Constrained Willmore Tori in the 4–Sphere

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Constrained Willmore Surfaces

Definition

A conformal immersion $f\colon M \to S^4 = \mathbb{R}^4 \cup \{\infty\}$ of a Riemann surface M is *constrained Willmore* if it is a critical point of the Willmore functional $\mathcal{W} = \int |\mathring{\mathbf{I}}|^2 dA$ under **conformal** variations.

(Willmore surfaces "=" critical pts. of $\mathcal{W}$ under **all** variations.)

Functional and constraint are conformally invariant

$\rightsquigarrow$ Möbius geometric treatment, e.g. in framework of quaternionic model of conformal 4–sphere $S^4 = \mathbb{HP}^1$

Examples

- CMC in 3D space–forms $\rightsquigarrow$ constrained Willmore
- Minimal in 4D space–forms $\rightsquigarrow$ Willmore
- Hamiltonian Stationary Lagrangian in $\mathbb{R}^4$ $\rightsquigarrow$ constr. Willmore

Prototype for Main Result: Harmonic Tori in S^2

Theorem

A harmonic map $f\colon T^2 \to S^2 = \mathbb{CP}^1$ *is either*

- *holomorphic or*
- *of finite type.*

More precisely:

If $deg(f) \neq 0$, then f is (anti–)holomorphic (Eells/Wood)
If $deg(f) = 0$, then f is of finite type (Pinkall/Sterling)

Finite type "$=$"

- attached to f is a Riemann surface Σ of finite genus called the *spectral curve* and
- the map f is obtained by *"algebraic geometric"* or *"finite gap" integration*

The Main Result: Constrained Willmore Tori in S^4

Theorem

A constrained Willmore immersion $f\colon T^2 \to S^4 = \mathbb{HP}^1$ is either

- *"holomorphic" (i.e., super–conformal or Euclidean minimal) or*
- *of finite type.*

Where:

- super–conformal "=" f is obtained by Twistor projection $\mathbb{CP}^3 \to \mathbb{HP}^1$ from holomorphic curve in $\mathbb{CP}^3$
- Euclidean minimal "=" there is a point $\infty \in S^4$ such that $f\colon T^2 \backslash \{p_1, \dots, p_n\} \to \mathbb{R}^4 = S^4 \backslash \{\infty\}$ is an Euclidean minimal surface with planar ends $p_1, \dots, p_n$.

Holomorphic Case versus Finite Type Case

Theorem implies that all constrained Willmore tori admit explicit parametrization by methods of complex algebraic geometry.

Holomorphic case (e.g. twistor case):

$$
\begin{tikzcd}
 & \mathbb{CP}^3 \arrow[d, "twistor"] \\
T^2 \arrow[r, "f"] \arrow[ur, dotted, "hol."] & \mathbb{HP}^1
\end{tikzcd}
$$

Finite type case:

$$
\begin{tikzcd}
\widehat{Jac}(\Sigma) \arrow[r, "hol."] & \mathbb{CP}^3 \arrow[d, "twistor"] \\
T^2 \arrow[r, "f"] \arrow[u, dotted, "linear"] & \mathbb{HP}^1
\end{tikzcd}
$$

Previous Results

- CMC tori are of finite type (Pinkall,Sterling; 1989)
  (CMC $\Leftrightarrow$ Gauss map $N\colon T^2 \to S^2$ harmonic)
- 1.) Burstall, Ferus, Hitchin, Pedit, Pinkall, Sterling ($\approx$ 90)
  S^2–result generalizes to various symmetric target spaces
  2.) Willmore $\Leftrightarrow$ conformal Gauss map harmonic

  1.)+2.) $\rightsquigarrow$ Conjecture: Willmore tori in S^3 are of finite type

  Schmidt 2002: constrained Willmore in S^3 are of finite type
- Willmore tori in S^4 with non–trivial normal bundle are of "holomorphic" type (Leschke, Pedit, Pinkall; 2003)

Strategy: Adopt Hitchin Approach to Harmonic Tori in S^2

0.) Formulate as zero–curvature equation with spectral parameter $\rightsquigarrow$ associated family ∇^μ of flat connections depending on spectral parameter $\mu \in \mathbb{C}^*$

- Harmonic maps to $S^2 = \mathbb{CP}^1$: complex rank 2 bundle
- constrained Willmore in $S^4 = \mathbb{HP}^1$: complex rank 4 bundle

1.) Which holonomy representations $H^\mu \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SL}_4(\mathbb{C})$ can occur for ∇^μ if underlying surface is torus $T^2 = \mathbb{C}/\Gamma$?

