Absence and existence of phase transitions in piecewise expanding coupled map lattices

Gerhard Keller, Universität Erlangen-Nürnberg

Coauthors:
Carlangelo Liverani, Università di Roma "Tor Vergata"
Jean-Baptiste Bardet, Université de Rennes 1

Durham, July 8, 2006

Definition: Coupled map lattice (CML)

- lattice: $\Lambda = \mathbb{Z}^d$ or $(\mathbb{Z}/L\mathbb{Z})^d$
- local systems: $\tau : I \to I$ (p.w. C^2, p.w. expanding, mixing)
  - Annihilation of two initial probability densities at exponential speed
    Spectral gap for Perron-Frobenius operator acting on BV(I)
    (K., C.R.Acad.Sc. Paris (1980))

Definition: Coupled map lattice (CML)

- lattice: $\Lambda = \mathbb{Z}^d$ or $(\mathbb{Z}/L\mathbb{Z})^d$
- local systems: $\tau : I \to I$ (p.w. C^2, p.w. expanding, mixing)
- global system: $\Omega = I^\Lambda$,

$$T : \Omega \to \Omega, \qquad (Tx)_p = \tau(x_p) \quad (p \in \Lambda)$$

$$\Phi_\epsilon : \Omega \to \Omega, \qquad \text{“}\epsilon\text{-close to } \mathrm{Id}_\Omega \text{ in } C^2\text{”}$$

$$T_\epsilon := T \circ \Phi_\epsilon \qquad \text{or} \qquad \Phi_\epsilon \circ T$$

Definition: Coupled map lattice (CML)

- lattice: $\Lambda = \mathbb{Z}^d$ or $(\mathbb{Z}/L\mathbb{Z})^d$
- local systems: $\tau : I \to I$ (p.w. C^2, p.w. expanding, mixing)
- global system: $\Omega = I^\Lambda$,

$$T : \Omega \to \Omega, \qquad (Tx)_p = \tau(x_p) \quad (p \in \Lambda)$$

$$\Phi_\epsilon : \Omega \to \Omega, \qquad \text{“}\epsilon\text{-close to } \mathrm{Id}_\Omega \text{ in } C^2\text{”}$$

$$T_\epsilon := T \circ \Phi_\epsilon \qquad \text{or} \qquad \Phi_\epsilon \circ T$$

Example: Diffusive nearest neighbour coupling

$$(\Phi_\epsilon x)_p = (1-\epsilon)x_p + \frac{\epsilon}{2d} \sum_{q \in \mathcal{N}(p)} x_q$$

Definition: Coupled map lattice (CML)

- lattice: $\Lambda = \mathbb{Z}^d$ or $(\mathbb{Z}/L\mathbb{Z})^d$
- local systems: $\tau : I \to I$ (p.w. C^2, p.w. expanding, mixing)
- global system: $\Omega = I^\Lambda$,

$$\begin{aligned} T &: \Omega \to \Omega, && (Tx)_p = \tau(x_p) \quad (p \in \Lambda) \\ \Phi_\epsilon &: \Omega \to \Omega, && \text{“}\epsilon\text{-close to } \mathrm{Id}_\Omega \text{ in } C^2\text{”} \\ T_\epsilon &:= T \circ \Phi_\epsilon \quad \text{or} && \Phi_\epsilon \circ T \end{aligned}$$

Example: Diffusive nearest neighbour coupling

$$(\Phi_\epsilon x)_p = (1-\epsilon)x_p + \frac{\epsilon}{2d} \sum_{q \in \mathcal{N}(p)} x_q$$

$\Phi_\epsilon : \Omega \to \Omega$ differentiable but **not diffeomorphism!**

SRB measures (also: physical, natural,. . . measures)

SRB measures (also: physical, natural,. . . measures)

- What are SRB measures?
  - Law of large numbers for “a.a.” initial conditions
  - Stability under “smooth” random perturbations

SRB measures (also: physical, natural,... measures)

- What are SRB measures?
  - Law of large numbers for "a.a." initial conditions
  - Stability under "smooth" random perturbations
- Existence of SRB measures?
- Uniqueness of SRB measures? Phase transitions?
- Exponential decay of correlations?

