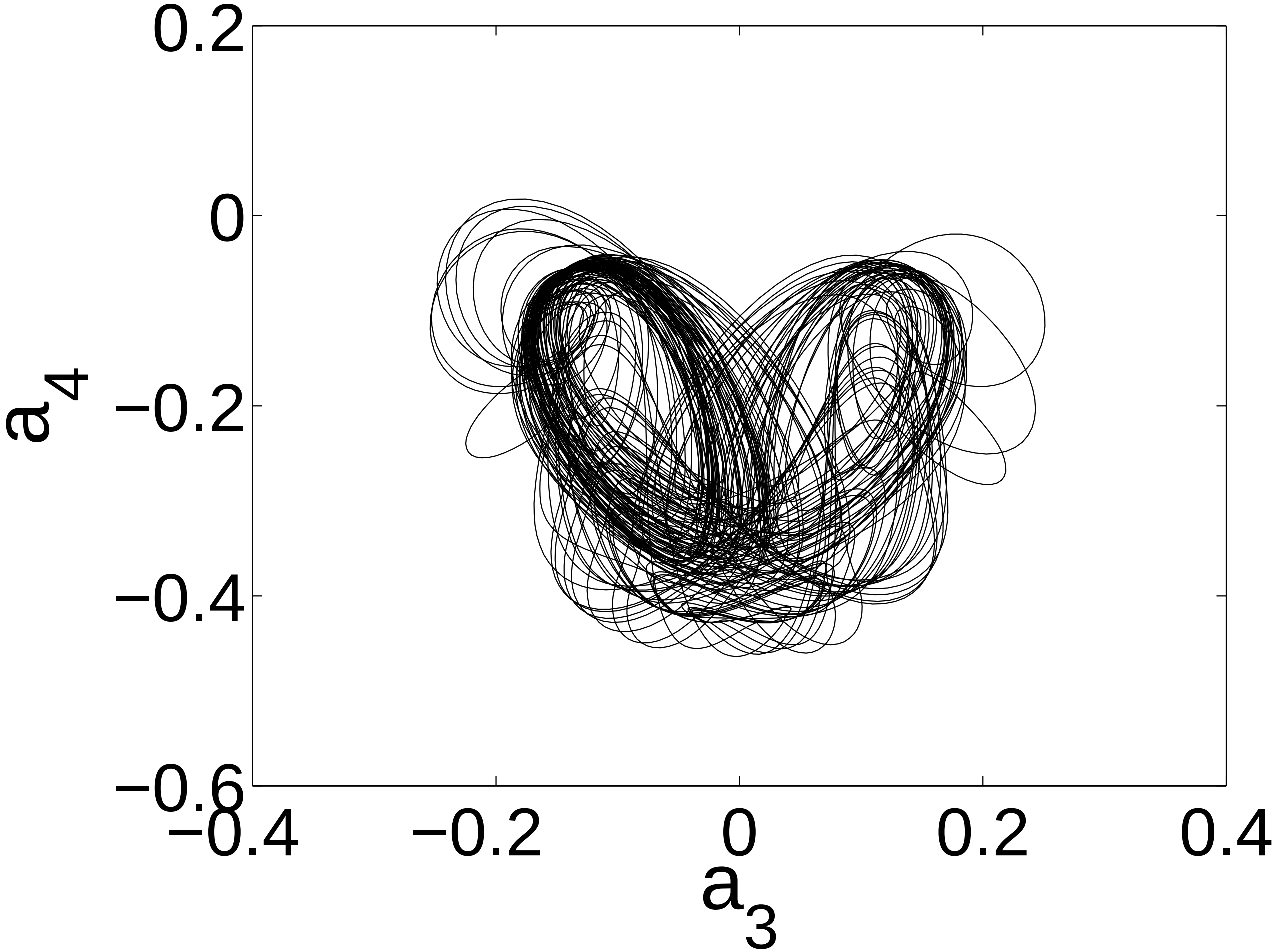
$$\boxed{\dot{a}_k = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m} .}$$

minimal number of modes:

1-d Kuramoto-Sivashinsky system: $16 - 10^3$

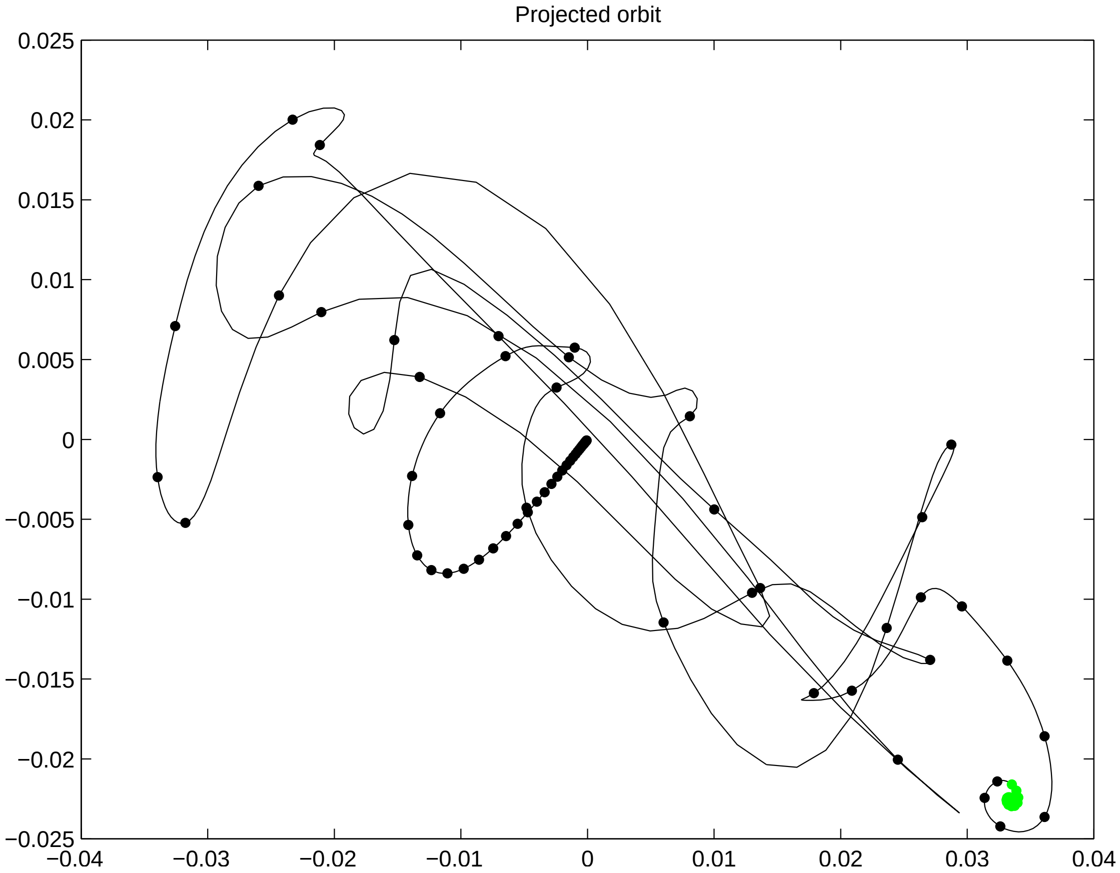
3-d plane Couette: $10^4 - 10^5$

A "turbulent KS" trajectory



long-time numerical run of the dynamics Kuramoto-Sivashinsky example (a two Fourier modes projection)