2.) Non–trivial holonomy
$\Longrightarrow$ existence of polynomial Killing field
$\Longrightarrow$ finite type

3.) Trivial holonomy $\cong$ "holomorphic" case

Implementation of these ideas in constrained Willmore Case needs results from quaternionic holomorphic geometry.

The Quaternionic Model of Surface Theory in S^4

Immersion $f\colon M \to S^4 = \mathbb{HP}^1 \quad \longleftrightarrow \quad$ line subbundle $L \subset \mathbb{H}^2$

Mean curvature sphere congruence $\quad \longleftrightarrow$

complex structure $S \in \Gamma(\mathrm{End}(\mathbb{H}^2))$ with $S^2 = -\mathrm{Id}$

2–sphere at $p \in M \quad \longleftrightarrow \quad$ eigenlines of S_p in $\mathbb{HP}^1$

S induced decomposition of trivial connection d

$$d = \underbrace{\partial + \bar\partial}_{S \text{ commuting}} + \underbrace{A + Q}_{S \text{ anti–comm.}}$$

∂ and A are of type K, i.e., complex str. on M acts by $*\omega = S\omega$
$\bar\partial$ and Q are of type $\bar K$, i.e., complex str. on M acts by $*\omega = -S\omega$

The Hopf Fields of a Conformal Immersion

A and Q are tensor fields called the *Hopf fields* of f.

- the Hopf fields measure the local change of S along immersion
- Willmore functional measures “global change of S”

$$\mathcal{W} = \int_M A \wedge *A = \int_M Q \wedge *Q$$

- Euler–Lagrange Equation of constrained Willmore surfaces (for compact M) is

$$d(2*A+\eta) = 0 \qquad \text{for} \qquad \begin{array}{r} \eta \in \Omega^1(\mathrm{End}(\mathbb{H}^2)) \\ \ker(\eta) = \mathrm{im}(\eta) = L \end{array}$$

Lagrange–multiplier η “is” holomorphic quadratic differential
Willmore surface $\longleftrightarrow \eta = 0$

The Associated Family of Constrained Willmore Surfaces

The associated family of a constrained Willmore immersion is the family of flat complex connections on the trivial complex rank 4 bundle $\mathbb{C}^4 = (\mathbb{H}^2, \mathbf{i})$

$$\nabla^\mu = d + (\mu - 1)A_\circ^{(1,0)} + (\mu^{-1} - 1)A_\circ^{(0,1)} \qquad \mu \in \mathbb{C}^*$$

where

- $A_\circ$ is defined by $2 * A_\circ = 2 * A + \eta$ and where
- $(1,0)$ and $(0,1)$ denote the decomposition into forms satisfying $*\omega = \omega\mathbf{i}$ and $*\omega = -\omega\mathbf{i}$.

Eigenlines of the Holonomy of ∇^μ on Torus

Flat connections on torus $\rightsquigarrow$ study holonomy and its eigenlines

AIM: if possible, define eigenline spectral curve Σ_{hol}
"=" unique Riemann surface $\Sigma_{hol} \xrightarrow{\mu} \mathbb{C}^*$ parametrizing non–trivial eigenlines of $H^\mu(\gamma)$

Eigenvalue of holonomy $H^\mu(\gamma)$ for one $\gamma \in \Gamma$

$\xrightarrow{\Gamma \text{abelian}}$ simultaneous eigenline of $H^\mu(\gamma)$ for all $\gamma \in \Gamma$

$\longrightarrow$ section $\psi \in \Gamma(\tilde{\mathbb{H}}^2)$ on universal cover $\mathbb{C}$ of torus with

- $\nabla^\mu \psi = 0$ and
- $\gamma^* \psi = \psi h_\gamma$ for all $\gamma \in \Gamma$ and some $h \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$

Be definition, such solution to $\nabla^\mu \psi = 0$ satisfies

$$d\psi = (1-\mu)A_\circ^{(1,0)} + (1-\mu^{-1})A_\circ^{(0,1)} \in \Omega^1(\tilde{L})$$

Link to Quaternionic Holomorphic Geometry

Immersion $f \rightsquigarrow$ quaternionic holomorphic structure on $\mathbb{H}^2/L$ (operator D whose kernel contains projections of all $v \in \mathbb{H}^2$)

Lemma

For every $h \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$, there is 1–1-correspondence between

- *holomorphic sections φ of $\mathbb{H}^2/L$ with monodromy h and*
- *sections $\psi \in \Gamma(\tilde{\mathbb{H}}^2)$ with*

$$d\psi \in \Omega^1(\tilde{L}) \qquad \textit{and} \qquad \gamma^*\psi = \psi h_\gamma \quad \textit{for all} \quad \gamma \in \Gamma.$$

The correspondence is given by $\psi \mapsto \varphi = [\psi]$.