SRB measures (also: physical, natural,. . . measures)

- What are SRB measures?
  - Law of large numbers for “a.a.” initial conditions
  - Stability under “smooth” random perturbations
- Existence of SRB measures?
- Uniqueness of SRB measures? Phase transitions?
- Exponential decay of correlations?

The beginnings

- Kaneko ’83+

SRB measures (also: physical, natural,. . . measures)

- What are SRB measures?
  - Law of large numbers for "a.a." initial conditions
  - Stability under "smooth" random perturbations
- Existence of SRB measures?
- Uniqueness of SRB measures? Phase transitions?
- Exponential decay of correlations?

The beginnings

- Kaneko '83+
- Bunimovich/Sinai '88

Small $|\epsilon|$, C^2 map (expanding or hyperbolic), diffeomorphic Φ_ϵ

Baladi, Bricmont, Bunimovich, Degli Eposti, Fischer, Isola, Järvenpää, Jiang, Kupiainen, Pesin, Rugh, Sinai, Volevich,

Small $|\epsilon|$, C^2 map (expanding or hyperbolic), diffeomorphic Φ_ϵ

Baladi, Bricmont, Bunimovich, Degli Eposti, Fischer, Isola, Järvenpää, Jiang, Kupiainen, Pesin, Rugh, Sinai, Volevich,

Small $|\epsilon|$, p.w. C^2 exp. interval map, C^2 but non-diffeomorphic Φ_ϵ

Small $|\epsilon|$, C^2 map (expanding or hyperbolic), diffeomorphic Φ_ϵ

Baladi, Bricmont, Bunimovich, Degli Eposti, Fischer, Isola, Järvenpää, Jiang, Kupiainen, Pesin, Rugh, Sinai, Volevich,

Small $|\epsilon|$, p.w. C^2 exp. interval map, C^2 but non-diffeomorphic Φ_ϵ

- K./Künzle '92, Künzle '93:
  Existence of invariant measures with absolutely continuous finite-dimensional marginals: $\mu_\epsilon \in \mathrm{AC}$.

Small $|\epsilon|$, C^2 map (expanding or hyperbolic), diffeomorphic Φ_ϵ

Baladi, Bricmont, Bunimovich, Degli Eposti, Fischer, Isola, Järvenpää, Jiang, Kupiainen, Pesin, Rugh, Sinai, Volevich,

Small $|\epsilon|$, p.w. C^2 exp. interval map, C^2 but non-diffeomorphic Φ_ϵ

- K./Künzle '92, Künzle '93:
  Existence of invariant measures with absolutely continuous finite-dimensional marginals: $\mu_\epsilon \in$ AC.
- Schmitt '03: Uniqueness, spectral gap if $\Lambda = \mathbb{Z}$ and $|\tau'|$ large

Small $|\epsilon|$, C^2 map (expanding or hyperbolic), diffeomorphic Φ_ϵ

Baladi, Bricmont, Bunimovich, Degli Eposti, Fischer, Isola, Järvenpää, Jiang, Kupiainen, Pesin, Rugh, Sinai, Volevich,

Small $|\epsilon|$, p.w. C^2 exp. interval map, C^2 but non-diffeomorphic Φ_ϵ

- K./Künzle '92, Künzle '93:
  Existence of invariant measures with absolutely continuous finite-dimensional marginals: $\mu_\epsilon \in \mathrm{AC}$.
- Schmitt '03: Uniqueness, spectral gap if $\Lambda = \mathbb{Z}$ and $|\tau'|$ large
- K./Liverani '04: Uniqueness, spectral gap, SRB if $\Lambda = \mathbb{Z}$
- K./Liverani '04/05: Uniqueness, spectral gap, SRB if $\Lambda = \mathbb{Z}^d$

Small $|\epsilon|$, C^2 map (expanding or hyperbolic), diffeomorphic Φ_ϵ

Baladi, Bricmont, Bunimovich, Degli Eposti, Fischer, Isola, Järvenpää, Jiang, Kupiainen, Pesin, Rugh, Sinai, Volevich,

Small $|\epsilon|$, p.w. C^2 exp. interval map, C^2 but non-diffeomorphic Φ_ϵ

- K./Künzle '92, Künzle '93:
  Existence of invariant measures with absolutely continuous finite-dimensional marginals: $\mu_\epsilon \in$ AC.
- Schmitt '03: Uniqueness, spectral gap if $\Lambda = \mathbb{Z}$ and $|\tau'|$ large
- K./Liverani '04: Uniqueness, spectral gap, SRB if $\Lambda = \mathbb{Z}$
- K./Liverani '04/05: Uniqueness, spectral gap, SRB if $\Lambda = \mathbb{Z}^d$
- Bardet/K. '06: Example for phase transition with $\Lambda = \mathbb{Z}^2$