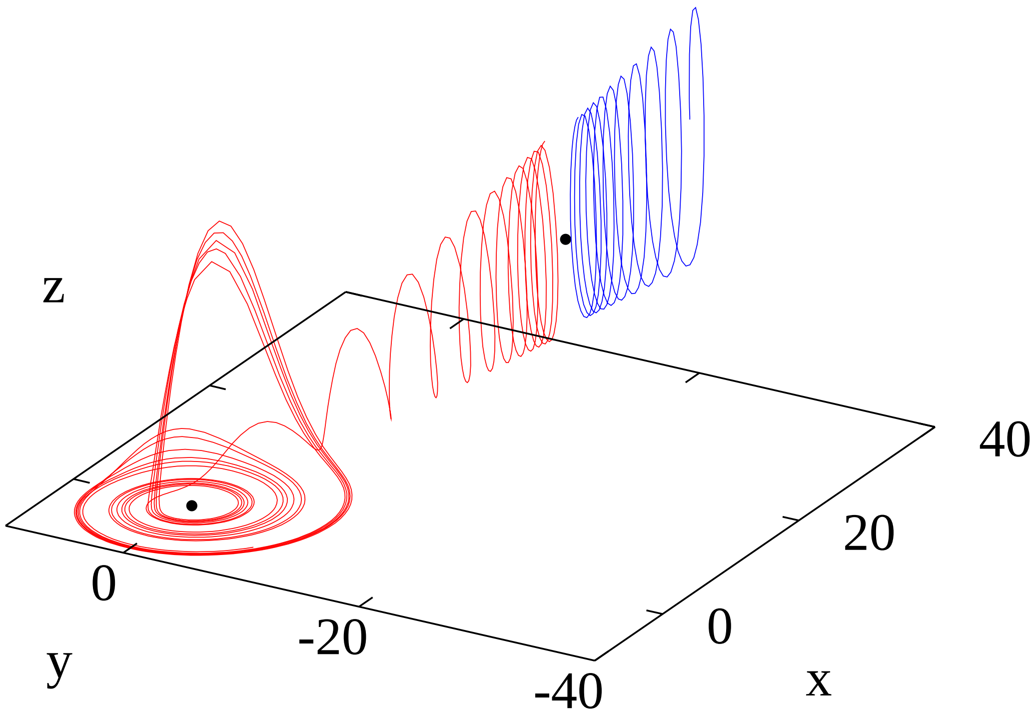
A "turbulent Plane Couette" trajectory Re = 400



a transient starting close to the upper branch, ending in the laminar state (30K modes 3-D Navier-Stokes DNS, a projection from Fourier × Fourier × Chebyshev → unstable spiral plane of Waleffe's upper branch)

Equilibria / Traveling waves

Role of Rössler flow equilibria



``+'' equilibrium point stable manifold
= basin boundary

right of the ``+'' trajectories escape

left of the ``+'' fall into chaotic attractor circling the ``−'' equilibrium point

Kuramoto-Sivashinsky equilibria

find $u(x,t) = u(x+L,t)$ spatially periodic Kuramoto-Sivashinsky equilibria using the variational method for ODE with "time" x

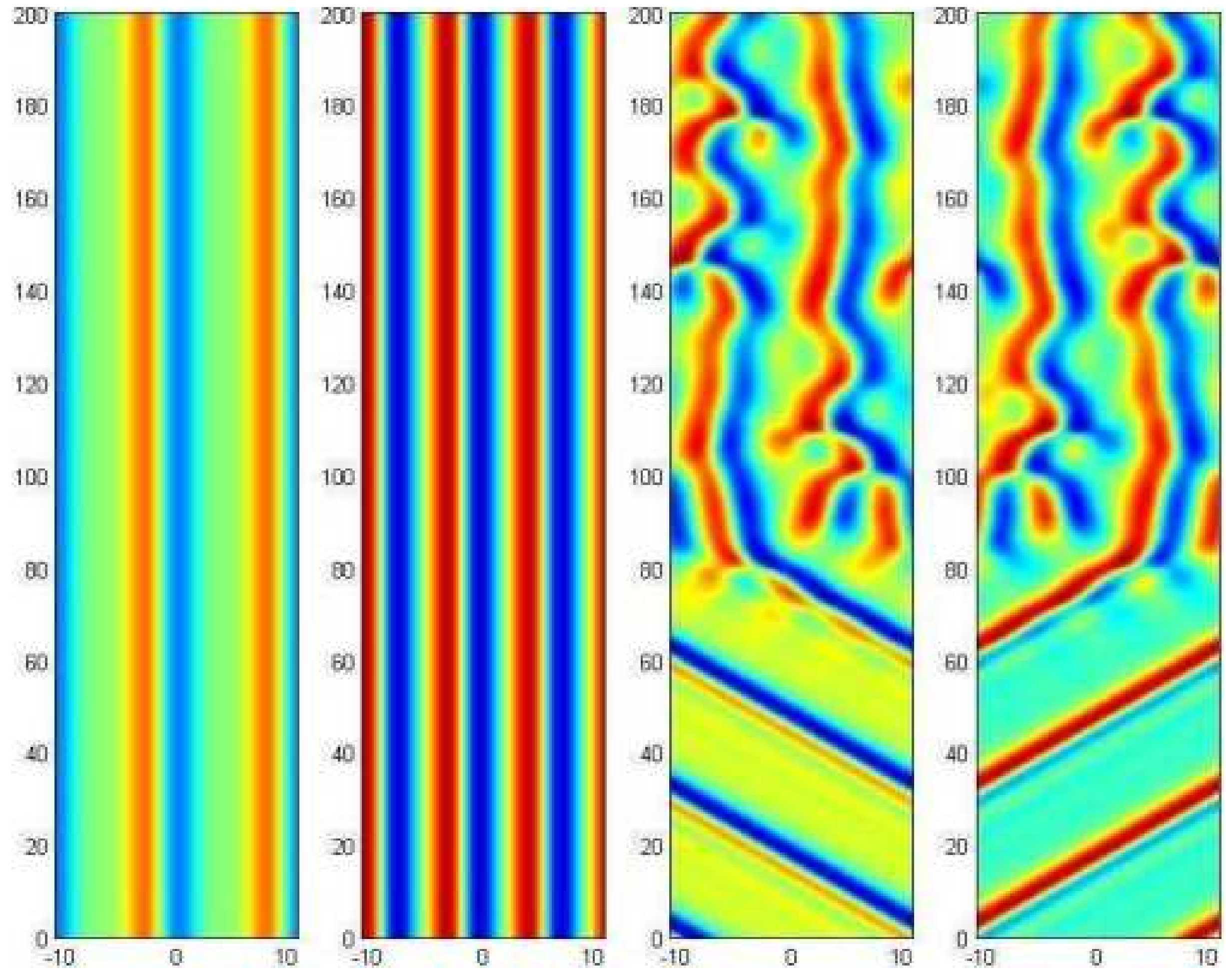
$$(u^2)_x - u_{xx} - \nu u_{xxxx} = 0 \,.$$

number of equilibria increases rapidly with the system size L.

need to classify them according to their importance for asymptotic dynamics

Important Kuramoto-Sivashinsky equilibria

The non-wondering set dynamics for $L = 22$ is qualitatively controled by unstable 2-wavelength and 3-wavelength equilibria, and a dual pair of discrete symmetry related unstable 1-wavelength relative equilibria/travelling waves.