Definition

The section ψ is called *prolongation* of the holomorphic section φ.
The map $L^\# := \psi\mathbb{H}$ is called a *Darboux transform* of f.

Taimanov–Schmidt Spectral Curve of Degree 0 Tori

Definition

Taimanov–Schmidt spectral curve Σ_{mult} of a conformally immersed torus f in $S^4 = \mathbb{HP}^1$ with trivial normal bundle is normalization of

$$\{h \in \mathrm{Hom}(\Gamma, \mathbb{C}^*) \mid \text{ monodromy of holomorphic section of } \mathbb{H}^2/L\}$$

Theorem

The set $\{h \in ...\}$ *is a 1–dimensional complex analytic subset of* $\mathrm{Hom}(\Gamma, \mathbb{C}^*) \cong \mathbb{C}^* \times \mathbb{C}^*$. *Moreover, for generic* $h \in \Sigma_{mult}$, *the space of holomorphic sections is complex 1–dimensional.*

This implies that generic holonomies $H^\mu(\gamma)$ have

- an even number of simple eigenvalues that are non–constant as functions of μ (called *non–trivial eigenvalues*) and
- $\lambda = 1$ as an eigenvalue of even multiplicity (*trivial eigenvalue*).

The non–trivial Normal Bundle Case (Degree $\neq 0$)

In the case of non–trivial normal bundle, the quaternionic Plücker formula implies that the only possible eigenvalue of the holonomies $H^{\mu}(\gamma)$ is $\lambda = 1$.

List of possible Types of Holonomy Representations

Lemma

For constrained Willmore tori in S^4, the holonomy $H^\mu(\gamma)$ of the associated family ∇^μ belongs to one of the following cases:

- I. *generically $H^\mu(\gamma)$ has 4 different eigenvalues,*
- II. *generically $H^\mu(\gamma)$ has $\lambda = 1$ as an eigenvalue of multiplicity 2 and 2 non–trivial eigenvalues,*
- IIIa. *all holonomies $H^\mu(\gamma)$ are trivial, or*
- IIIb. *all holonomies $H^\mu(\gamma)$ are of Jordan type with eigenvalue 1 (and have 2×2 Jordan blocks).*

Non–trivial normal bundle $\rightsquigarrow$ holonomy belongs to Case III

Non–trivial Holonomy, Polynomial Killing Field (Case I)

Can define eigenline curve $\Sigma_{hol} \xrightarrow[4:1]{\mu} \mathbb{C}^*$ of $\mu \mapsto H^\mu(\gamma)$

- Γ abelian $\rightsquigarrow$ independent of choice of $\gamma \in \Gamma \backslash \{0\}$
- map $\Sigma_{hol} \to \Sigma_{mult}$ is (essentially) biholomorphic

AIM: construct polynomial Killing field, i.e., a family of sections of $\mathrm{End}_{\mathbb{C}}(\mathbb{H}^2, \mathbf{i})$ that is polynomial in μ and satisfies $\nabla^\mu \xi(\mu, .) = 0$ or, equivalently, a solution $\xi(\mu, p) = \sum_{j=0}^k \xi_j(p)\mu^j$ to Lax–equation

$$d\xi = [(1-\mu)A_\circ^{(1,0)} + (1-\mu^{-1})A_\circ^{(0,1)}, \xi].$$

Such ξ commutes with all $H^\mu(\gamma)$

$\rightsquigarrow$ same eigenline curve

$\rightsquigarrow$ Σ_{hol} can be compactified by filling in points over $\mu = 0, \infty$

Hitchin Trick

The $(1,0)$ and $(0,1)$–parts of ∇^μ extend to $\mathbb{C}$ and $\mathbb{C}^* \cup \{\infty\}$:

$$\partial^\mu = (\nabla^\mu)^{(1,0)} = \partial + (\mu - 1)A_\circ^{(1,0)}$$
$$\bar\partial^\mu = (\nabla^\mu)^{(0,1)} = \bar\partial + (\mu^{-1} - 1)A_\circ^{(0,1)}$$

Theorem

For a holomorphic family $F(\lambda)$, $\lambda \in U \subset \mathbb{C}$ of Fredholm operators,

- *the minimal kernel dimension of $F(\lambda)$, $\lambda \in U$ is generic and*
- *the holomorphic bundle $\mathcal{V}_\lambda = \ker(F(\lambda))$ defined at generic points holomorphically extends through the isolated points of higher dimensional kernel.*

Apply to ∂^μ and $\bar\partial^\mu$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^4) = \mathrm{End}_{\mathbb{C}}(\mathbb{H}^2, \mathbf{i}) \rightsquigarrow$ rank 4 bundle $\mathcal{V} \to \mathbb{CP}^1$ whose fiber $\mathcal{V}_\mu$, $\mu \in \mathbb{C}^*$ is $\{\nabla^\mu - \text{parallel sections}\}$ and whose meromorphic sections are polynomial Killing fields.