Notation: the measures

- $\mathcal{M}(\Omega)$: finite signed Borel measures on $\Omega = I^\Lambda$,

$$|\mu| := \sup_{|\varphi|_\infty \le 1} \mu(\varphi), \qquad \text{“total variation”, “total mass”}$$

Notation: the measures

- $\mathcal{M}(\Omega)$: finite signed Borel measures on $\Omega = I^{\Lambda}$,

$$|\mu| := \sup_{|\varphi|_\infty \le 1} \mu(\varphi), \quad \text{"total variation", "total mass"}$$

- $\mathcal{B}(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \mathrm{Var}(\mu) < \infty\}$,

$$\mathrm{Var}(\mu) := \sup_{|\varphi|_\infty \le 1} \sup_{p \in \Lambda} \mu(\partial_p \varphi), \quad \text{"variation"}$$

$\varphi : \mathbb{R}^{\Lambda} \to \mathbb{R}$ test functions with continuous partial derivatives

Notation: the measures

- $\mathcal{M}(\Omega)$: finite signed Borel measures on $\Omega = I^\Lambda$,

$$|\mu| := \sup_{|\varphi|_\infty \leq 1} \mu(\varphi), \qquad \text{“total variation”, “total mass”}$$

- $\mathcal{B}(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \mathrm{Var}(\mu) < \infty\}$,

$$\mathrm{Var}(\mu) := \sup_{|\varphi|_\infty \leq 1} \sup_{p \in \Lambda} \mu(\partial_p \varphi), \qquad \text{“variation”}$$

$\varphi : \mathbb{R}^\Lambda \to \mathbb{R}$ test functions with continuous partial derivatives

$$d\mu = f\, d\lambda^n : \quad |\mu(\partial_p \varphi)| = \left| \int \partial_p f \cdot \varphi \, d\lambda^n \right| \leq \|\partial_p f\|_{L^1}$$

Notation: the measures

- $\mathcal{M}(\Omega)$: finite signed Borel measures on $\Omega = I^\Lambda$,

$$|\mu| := \sup_{|\varphi|_\infty \leq 1} \mu(\varphi), \quad \text{“total variation”, “total mass”}$$

- $\mathcal{B}(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \mathrm{Var}(\mu) < \infty\}$,

$$\mathrm{Var}(\mu) := \sup_{|\varphi|_\infty \leq 1} \sup_{p \in \Lambda} \mu(\partial_p \varphi), \quad \text{“variation”}$$

  $\varphi : \mathbb{R}^\Lambda \to \mathbb{R}$ test functions with continuous partial derivatives
- $|\mu| \leq \frac{1}{2} \mathrm{Var}(\mu)$, so “Var” is a norm.

Notation: the measures

- $\mathcal{M}(\Omega)$: finite signed Borel measures on $\Omega = I^\Lambda$,

$$|\mu| := \sup_{|\varphi|_\infty \le 1} \mu(\varphi), \quad \text{"total variation", "total mass"}$$

- $\mathcal{B}(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \mathrm{Var}(\mu) < \infty\}$,

$$\mathrm{Var}(\mu) := \sup_{|\varphi|_\infty \le 1} \sup_{p \in \Lambda} \mu(\partial_p \varphi), \quad \text{"variation"}$$

$\varphi : \mathbb{R}^\Lambda \to \mathbb{R}$ test functions with continuous partial derivatives

- $|\mu| \le \frac{1}{2} \mathrm{Var}(\mu)$, so "Var" is a norm.
- $\mathcal{B}(\Omega) \subsetneq \mathrm{AC}(\Omega)$
  (absolutely continuous finite-dimensional marginals)

Notation: the measures

- $\mathcal{M}(\Omega)$: finite signed Borel measures on $\Omega = I^\Lambda$,

$$|\mu| := \sup_{|\varphi|_\infty \le 1} \mu(\varphi), \quad \text{“total variation”, “total mass”}$$