2-wavelength equilibrium 3-wavelength equilibrium
on the interval [0, L]

a typical instantaneous ``turbulent'' Kuramoto-Sivashinsky profile bears resemblance to one of these equilibria.

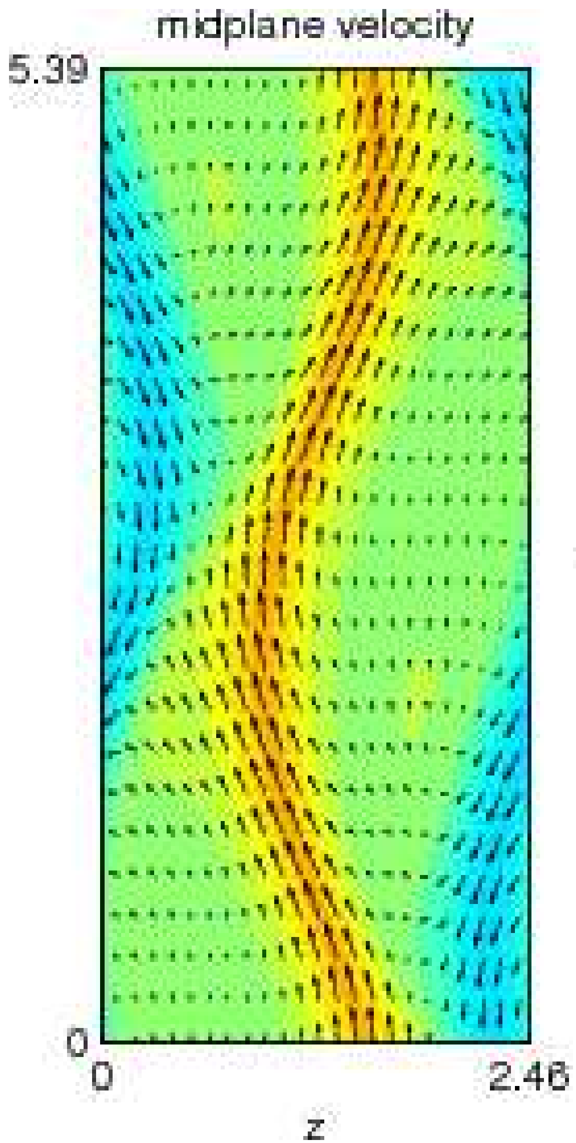
F. Wallefe Exact Coherent Structure[10], plotted by John F. Gibson[11].

Plane Couette at Re = 400

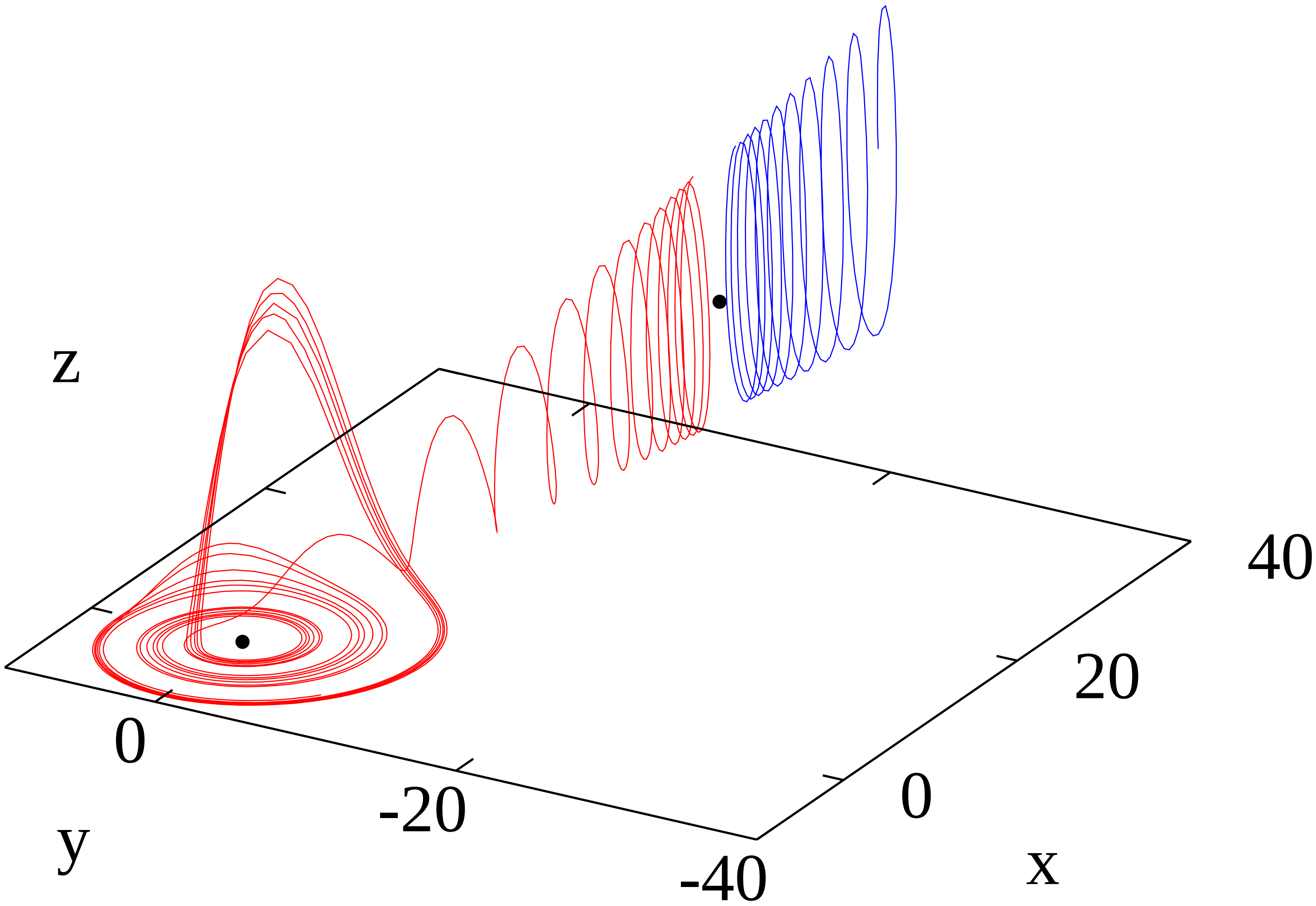
"upper branch"
unstable equilibrium

Periodic [$L_x = 2\pi/1.14$, $L_z = 2\pi/2.5$] box
in x (streamwise) and z (spanwise),

Chebyshev wall normal.