The Case of Trivial Holonomy

Case IIIa: apply Hitchin trick to ∂^μ and $\bar{\partial}^\mu$ on $\mathbb{C}^4 = (\mathbb{H}^2, \mathbf{i})$
$\rightsquigarrow$ rank 4 bundle $\mathcal{V} \to \mathbb{CP}^1$ whose fiber $\mathcal{V}_\mu$, $\mu \in \mathbb{C}^*$ is
$\{\nabla^\mu - \text{parallel sections of } \mathbb{C}^4\}$

Investigating the asymptotics of holomorphic sections of $\mathcal{V}$ at $\mu = 0$ or ∞ shows that Case IIIa is only possible if f is super–conformal or Euclidean minimal

Case IIIb: Hitchin trick $\rightsquigarrow$ existence of polynomial Killing field ξ with $\xi^2 = 0$

Investigating the asymptotics of ξ at $\mu = 0$ or ∞ shows that Case IIIb is only possible if f is Euclidean minimal.

Theorem

Let $f\colon T^2 \to S^4$ be constrained Willmore. Then either

I. f is of finite type and μ extends to covering $\Sigma \xrightarrow[4:1]{\mu} \mathbb{CP}^1$ or

II. f is of finite type and μ extends to covering $\Sigma \xrightarrow[2:1]{\mu} \mathbb{CP}^1$ or

IIIa. all holonomies are trivial and f is super–conformal or an algebraic Euclidean minimal surface or

IIIb. all holonomies are of Jordan type and f is a non–algebraic Euclidean minimal surface.

Non–trivial normal bundle $(deg(\perp_f) \neq 0)$

$\rightsquigarrow$ "Holomorphic" Case IIIa or IIIb $\rightsquigarrow$ Willmore

Trivial normal bundle $(deg(\perp_f) = 0)$ and not Euclidean minimal

$\rightsquigarrow$ Finite type Cases I or II

Willmore Case

- Let $f\colon T^2 \to S^4$ be Willmore and not Euclidean minimal. Then either
  - $deg(\perp_f) = 0$ and f belongs to Case I (and is of finite type with $\Sigma \xrightarrow[4:1]{\mu} \mathbb{CP}^1$) or
  - $deg(\perp_f) \neq 0$ and f belongs to Case IIIa (and is super–conformal).
- Euclidean minimal tori belong to Case IIIa or IIIb. In case that the normal bundle is trivial (as it is for minimal tori with planar ends in $\mathbb{R}^3 = S^3 \backslash \{\infty\}$)
  - one cannot define Σ_{hol} using ∇^μ,
  - but Taimanov–Schmidt spectral curve Σ_{mult} is well defined.

  Question: can Σ_{mult} be compactified? is Σ_{mult} reducible?
- Case II does not occur for Willmore tori (with $\eta = 0$).

Tori with Harmonic Normal Vectors

Theorem

If a conformal immersion $f\colon T^2 \to S^4 = \mathbb{HP}^1$ has the property that, for some point $\infty \in S^4$, one factor of the Gauss map

$$M \to Gr^+(2,4) = S^2 \times S^2$$

is harmonic, then f is constrained Willmore and belongs to

- *Case II of the Main Theorem if the harmonic factor is not holomorphic and to*
- *Case III if the factor is holomorphic.*

In Case III of the Main Theorem, there always exists $\infty \in S^4$ such that one factor of the Gauss map is holomorphic.
In Case II, if $\mathcal{W} < 8\pi$, there always exists $\infty \in S^4$ such that one factor of the Gauss map is harmonic.

Examples of Tori with Harmonic Normal Vectors

- CMC tori in $\mathbb{R}^3$
  ($H \neq 0$ case: Bobenko $\rightsquigarrow$ arbitrary genus)
- CMC tori in S^3
  (Bobenko $\rightsquigarrow$ arbitrary genus)
- Hamiltonian stationary tori
  (Helein, Romon $\rightsquigarrow$ harmonic map takes values in a great circle $\rightsquigarrow g = 0$)
- Lagrangian tori with conformal Maslov form
  (Castro, Urbano $\rightsquigarrow$ harmonic map is equivariant $\rightsquigarrow g \leq 1$)