- $\mathcal{B}(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \mathrm{Var}(\mu) < \infty\}$,

$$\mathrm{Var}(\mu) := \sup_{|\varphi|_\infty \le 1} \sup_{p \in \Lambda} \mu(\partial_p \varphi), \quad \text{“variation”}$$

  $\varphi : \mathbb{R}^\Lambda \to \mathbb{R}$ test functions with continuous partial derivatives

- $|\mu| \le \frac{1}{2}\mathrm{Var}(\mu)$, so “Var” is a norm.
- $\mathcal{B}(\Omega) \subsetneq \mathrm{AC}(\Omega)$
  (absolutely continuous finite-dimensional marginals)
- $\mu \in \mathcal{B}(\Omega) \Rightarrow \mu$ has finite entropy density

Notation: "good" (a_1, a_2)-coupling

$\Phi_\epsilon : \Omega \to \Omega$, $\Phi_\epsilon(x) = x + A_\epsilon(x)$

Notation: "good" (a_1, a_2)-coupling

$\Phi_\epsilon : \Omega \to \Omega$, $\Phi_\epsilon(x) = x + A_\epsilon(x)$

- $$|(A_\epsilon(x))_p| \leq 2|\epsilon|$$
- $$|\partial_q (A_\epsilon(x))_p| \leq 2|\epsilon|\, A'_{qp}, \quad A' \text{ a } \Lambda \times \Lambda\text{-matrix}, \; \|A'\|_1 = a_1$$
- $$|\partial_k \partial_q (A_\epsilon(x))_p| \leq 2|\epsilon|\, A''_{qp}, \quad A'' \text{ a } \Lambda \times \Lambda\text{-matrix}, \; \|A''\|_1 = a_2$$

Notation: “good” (a_1, a_2)-coupling

$\Phi_\epsilon : \Omega \to \Omega$, $\Phi_\epsilon(x) = x + A_\epsilon(x)$

- $|(A_\epsilon(x))_p| \leq 2|\epsilon|$
- $|\partial_q (A_\epsilon(x))_p| \leq 2|\epsilon|\, A'_{qp}$, A' a $\Lambda \times \Lambda$-matrix, $\|A'\|_1 = a_1$
- $|\partial_k \partial_q (A_\epsilon(x))_p| \leq 2|\epsilon|\, A''_{qp}$, A'' a $\Lambda \times \Lambda$-matrix, $\|A''\|_1 = a_2$

Example Diffusive nearest neighbour coupling is a $(1, 0)$-coupling

Notation: "good" (a_1, a_2)-coupling

$\Phi_\epsilon : \Omega \to \Omega$, $\Phi_\epsilon(x) = x + A_\epsilon(x)$

- $|(A_\epsilon(x))_p| \leq 2|\epsilon|$
- $|\partial_q (A_\epsilon(x))_p| \leq 2|\epsilon|\, A'_{qp}$, A' a $\Lambda \times \Lambda$-matrix, $\|A'\|_1 = a_1$
- $|\partial_k \partial_q (A_\epsilon(x))_p| \leq 2|\epsilon|\, A''_{qp}$, A'' a $\Lambda \times \Lambda$-matrix, $\|A''\|_1 = a_2$

Example Diffusive nearest neighbour coupling is a $(1, 0)$-coupling

Proposition: Existence (K./Künzle '92, Künzle '93)

Given τ p.w. C^2 expanding and a good coupling,

$$\exists \epsilon_1 > 0 \text{ s.t. } \forall |\epsilon| < \epsilon_1 \; \exists \mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$$

Notation: "good" (a_1, a_2)-coupling

$\Phi_\epsilon : \Omega \to \Omega$, $\Phi_\epsilon(x) = x + A_\epsilon(x)$

- $|(A_\epsilon(x))_p| \leq 2|\epsilon|$
- $|\partial_q(A_\epsilon(x))_p| \leq 2|\epsilon|\, A'_{qp}$, A' a $\Lambda \times \Lambda$-matrix, $\|A'\|_1 = a_1$
- $|\partial_k\partial_q(A_\epsilon(x))_p| \leq 2|\epsilon|\, A''_{qp}$, A'' a $\Lambda \times \Lambda$-matrix, $\|A''\|_1 = a_2$

Example Diffusive nearest neighbour coupling is a $(1, 0)$-coupling

Proposition: Existence (K./Künzle '92, Künzle '93)