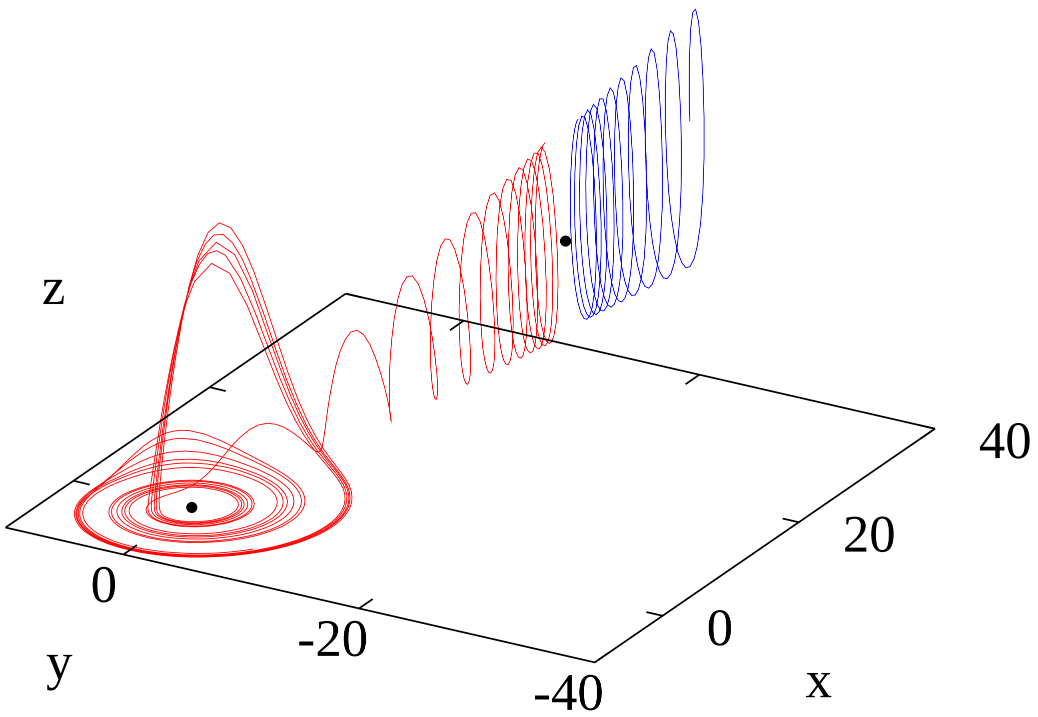
[10] www.math.wisc.edu/~waleffe/ECS/RRC-data.html
[11] www.nongnu.org/channelflow


z
40
20
0
x
0
-20
-40
y

Stability of Rössler flow equilibria

two equilibrium points (x^-, y^-, z^-) (x^+, y^+, z^+)

stable manifold of "+" equilibrium point = attraction basin boundary:



right of the "+" equilibrium trajectories escape,

left of the "+" spiral toward the "−" equilibrium point

$\rightarrow$ seem to wander chaotically for all times.

linearized stability exponents

$$(\lambda_1^-, \lambda_2^- \pm i\vartheta_2^-) = (-5.686, \quad 0.0970 \pm i0.9951)$$
$$(\lambda_1^+, \lambda_2^+ \pm i\vartheta_2^+) = (0.1929, \quad -4.596 \times 10^{-6} \pm i5.428)$$

The $\lambda_2^- \pm i\vartheta_2^-$ eigenvectors span a plane
this plane rotates with angular period $T_- \approx \left|2\pi/\vartheta_2^-\right| = 6.313$

a trajectory that starts near the "–" equilibrium point spirals away
per one rotation with multiplier $\Lambda_{radial} \approx \exp(\lambda_2^- T_-) = 1.84$

each Poincaré section return, contracted into the stable manifold
by amazing factor of $\Lambda_1 \approx \exp(\lambda_1^- T_-) = 10^{-15.6}$ (!)

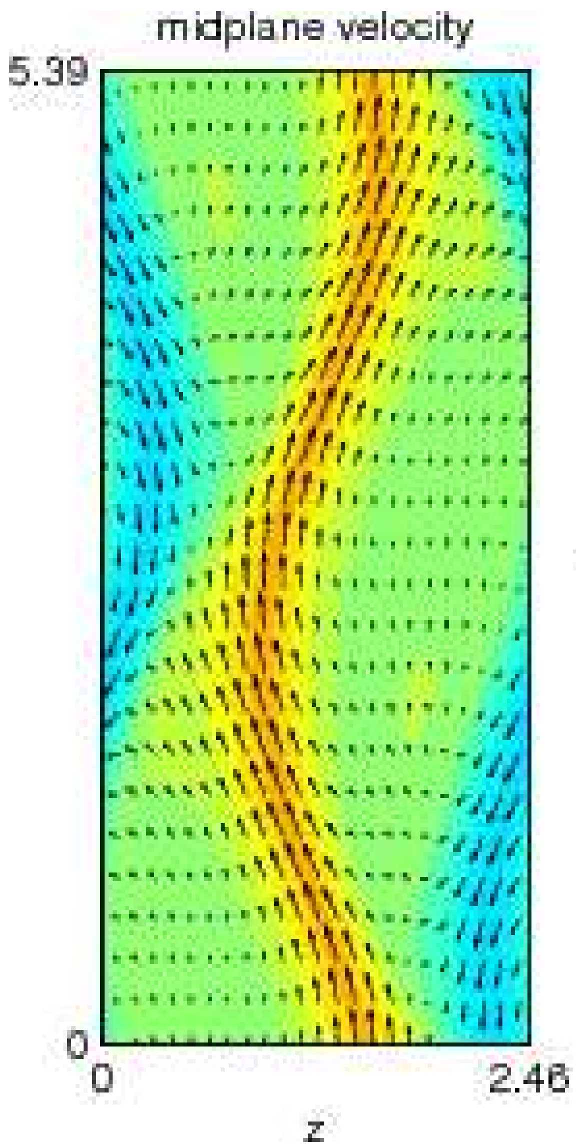
Important Kuramoto-Sivashinsky equation equilibria:
the first few stability exponents

S	$\lambda_1 \pm i\vartheta_1$	$\lambda_2 \pm i\vartheta_2$	$\lambda_3 \pm i\vartheta_3$
C_1	$0.04422 \pm i\,0.26160$	$-0.255 \pm i\,0.431$	$-0.347 \pm i\,0.463$
C_2	0.33053	$0.097 \pm i\,0.243$	$-0.101 \pm i\,0.233$
R_1	$0.01135 \pm i\,0.79651$	$-0.215 \pm i\,0.549$	$-0.358 \pm i\,0.262$
R_2	0.33223	$-0.001 \pm i\,0.703$	$-0.281 \pm i\,0.399$
T	0.25480	$-0.07 \pm i\,0.645$	-0.264

spiraling out in a plane, all other directions contracting

F. Wallefe Exact Coherent Structure[12], plotted by John F. Gibson[13].

Plane Couette at Re = 400



"upper branch"
unstable equilibrium

Periodic [$L_x = 2\pi/1.14$, $L_z = 2\pi/2.5$] box
in x (streamwise) and z (spanwise),

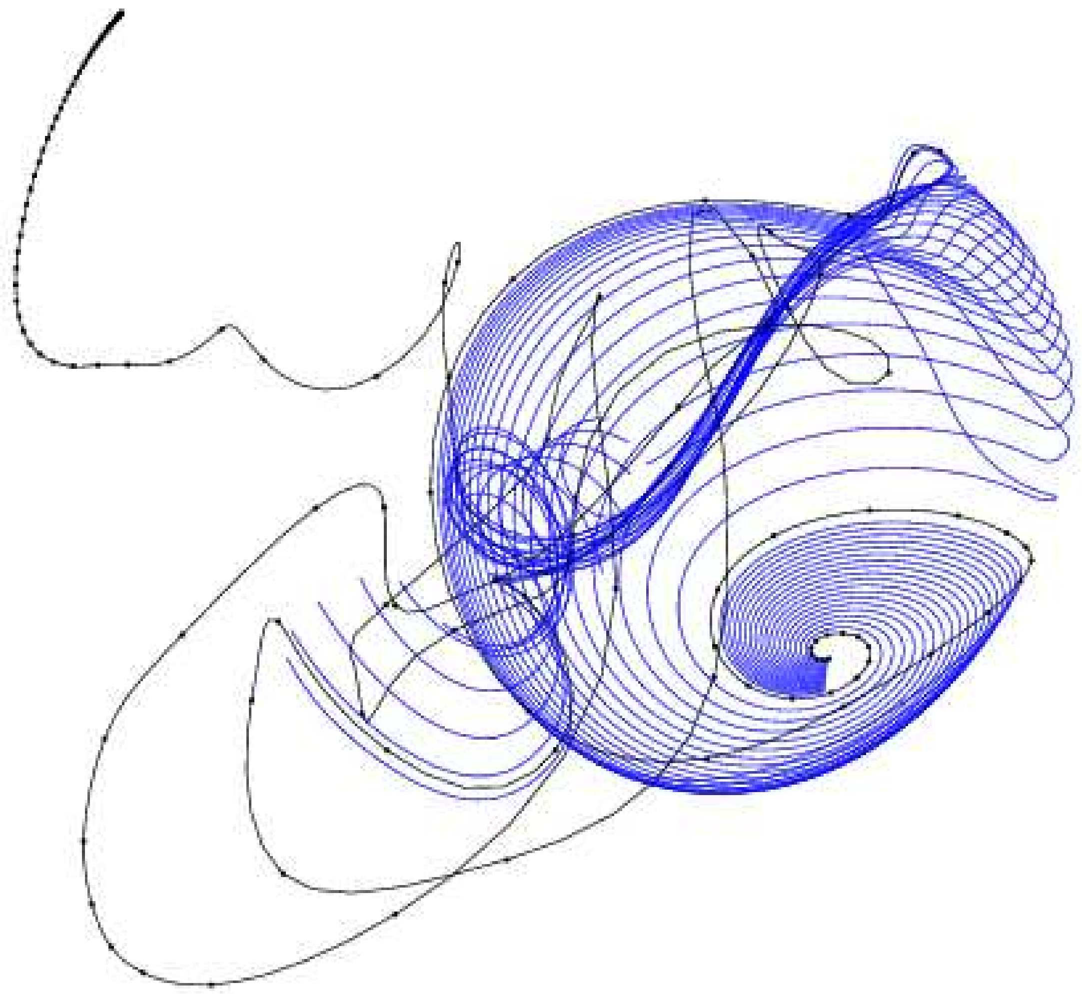
Chebyshev wall normal.

[12] www.math.wisc.edu/~waleffe/ECS/RRC-data.html
[13] www.nongnu.org/channelflow

Unstable manifold, upper-branch equilibrium

Black: from ``upper branch'' to laminar fixed point

Blue: trajectories started near unstable equilibrium → 2-d unstable manifold over large region of phase space



R = 400 plane Coutte phase space, projection $30 \times 10^3 \rightarrow 2$ dimensions

[J.F. Gibson]

Equivaraint trace formulae

Dynamical systems

state space $\mathcal{M}$

representative point $x(t) \in \mathcal{M}$: a physical system at instant in time

dynamics: $f^t(x_0)$ = representative point time t later

deterministic dynamics: evolution rule f maps a point into exactly one point at time t.

dynamical system: the pair $(\mathcal{M}, f)$

$\mathcal{M} \approx \mathbb{R}^d$, d numbers determine next state.

Flows

For infinitesimal times, flows can be defined by differential equations – a generalized vector field

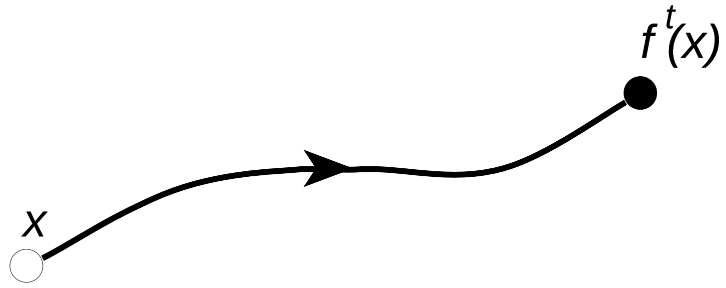
$$v(x) = \dot{x}(t)\,.$$

Examples:

Newton's laws for a mechanical system

general flows, mechanical or not, defined by a time-independent vector field $v(x)$

Trajectories



trajectory: evolution rule f^t traces out curve $x(t) = f^t(x_0)$, through the point $x_0 = x(0)$:

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau)) , \qquad x(0) = x_0 .$$

Types of trajectories?

stationary: $f^t(x) = x$ for all t

periodic: $f^t(x) = f^{t+T_P}(x)$ for a given minimum period T_P

aperiodic: $f^t(x) \neq f^{t'}(x)$ for all $t \neq t'$.

A periodic orbit corresponds to a trajectory that returns exactly to the initial point in a finite time.

Periodic orbits: a very small subset of the phase space, in the same sense that rational numbers are a set of zero measure on the unit interval.

for a generic dynamical system most motions are aperiodic

Evolution operators

rewrite as

$$\left\langle e^{\beta\cdot A^t}\right\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} dx \int_{\mathcal{M}} dy\, \delta\big(y - f^t(x)\big)\, e^{\beta\cdot A^t(x)} .$$

$\delta\big(y - f^t(x)\big)$ is the Dirac delta function.

evolution operator

$$\mathcal{L}^t(y,x) = \delta\big(y - f^t(x)\big)\, e^{\beta\cdot A^t(x)} .$$

replaces individual trajectories $f^t(x)$ by evolution of a density of the totality of initial conditions:

probe the entire phase space with finite time pieces of trajectories originating from every point in $\mathcal{M}$.

leading eigenvalue

$$\mathcal{L}^t(y,x) \to e^{s_0} \to \text{expectation values}$$

Trace formula for a deterministic flow

The classical trace formula for flows:

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{\left| \det \left(\mathbf{1} - \mathbf{J}_p^r \right) \right|} .$$

Hyperbolicity assumption

stabilities of all cycles exponentially bounded

$$|\Lambda_{p,e}| > e^{\lambda_e T_p} \qquad \text{any } p, \text{ any expanding } |\Lambda_{p,e}| > 1$$

$$|\Lambda_{p,c}| < e^{-\lambda_c T_p} \qquad \text{any } p, \text{ any contracting } |\Lambda_{p,c}| < 1\,,$$

$\lambda_e, \lambda_c > 0$ are strictly positive bounds on the expanding, contracting cycle Lyapunov exponents.

for long times, $t = rT_p \to \infty$, only the product of expanding eigenvalues matters:

$$\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right| \to |\Lambda_p|^r$$

Local trace

Trace over prime cycle p of period n_p, neighborhood $\mathcal{M}_p$

$$\mathrm{tr}_p \mathcal{L}^{n_p} = \int_{\mathcal{M}_p} dx\, \delta\left(x - f^{n_p}(x)\right) = \frac{n_p}{\left|\det\left(\mathbf{1} - \mathbf{J}_p\right)\right|}$$