Given τ p.w. C^2 expanding and a good coupling,

$$\exists \epsilon_1 > 0 \text{ s.t. } \forall |\epsilon| < \epsilon_1 \; \exists \mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$$

Example In case of diffusive nearest neighbour coupling:

$$\epsilon_1 = \frac{1}{2} - \frac{1}{\kappa_1} \quad \text{where} \quad \kappa_1 := \inf |\tau'| > 2.$$

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz),

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz), $\exists \epsilon_0 \in (0, \epsilon_1)$ s.t. $\forall |\epsilon| < \epsilon_0$

- $\exists$ unique! $\mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz), $\exists \epsilon_0 \in (0, \epsilon_1)$ s.t. $\forall |\epsilon| < \epsilon_0$

- $\exists$ unique! $\mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$
- μ_ϵ exponentially mixing in time and space

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz), $\exists \epsilon_0 \in (0, \epsilon_1)$ s.t. $\forall |\epsilon| < \epsilon_0$

- $\exists$ unique! $\mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$
- μ_ϵ exponentially mixing in time and space
- $\mu_\epsilon = \text{weak-}\lim_{L\to\infty} \mu_{\epsilon,L}$, (a.c. inv. measure on $I^{(\mathbb{Z}/(L\mathbb{Z}))^d}$)

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz), $\exists \epsilon_0 \in (0, \epsilon_1)$ s.t. $\forall |\epsilon| < \epsilon_0$

- $\exists$ unique! $\mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$
- μ_ϵ exponentially mixing in time and space
- $\mu_\epsilon = \text{weak-}\lim_{L\to\infty} \mu_{\epsilon,L}$, (a.c. inv. measure on $I^{(\mathbb{Z}/(L\mathbb{Z}))^d}$)
- μ_ϵ stable under smooth random perturbations of the system

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz), $\exists\epsilon_0 \in (0,\epsilon_1)$ s.t. $\forall|\epsilon| < \epsilon_0$

- $\exists$ unique! $\mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$
- μ_ϵ exponentially mixing in time and space
- $\mu_\epsilon =$ weak-$\lim_{L\to\infty} \mu_{\epsilon,L}$, (a.c. inv. measure on $I^{(\mathbb{Z}/(L\mathbb{Z}))^d}$)
- μ_ϵ stable under smooth random perturbations of the system
- **Strong law of large numbers:**

  Let $\psi \in C(\Omega,\mathbb{R})$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(T_\epsilon^k x) = \mu_\epsilon(\psi) \text{ for } \lambda \text{ -a.e. } x \in \Omega$$

Theorem (K./Liverani, Commun. Math. Phys. **262**, 2006)

Given τ p.w. C^2 expanding and a good finite range coupling (short range works if τ is Lipschitz), $\exists\epsilon_0 \in (0, \epsilon_1)$ s.t. $\forall|\epsilon| < \epsilon_0$

- $\exists$ unique! $\mu_\epsilon = T_\epsilon^* \mu_\epsilon \in \mathcal{B}(\Omega)$
- μ_ϵ exponentially mixing in time and space
- $\mu_\epsilon = \text{weak-}\lim_{L\to\infty} \mu_{\epsilon,L}$, (a.c. inv. measure on $I^{(\mathbb{Z}/(L\mathbb{Z}))^d}$)
- μ_ϵ stable under smooth random perturbations of the system
- **Strong law of large numbers:**
  $f : I \to \mathbb{R}$ probab. density of bd. variation, $\lambda_f = (fm)^\Lambda$.
  Let $\psi \in C(\Omega, \mathbb{R})$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(T_\epsilon^k x) = \mu_\epsilon(\psi) \text{ for } \lambda_f\text{-a.e. } x \in \Omega$$

SRB measure!