Assume that no marginal eigenvalue

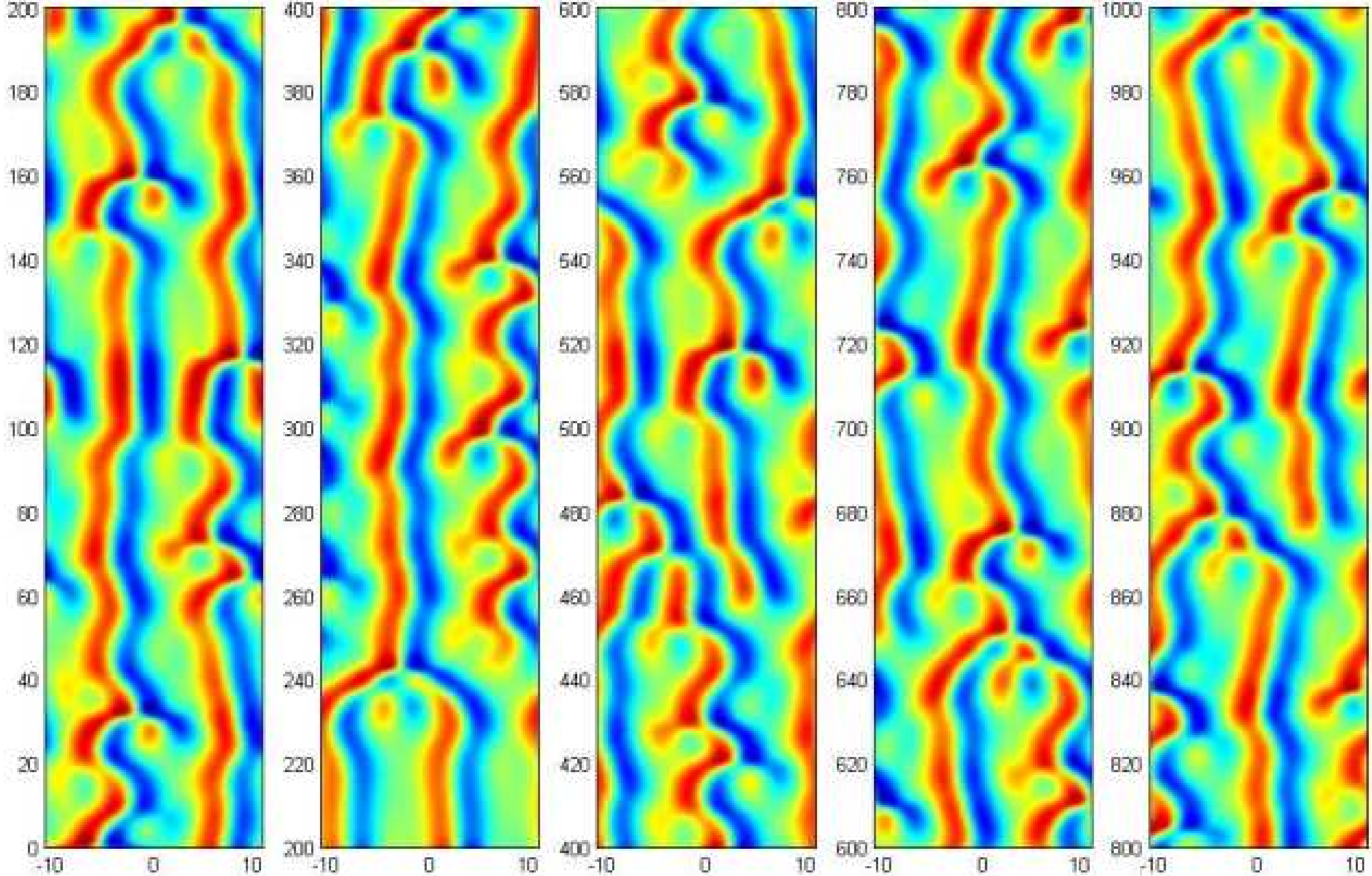
factor eigenvalues of Jacobian matrix $\mathbf{J}_p$ into expanding and contracting sets $\{e, c\}$:

$$\left|\det\left(\mathbf{1} - \mathbf{J}_p\right)\right|^{-1} = \frac{1}{|\Lambda_p|} \prod_e \frac{1}{1 - 1/\Lambda_{p,e}} \prod_c \frac{1}{1 - \Lambda_{p,c}} \,,$$

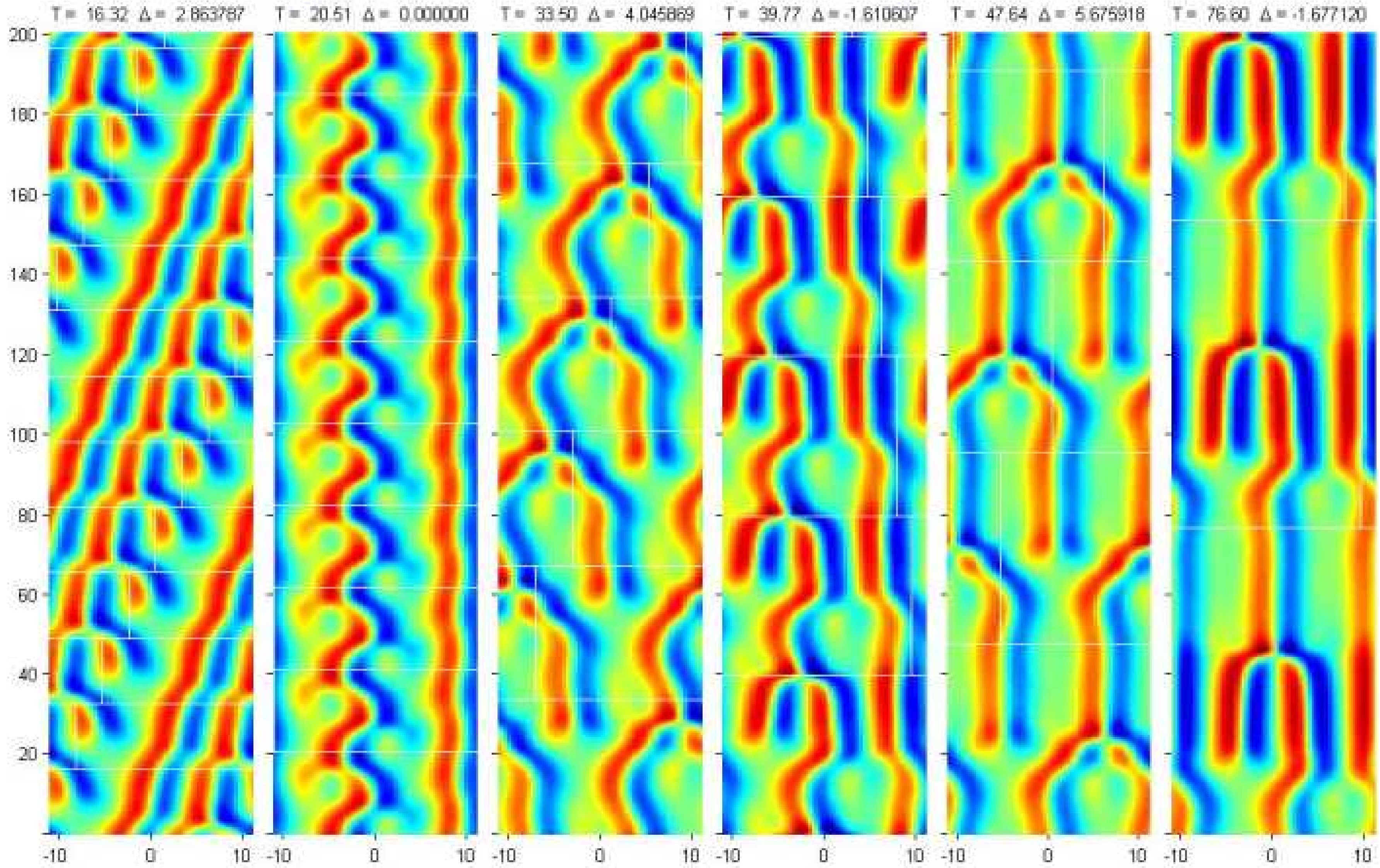
$\Lambda_p = \prod_e \Lambda_{p,e}$ = product of expanding eigenvalues.