Ingredients of the proof

Lasota-Yorke type estimate

$$\mathrm{Var}(T_\epsilon^{*n}\mu) \le C \cdot \rho^n \cdot \mathrm{Var}(\mu) + B \cdot |\mu| \qquad (0 < \rho < 1)$$

Ingredients of the proof

Lasota-Yorke type estimate

$$\mathrm{Var}(T_\epsilon^{*n}\mu) \le C \cdot \rho^n \cdot \mathrm{Var}(\mu) + B \cdot |\mu| \qquad (0 < \rho < 1)$$

Decoupling estimate

For $p \in \Lambda$ define $\Phi_{\epsilon,p} : \Omega \to \Omega$ as

"Φ_ϵ with p decoupled from all other $q \in \Lambda$"

Let $T_{\epsilon,p} = \Phi_{\epsilon,p} \circ T$. Then

$$|T_\epsilon^{*N}\mu - T_{\epsilon,p}^{*N}\mu| \le CN\epsilon\,\mathrm{Var}(\mu)$$

Ingredients of the proof

Lasota-Yorke type estimate

$$\mathrm{Var}(T_\epsilon^{*n}\mu) \le C \cdot \rho^n \cdot \mathrm{Var}(\mu) + B \cdot |\mu| \qquad (0 < \rho < 1)$$

Decoupling estimate

For $p \in \Lambda$ define $\Phi_{\epsilon,p} : \Omega \to \Omega$ as

"Φ_ϵ with p decoupled from all other $q \in \Lambda$"

Let $T_{\epsilon,p} = \Phi_{\epsilon,p} \circ T$. Then

$$|T_\epsilon^{*N}\mu - T_{\epsilon,p}^{*N}\mu| \le CN\epsilon\,\mathrm{Var}(\mu)$$

Observe: Switching on/off the coupling in a lattice of size L is a "perturbation" of size $NL\epsilon$. Here each μ_p is treated separately, the perturbation is of size $N\epsilon$.

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p \in \Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \qquad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p \in \Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \qquad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

Example: $\Lambda = \{1, 2, 3\}$, $d\mu(x_1, x_2, x_3) = h(x_1, x_2, x_3) dx_1 dx_2 dx_3$,

$$h_1(x_1, x_2, x_3) := \int h(u, x_2, x_3)\, du$$
$$h_2(x_1, x_2, x_3) := \int h(u, v, x_3)\, dudv$$
$$h_3(x_1, x_2, x_3) := \int h(u, v, w)\, dudvdw = 0$$

Then

$$h = (h - h_1) + (h_1 - h_2) + (h_2 - h_3)$$

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p \in \Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \quad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

$$\text{Let } \bar{\mu} := (\mu_p)_{p \in \Lambda}, \qquad \|\bar{\mu}\| := \sup_p \|\mu_p\|$$

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p \in \Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \quad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

$$\text{Let } \bar{\mu} := (\mu_p)_{p \in \Lambda}, \qquad \|\bar{\mu}\| := \sup_p \|\mu_p\|$$

Conclusion

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p\in\Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \quad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

$$\text{Let } \bar{\mu} := (\mu_p)_{p\in\Lambda}, \qquad \|\bar{\mu}\| := \sup_p \|\mu_p\|$$

Conclusion

$$\|T_{\epsilon,p}^{*N}\mu_p\| \le C\cdot\sigma_0^N\cdot\|\mu_p\|, \qquad \|\bar{T}_\epsilon^{*N}\bar{\mu}\| \le C\cdot(N^d\sigma_0^N + N^{d+1}\epsilon)\cdot\|\bar{\mu}\|$$

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p\in\Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \quad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

$$\text{Let } \bar\mu := (\mu_p)_{p\in\Lambda}, \qquad \|\bar\mu\| := \sup_p \|\mu_p\|$$

Conclusion

$$\|T^{*N}_{\epsilon,p}\mu_p\| \le C\cdot\sigma_0^N\cdot\|\mu_p\|, \qquad \|\bar T^{*N}_{\epsilon}\bar\mu\| \le \underbrace{C\cdot(N^d\sigma_0^N + N^{d+1}\epsilon)}_{<\frac{1}{2}\text{ by choice of } N \text{ and } \epsilon}\cdot\|\bar\mu\|$$

Ingredients of the proof

Telescoping Let $\mu = \mu' - \mu''$.

$$\mu = \sum_{p\in\Lambda} \underbrace{\mu_p}_{\text{signed measure}} \quad \text{where} \quad \mu_p(f) = 0 \text{ if } \partial_p f = 0$$

$$\text{Let } \bar{\mu} := (\mu_p)_{p\in\Lambda}, \qquad \|\bar{\mu}\| := \sup_p \|\mu_p\|$$