Relative periodic orbits: how to find them

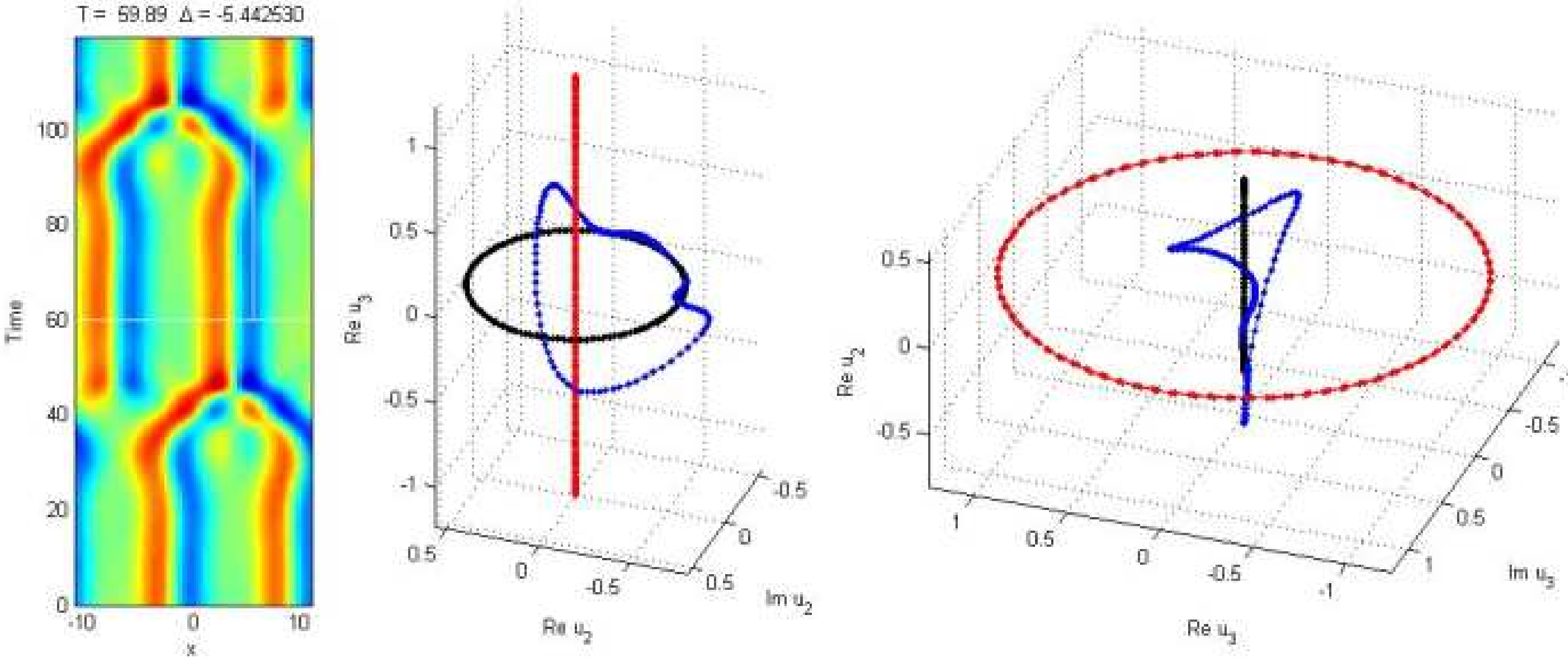
ask Ruslan Davidchack



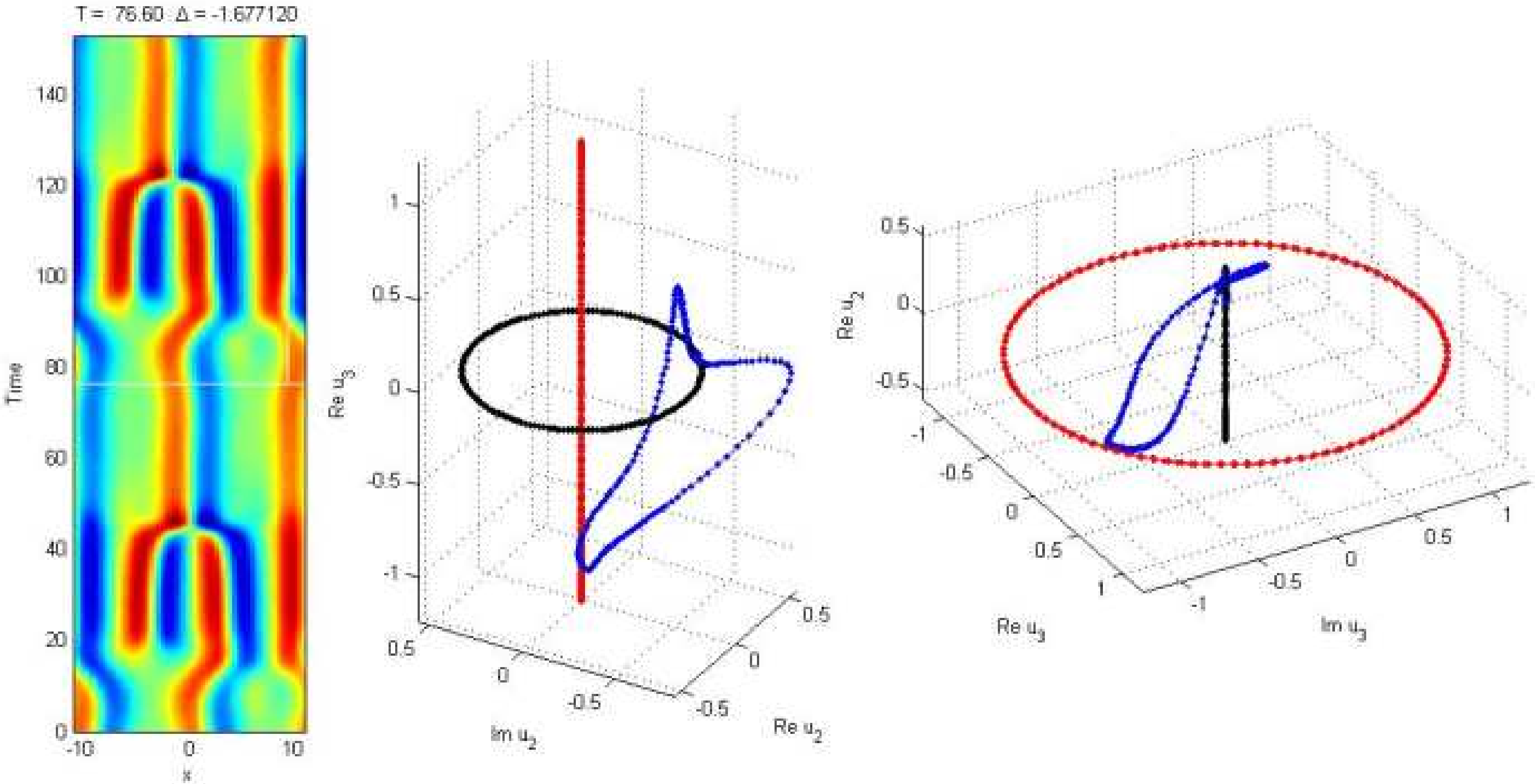
200
180
160
140
120
100
80
60
40
20
0
-10
0
10
400
380
360
340
320
300
280
260
240
220
200
600
580
560
540
520
500
480
460
440
420
400
800
780
760
740
720
700
680
660
640
620
600
1000
980
960
940
920
900
880
860
840
820
800



T = 16.32 Δ = 2.863787
T = 20.51 Δ = 0.000000
T = 33.50 Δ = 4.045869
T = 39.77 Δ = -1.610607
T = 47.64 Δ = 5.675918
T = 76.60 Δ = -1.677120



T = 59.89 Δ = -5.442530
Time
x
Re u3
Re u2
Im u2
Re u2
Re u3
Im u3

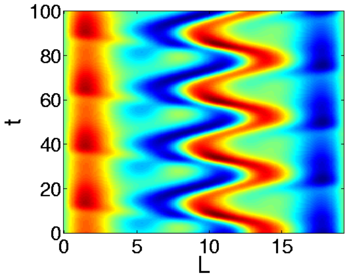


T = 76.60 Δ = -1.677120
Time
x
Re u3
Im u2
Re u2
Re u2
Re u3
Im u3

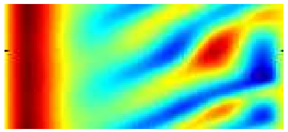
Kuramoto-Sivashinsky: Hopf's vision

A long time series:

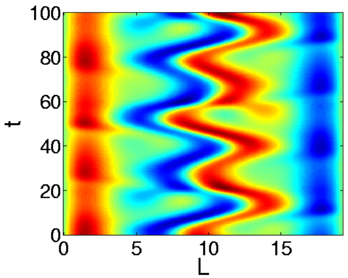
jumps between



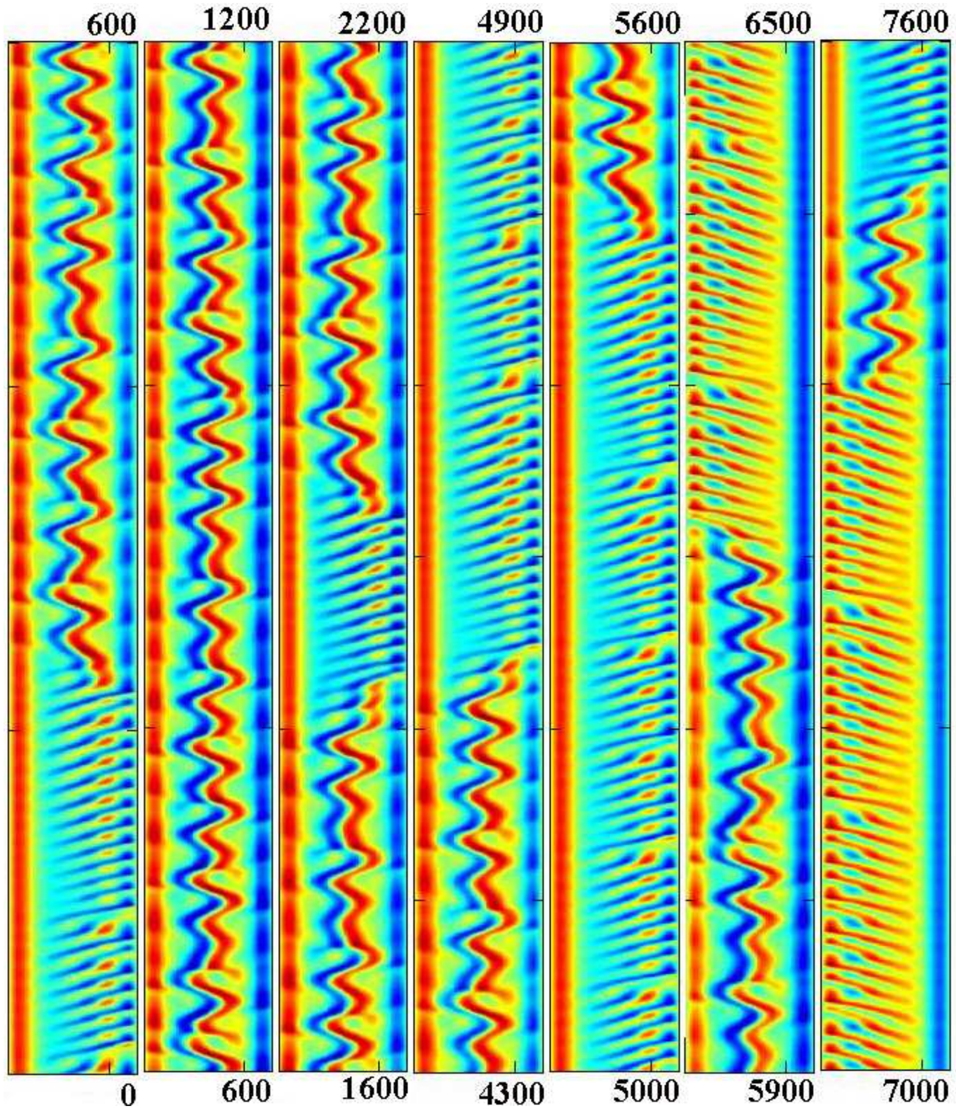
→



→



→ etc.



.

Future looks bright

Recurrent patterns vs. models of turbulence

What new does recurrent patterns program offer?

Normal form models of applied mathematics – such as the Lorenz model – capture qualitatively some bifurcations and chaos similar to those observed in hydrodynamics

Periodic orbit theory provides accurate quantitative predictions for given flow, given boundary conditions, given "Reynolds" and other parameters.

Conclusion: Hof is hope renewed for
Hopf's Last Hope for a Theory of Turbulence

Hopf's vision: repertoire of recurrent spatio-temporal patterns explored by turbulent dynamics

detailed dynamics horrible, but much less so than feared:
pieced together from 1-d return maps (!)

"To do" list:

Q: plane Couette-Taylor shear flow?
Waleffe; Kawahara & Kida: it can be done!

.

> In theory there is no difference between theory and practice.
> In practice there is. Yogi Berra

not Snepscheut! appologies to Lyonia,
thanks to Mason Porter.