Conclusion

$$\|T_{\epsilon,p}^{*N}\mu_p\| \leq C\cdot\sigma_0^N\cdot\|\mu_p\|, \qquad \|\bar{T}_\epsilon^{*N}\bar{\mu}\| \leq \underbrace{C\cdot(N^d\sigma_0^N + N^{d+1}\epsilon)}_{<\frac{1}{2} \text{ by choice of } N \text{ and } \epsilon}\cdot\|\bar{\mu}\|$$

$$\|\bar{T}_\epsilon^n\bar{\mu}\| \leq C\cdot\sigma^n\cdot\|\bar{\mu}\| \quad \text{for all } n$$

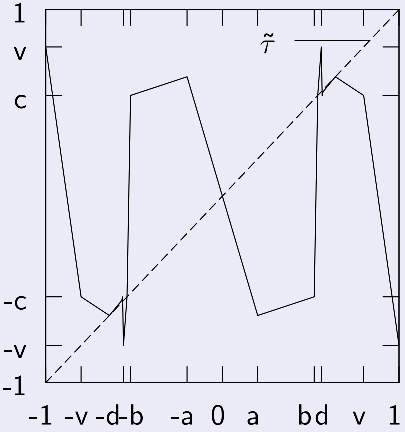
Attention! $|T_\epsilon^n\mu| \not\to 0$

Example for a phase transition

- $\Lambda = \mathbb{Z}^2, \quad (\Phi_\epsilon x)_p = (1-\epsilon)x_p + \frac{\epsilon}{2}(x_{p+e_1} + x_{p+e_2})$

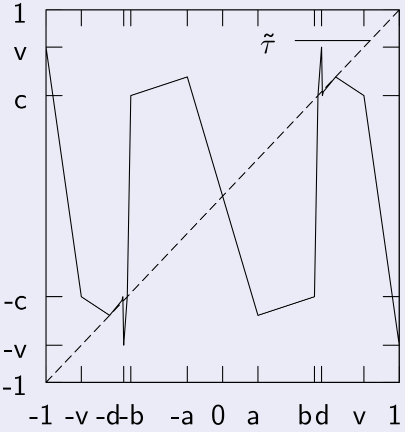
Example for a phase transition

- $\Lambda = \mathbb{Z}^2, \quad (\Phi_\epsilon x)_p = (1-\epsilon)x_p + \frac{\epsilon}{2}(x_{p+e_1} + x_{p+e_2})$
- $\tau = \frac{1}{v}\tilde{\tau}^k$



Example for a phase transition

- $\Lambda = \mathbb{Z}^2, \quad (\Phi_\epsilon x)_p = (1-\epsilon)x_p + \frac{\epsilon}{2}(x_{p+e_1} + x_{p+e_2})$
- $\tau = (\frac{1}{v}\tilde{\tau}^k)^3$



Theorem (Bardet/K., to appear in Nonlinearity)

There are $0 < \epsilon_1 < \epsilon_2 < \eta < \frac{1}{4}$ such that the following hold:

a) For $\epsilon \in [0, \frac{1}{4}]$, the map T_ϵ has at least one invariant probability measure in $\mathcal{B}(\Omega)$ which is also translation invariant.

Theorem (Bardet/K., to appear in Nonlinearity)

There are $0 < \epsilon_1 < \epsilon_2 < \eta < \frac{1}{4}$ such that the following hold:

a) For $\epsilon \in [0, \frac{1}{4}]$, the map T_ϵ has at least one invariant probability measure in $\mathcal{B}(\Omega)$ which is also translation invariant.

b) For $\epsilon \in [0, \epsilon_1]$, the map T_ϵ has a unique invariant probability measure in $\mathcal{B}(\Omega)$. (This measure is necessarily also translation invariant.)

Theorem (Bardet/K., to appear in Nonlinearity)

There are $0 < \epsilon_1 < \epsilon_2 < \eta < \frac{1}{4}$ such that the following hold:

a) For $\epsilon \in [0, \frac{1}{4}]$, the map T_ϵ has at least one invariant probability measure in $\mathcal{B}(\Omega)$ which is also translation invariant.

b) For $\epsilon \in [0, \epsilon_1]$, the map T_ϵ has a unique invariant probability measure in $\mathcal{B}(\Omega)$. (This measure is necessarily also translation invariant.)

c) For $\epsilon \in [\epsilon_2, \eta]$, the map T_ϵ has at least two invariant probability measures μ_ϵ^+ and μ_ϵ^- in $\mathcal{B}(\Omega)$.

Theorem (Bardet/K., to appear in Nonlinearity)

There are $0 < \epsilon_1 < \epsilon_2 < \eta < \frac{1}{4}$ such that the following hold:

a) For $\epsilon \in [0, \frac{1}{4}]$, the map T_ϵ has at least one invariant probability measure in $\mathcal{B}(\Omega)$ which is also translation invariant.

b) For $\epsilon \in [0, \epsilon_1]$, the map T_ϵ has a unique invariant probability measure in $\mathcal{B}(\Omega)$. (This measure is necessarily also translation invariant.)

c) For $\epsilon \in [\epsilon_2, \eta]$, the map T_ϵ has at least two invariant probability measures μ_ϵ^+ and μ_ϵ^- in $\mathcal{B}(\Omega)$.

d) For $\epsilon \in [0, \frac{1}{4}]$ and each $L \in \mathbb{N}$ there is a unique a.c. $\mu_{\epsilon,L}$.

Theorem (Bardet/K., to appear in Nonlinearity)

There are $0 < \epsilon_1 < \epsilon_2 < \eta < \frac{1}{4}$ such that the following hold:

a) For $\epsilon \in [0, \frac{1}{4}]$, the map T_ϵ has at least one invariant probability measure in $\mathcal{B}(\Omega)$ which is also translation invariant.

b) For $\epsilon \in [0, \epsilon_1]$, the map T_ϵ has a unique invariant probability measure in $\mathcal{B}(\Omega)$. (This measure is necessarily also translation invariant.)

c) For $\epsilon \in [\epsilon_2, \eta]$, the map T_ϵ has at least two invariant probability measures μ_ϵ^+ and μ_ϵ^- in $\mathcal{B}(\Omega)$.

d) For $\epsilon \in [0, \frac{1}{4}]$ and each $L \in \mathbb{N}$ there is a unique a.c. $\mu_{\epsilon,L}$.

Remark: τ can be chosen to be an analytic circle endomorphism.

Theorem (Bardet/K., to appear in Nonlinearity)

There are $0 < \epsilon_1 < \epsilon_2 < \eta < \frac{1}{4}$ such that the following hold:

a) For $\epsilon \in [0, \frac{1}{4}]$, the map T_ϵ has at least one invariant probability measure in $\mathcal{B}(\Omega)$ which is also translation invariant.

b) For $\epsilon \in [0, \epsilon_1]$, the map T_ϵ has a unique invariant probability measure in $\mathcal{B}(\Omega)$. (This measure is necessarily also translation invariant.)

c) For $\epsilon \in [\epsilon_2, \eta]$, the map T_ϵ has at least two invariant probability measures μ_ϵ^+ and μ_ϵ^- in $\mathcal{B}(\Omega)$.

d) For $\epsilon \in [0, \frac{1}{4}]$ and each $L \in \mathbb{N}$ there is a unique a.c. $\mu_{\epsilon,L}$.

Remark: τ can be chosen to be an analytic circle endomorphism.

Proof by approximating Toom's PCA (cf. Gielis/MacKay)

Combinatorics: Lebowitz/Maes/Speer,
Analytic estimates: transfer operator, bounded variation

Summary

- For locally coupled piecewise expanding interval maps we proved
  - Uniqueness of an SRB measure for small coupling
  - Possibility of phase transition on $\Lambda = \mathbb{Z}^2$

Summary

- For locally coupled piecewise expanding interval maps we proved
  - Uniqueness of an SRB measure for small coupling
  - Possibility of phase transition on $\Lambda = \mathbb{Z}^2$
- Method extends to other "systems" where the local system can be described in terms of a linear operator with spectral gap.

Summary

- For locally coupled piecewise expanding interval maps we proved
  - Uniqueness of an SRB measure for small coupling
  - Possibility of phase transition on $\Lambda = \mathbb{Z}^2$
- Method extends to other "systems" where the local system can be described in terms of a linear operator with spectral gap.

Questions

- Uniqueness in $\mathcal{B}(\Omega)$, not in AC. Is ACC a good class?
- Invariant measures determined by restriction to spatial tail field?
- Phase transitions when also Φ_ϵ bi-analytic?
- Phase transitions on $\Lambda = \mathbb{Z}$?